Mathematics. — *Some topological problems.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

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In this paper I shall give a survey about some topological problems; for shortness I am not giving any proofs; these are to be published later on.

When mentioning a set or a space I always mean a separable set or space (regular space with countable base), i.e., topologically speaking, a subset of the HILBERT space R_ω or — when the space has a finite dimension — even a subset of the n-dimensional EUCLIDIAN space R_n.

1. *Topological classification of all microcompact countable spaces.* The topological classification of a certain family of sets — often called the chief problem of topology — has been achieved for but a small number of special families, for instance the well-known family of all closed surfaces.

It is, however, not very difficult to give all topological invariants of the family of all compact countable spaces A. By constructing the transfinite, but always countable sequence of the derived sets of A:

$$A = A_0 \supset A_1 \supset \ldots A_\omega \supset \ldots,$$

there is formed a last not-vacuous derived set A_λ, where λ is an ordinal number of the first or second class. A_λ consists of a finite number — say l — of points. Thus one may attach to each compact countable space A the pair of numbers (λ, l) which we call the *degree* of A; conversely with every degree (λ, l) there exists a compact countable space A (the degree of the vacuous set being $(0, 0)$). It may be proved that the degree is a topological invariant. Hence:

Theorem I. *Two compact countable spaces are different in a topological sense if, and only if, their degrees differ. By considering all possible degrees (λ, l) one has a topological classification of the compact countable spaces.*

Secondly, let us consider a microcompact, not necessarily compact, countable space B. Denote by μ the first ordinal number (of the first or second class) for which the derived set B_μ is compact. B_μ is permitted to be vacuous. Let (λ, l) be the degree of $A = B_\mu$; then we define (μ, λ, l) as the *degree* of B. Thus we may prove:

Theorem II. *Two microcompact countable spaces are different in a topological sense if, and only if, their degrees differ. By considering all possible degrees (μ, λ, l) one has a topological classification of the microcompact countable spaces.*

In particular *we have thus given a topological classification of all (e.g. in an R_n) closed countable sets.*

I have not been able, however, to give a topological classification of all countable sets, not even of all countable sets with a not-vacuous nucleus [1]).

2. *Continuous classification of all countable and all compact 0-dimensional sets.*

Let us consider the following property of a countable set A: "A is compact and the degree of A is $< (\lambda, l)$ [2])". Denote this property by $(< \lambda, l)$. We may prove:

Theorem III. *All continuous invariants of the family of all countable spaces are given by* $(< \lambda, l)$.

Thus we have shown in particular: 1⁰. given an arbitrary non-compact countable space A and an arbitrary countable space B, there exists a continuous mapping of A on B; 2⁰. there exists a continuous mapping of a countable compact space A on a countable compact space B if, and only if, the degree of B is the same as or less than that of A.

Considering compact 0-dimensional spaces one has

Theorem IV. *All continuous invariants of the family of all compact 0-dimensional sets are the properties "countability" and "countability and* $(< \lambda, l)$*".*

Thus we have shown in particular that every compact uncountable 0-dimensional set may be mapped continuously on every compact 0-dimensional set.

Meanwhile I cannot give a topological classification of all compact 0-dimensional sets.

3. *Topological characterisation of all subsets of the* R_1. An M.K.-set (MOORE–KLINE) is defined as a set which 1⁰. is compact, 2⁰. whose components are points or (closed) simple arcs, 3⁰. where no interior point of a simple arc-component S is limit element of the complement $K—S$.

It is known (KLINE) that the M.K.-sets are identical with the *compact* subsets of the R_1. Thus all compact subsets of the R_1 are characterised topologically.

We now more generally define an M.K.-set as a set which satisfies the conditions $2'$ and $3'$: $2'$. the components are points or open, half-open or closed simple arcs (i.e. the topological image of an interval $0 < x < 1$, or $0 \leqq x < 1$, or $0 \leqq x \leqq 1$); $3'$. no interior point of a component S which is not a single point is limit element of the complement $K—S$.

Using a compactification-theorem of FREUDENTHAL and myself [3]), saying

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1) The nucleus being the largest subset of the given space, which is dense in itself.

2) $(\lambda, l) < (\mu, m)$, if $\lambda < \mu$ or $\lambda = \mu$, $l < m$.

3) See my thesis "Topologische Studiën", Groningen (1942), hereafter denoted by [1].

that the semicompact [4]) spaces M are identical with the spaces M which may be compactified by an 0-dimensional set N (i.e. $\overline{M} = M + N$), one may prove:

Theorem V. *The (generalised) semicompact M.K.-spaces are identical — in a topological sense — with the subsets of the set of real numbers.*

4. *Topologically ordered sets.*

Sometimes a (topological) space S may be ordered (in the ordinary way), i.e. one may define a relation of order between the points of S "without destroying the topological structure" of S. An example gives an arbitrary 0-dimensional space S: S is homoeomorphic with a subset of R_1 and therefore ordered. More in general Theorem V gives the possibility of ordering the generalised semicompact M.K.-spaces.

On the other hand consider a given ordered set M. By defining the neighbourhoods of a point $m \subset M$ as the open "intervals" (ab), containing m $(a < m < b)$, this relation of order induces a topology in M. This topology of the ordered set M we call the *induced topology*.

There now arises the following question: is it possible to order a certain (topological) space S in such a way that the topology, induced (by the ordering) in S, concurs with the given topology of S? If such an ordering is possible we call S a *perfectly ordered space*. It is easy to see that every compact M.K.-space may be perfectly ordered. If a space can be ordered it is not yet always possible to determine a *perfect ordering:* e.g. the ordered space consisting of the points x with $0 < x < 1$, $x = 2$. The ordered space $0 \leqq x < 1$, $x = 2$ also is not perfectly ordered, but one may determine a perfect ordering by taking the point $x = -1$ instead of the point $x = 2$.

We may now also give an exact definition of the term used in the first sentence of this number: "without destroying the topological structure". We define therefore: a (topological) space S may be *ordered* if S is homoeomorphic with a subspace of a perfectly ordered space.

One may prove

Theorem VI. *Any 0-dimensional set may be perfectly ordered.*

On the other hand a set of dimension > 0 may not always be perfectly ordered, as was already shown above.

Considering well-ordered sets instead of ordered sets one may define *perfectly well-ordered* and *well-ordered* sets in just the same way. The theory of these sets is mainly exhausted by the following theorems.

Theorem VII. *The microcompact countable spaces are identical with those sets which may be perfectly well-ordered.*

Theorem VIII. *The countable spaces with a vacuous nucleus [1]) are identical with those sets which may be well-ordered.*

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[4]) A space is semicompact if every point has arbitrarily small neighbourhoods with compact boundaries (ZIPPIN).

5. *The problem of acontinuity.*

Two sets A and B we call *continuously different* if there does not exist a continuous mapping of A on B, and conversely. The family $\{A, B\}$ is called an acontinuous family. An arbitrary family of sets $\{A, B, \dots, X, Y, \dots\}$ is called an *acontinuous family* if any two sets X and Y of the family are continuously different. Then all sets of the family are called continuously different. A simple arc and the set consisting of two points give us a simple example of two continuously different sets.

It is easy to show that for every natural number n there exists an acontinuous family of n sets. To construct, however, infinite acontinuous families offers a far more difficult problem. Yet they exist, and by constructing well chosen sets one may prove

Theorem IX. *There exist acontinuous families of the potency* $\aleph$ [5]*, the sets of which are all compact, connected and of dimension n* (n *being* $\geqq 1$).

Theorem X. *There exist acontinuous families of the potency* $2^{\aleph}$*, the sets of which are all connected and of dimension n* ($n \geqq 1$).

Because the potency of all compact sets is $\aleph$, and the potency of all sets is $2^{\aleph}$, it is impossible in this respect to give further-reaching results. Theorems IX and X in particular show the existence of $\aleph$ topologically different compact and $2^{\aleph}$ topologically different sets respectively.

Take a given arbitrary family of sets F. We form all possible acontinuous families F_i, each F_i consisting of sets belonging to F. Let m_i be the potency of F_i. If there exists an m_j with $m_j \geqq m_i$ for every index i, then m_j is called the *degree of acontinuity* of the family F. If the potency of F_i is finite for every i, but there does not exist an m_j with $m_j \geqq m_i$ for every i, then the degree of acontinuity of F is ω by definition. We may construct families with degree of acontinuity ω. Therefore the degree of acontinuity of a family may be

$$1, 2, \dots, n, \dots, \omega, \aleph_0, \aleph, 2^{\aleph}.$$

We denote the degree of acontinuity

of the family of all	by
countable sets	δ_1
compact 0-dimensional sets	δ_2
countable or microcompact 0-dimensional sets	δ_3
compact connected sets	δ_4
compact sets	δ_5
connected sets	δ_6
sets	δ_7
compact sets with an uncountable number of components	δ_8

[5] $\aleph_0$, $\aleph$ are the potencies of the sets of all natural and all real numbers respectively.

Theorem XI.
$$\delta_1 = \delta_2 = \delta_8 = 1$$
$$\delta_3 = 2$$
$$\delta_4 = \delta_5 = \aleph$$
$$\delta_6 = \delta_7 = 2^{\aleph}$$

I cannot, however, determine the degree of acontinuity δ_0 of the family of all 0-dimensional sets. I only know $\delta_0 \geqq 2$.

Consider an arbitrary family F and all acontinuous families F_i, consisting of sets belonging to F, each of them containing the same set $A \subset F$. One may define like we did above, the *degree of acontinuity of A in respect to F: the local degree of acontinuity.*

It is not difficult to prove:

Theorem XII. *The local degree of acontinuity of a simple arc in respect to the family of all compact sets is* 2.

6. *The extension of mappings.*

The problem of the extension of continuous and topological mappings has been studied extensively in topology. We shall add some new results to this extension-theory.

Considering not only separable, but also more in general completely regular spaces we may prove:

Theorem XIII. *Any continuous mapping f of a completely regular (resp. separable) space R on a completely regular (resp. separable) space R' may be extended to a continuous mapping $\overline{f}$ of a bicompact normal (resp. (bi)compact separable) space $\overline{R}$ on a bicompact normal (resp. (bi)compact separable) space $\overline{R'}$; R and R' are dense in $\overline{R}$ and $\overline{R'}$.*

Moreover it is possible to construct for any bicompact normal (resp. (bi)compact separable) space $\overline{R'}$ (R' lying everywhere dense in $\overline{R'}$) a required $\overline{R}$ and $\overline{f}$.

The analogon of this theorem for RIESZ-spaces holds also true, but is less important.

By using the method of topological products (LEFSCHETZ), the proof of Theorem XIII is fairly simple.

This theorem does not apply for every bicompactification $\overline{R}$ of R; on the contrary: the bicompactification $\overline{R}$ of R depends on $\overline{R'}$ and f.

Therefore, if R is a subset of a given bicompact space $\overline{R}$ there arises the problem under what circumstances a continuous extension remains possible.

With regard to *metrical* compact spaces $\overline{R}$ and $\overline{R'}$ it is known that a continuous mapping f of R on R' may be extended to a continuous mapping $\overline{f}$ of $\overline{R}$ on $\overline{R'}$ if, and only if, f is a *uniformly continuous mapping*. But the property of uniform continuity is not a topological invariant and depends on the given metric; therefore it is of no use in not-metrical spaces and of little use in respect to topology at all.

We shall now give (topologically invariant) conditions for f, R, resp. R', with regard to the inbedding in $\overline{R}$, resp. $\overline{R'}$, on which continuous extension is possible. These conditions are:

1⁰. f is a *closed continuous mapping*, i.e. a continuous transformation, mapping any closed subset of R on a closed (sub)set of R';

2⁰. R *is quasiconnected in respect to* $\overline{R}$—R.

This implies: taking an arbitrary point $p \subset \overline{R}-R$, there exist arbitrarily small neighbourhoods $U(p/R)$ of p in R ($U(p/R)$ apparently does not contain the point p), in which any two subsets having p for limit element are quasiconnected. *Two subsets S_1 and S_2 of a space S are quasiconnected* (in this space), if any open and closed subset of S containing S_1 also contains at least one point of S_2.

3⁰. $\overline{R'}$—R' *is discontinuous*, i.e. $\overline{R'}$—R' does not contain a subset being a continuum.

One may prove

Theorem XIV. *In a (separable) space are given two compact subsets $\overline{R} = R + A$ and $\overline{R'} = R' + A'$. If R is quasiconnected in respect to $A = \overline{R} - R$ and if $A' = \overline{R'} - R'$ is discontinuous, any closed continuous mapping f of R on R' may be extended to a (closed) continuous mapping $\overline{f}$ of $\overline{R}$ on $\overline{R'}$.*

Remarks. The analogon of this theorem for bicompact normal spaces $\overline{R}$ and $\overline{R'}$ holds true as well; in this case however condition 3⁰. has to undergo a small change. — Because every topological mapping is a closed continuous mapping, the theorem holds true in particular for topological mappings f. — A special case of this theorem has already been proved in [1]. — In this theorem are given sufficient conditions for continuous extension. I have examined too how far these conditions are necessary. They appear to be fairly general, but not absolutely necessary. It appears in particular that it is necessary — topologically speaking — to involve special conditions 1⁰. for the continuous mapping f, 2⁰. for the inbedding of R in $\overline{R}$, 3⁰. for the inbedding of R' in $\overline{R'}$, if for *every* f of the (by the given conditions determined) class of special continuous mappings there may exist a continuous extension $\overline{f}$.

Let D be the discontinuum of CANTOR, in other words a perfect (i.e. compact, dense in itself) 0-dimensional set. D is sometimes called *homogeneous*, because for any two points d_1 and d_2 of D there exists a topological mapping of D on itself, which maps d_1 on d_2 (or conversely).

How far is it possible to determine a topological mapping of D on itself which maps a given subset D_1 of D on a given subset D_2 of D? Of course D_1 and D_2 must be chosen as homoeomorphic sets. The answer gives

Theorem XV. *D_1 and D_2 are homoeomorphic closed proper subsets of the discontinuum D. A topological mapping t of D_1 on D_2 may be extended to a topological mapping of D on itself, if and only if, the boundary $R[D_1]$ (of D_1 in respect to D) is mapped by t on the boundary $R[D_2]$.*

This theorem is particularly of importance if the boundaries are identical with D_1 and D_2 themselves. This is the case when D_1 and D_2 are countable or when D_1 and D_2 are nowhere dense in D. In these cases any topological mapping of D_1 on D_2 may be extended to D.

Finally we are giving a generalisation of an extension-theorem of STOÏLOW.

Theorem XVI. *$\overline{I} = I + A$ is a compact set. I is a set consisting of isolated points, the limit elements of which form the compact set A. The same conditions hold true for $\overline{K} = K + B$. Any topological mapping of A on B may now be extended to a topological mapping of $\overline{I}$ on $\overline{K}$.*