Mathematics. — *A theorem about ideals in commutative rings*. By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

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Let A and B be two ideals of a commutative ring O. The greatest common divisor (abbreviated g.c.d.) "in idealsense" of A and B is the ideal $(A; B)$ generated by the set of all elements $p + q$, if $p \subset A$ and $q \subset B$. The g.c.d. of two principal ideals (a) and (b) is to be written $(a; b)$. When the ring O is an integral domain with a unity element, of which the elements have the property of the Factor Theorem of Algebra, each pair of elements a and b of O has a g.c.d.; that means an element c, such that all common divisors of a and b are also divisors of c, and conversely all divisors of c are divisors of a and b. We write the g.c.d. in elementsense in this way: $\{a; b\} = c$.

Let O be an integral domain with unity element, whose elements satisfy the Factor Theorem of Algebra; if we suppose a and b to be two elements of O with $\{a; b\} = c$, then generally the g.c.d. in idealsense of (a) and (b) is not (c); in other words, the two notions of g.c.d. generally do not correspond with each other. (B. L. VAN DER WAERDEN, Moderne Algebra I, 2nd. ed. p. 66.)

Theorem. In a principal idealring both notions of g.c.d. correspond with each other; conversely, if O is an integral domain with unity element, for which the basistheorem of HILBERT is satisfied and for which both notions of g.c.d. correspond with each other, O is a principal idealring.

Proof. The first part is simple enough: let O be a principal idealring, then the g.c.d. of (a) and (b) is an ideal $(c) = (a; b)$, generated by an element c and so, that $c = r \cdot a + s \cdot b$, while $a = g \cdot c$ and $b = h \cdot c$; this means, that $\{a; b\} = c$ and both notions of g.c.d. correspond with each other. Conversely, if in a ring O both notions of g.c.d. correspond with each other, then for each pair of elements a and b of $O : (a; b) = (c)$. Thus for every three elements a_1, a_2 and $a_3 : (a_1; a_2; a_3) = ((a_1; a_2); a_3) = (c)$, and as this may obviously be extended to a finite set $a_1, a_2, \ldots, a_r$ of O: $(a_1; a_2; \ldots; a_r) = (c)$.

On account of the supposition that O satisfies the basistheorem of HILBERT, each ideal of O has a finite basis $(a_1; \ldots; a_r)$ and if moreover O is an integral domain with unity element, then O is a principal idealring.

Observation. It is possible to demonstrate by means of an example, if each pair of elements a and b of a ring O defines an ideal $(a; b) = (c)$, generated by an element c, that this information does not include the fact that each ideal of O must be a principal ideal.

Therefore we consider a field K and a function w on K to an ordered valuationgroup Γ. Then we define $w(a)$ for every element a of K; the function w is said to define a valuation of K if $w(a \cdot b) = w(a) + w(b)$, $w(a + b) \geqq \min(w(a); w(b))$; to each $\alpha \subset \Gamma$ corresponds at least one element $a \subset K$ with $w(a) = \alpha$. The set of elements a with $w(a) \geqq 0$ we call the valuationring O of K. To each ideal A of O we attach a valuation $w(A)$, being the lowest bound of the valuations of its elements. Furthermore we say, an ideal A of O has the symbol f or i according to whether A has elements with the value $w(A)$ or not. An ideal of the symbol f is always a principal ideal; on the other hand an ideal of the symbol i never has a finite basis. When we study a valuationring O of a valued field K with dense valuationgroup Γ, then it is evident that the ideal B of all elements b with $w(b) > 0$ is an ideal of the symbol i and B does not possess a finite basis. Of each pair of ideals in O at least one is a divisor of the other. Thus,when a and b are two elements of O with $w(a) > 0$ and $w(b) > 0$, then (a) must be divisor of (b) — or conversely.

But this means that $(a; b) = (a)$ or (b) for each pair a and b of O, while the prime-ideal B of all elements b with $w(b) > 0$ is not a principal ideal because B has no element with a lowest positive value. The valuationring O, notwithstanding the fact that O is an integral domain with unity element and satisfying for each pair of elements a and b of O the condition $(a; b) = (c)$, cannot be a principal idealring.