**Mathematics.** — *Hexagonal three-webs.* By J. HAANTJES. (Communicated by Prof. J. A. SCHOUTEN.)

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*Summary.* In "Geometrie der Gewebe" by BLASCHKE and BOL [1]) the necessary and sufficient condition is given, which has to be satisfied in order that a 3-web be hexagonal. But this condition is ony usable when the three vector fields of the tangent vectors of the curves are known. This paper deals with the case that the 3-web is given by a cubic differential form, which coefficients are functions of the coordinates, and gives the condition which has to be satisfied by these coefficients in order that the 3-web involved be hexagonal.

A 3-web in a two-dimensional space is defined as a triple infinite system of curves with the property that through every point of a definite region passes one curve of each system. Let the curves through an arbitrarily chosen point $P$ of the region belonging to the first, second and third system be denoted by $C_1$, $C_2$ and $C_3$ respectively. Consider a point $A$ of $C_1$. Let $B$ be the point of intersection of the (3)-curve through $A$ and $C_2$, $C$ the point of intersection of the (1)-curve through $B$ and $C_3$, $D$ the point of intersection of the (2)-curve through $C$ and $C_1$ and so on till we get the points $A, B, C, D, E, F$ and $G$, where $G$ lies just as $A$ on $C_1$. If this point $G$ coincides with $A$ for every choice of $P$ and $A$ the 3-web is called a *hexagonal 3-web.*

In the following we consider the 3-web given by the null directions of a cubic differential form

$$p_{hij}\, dy^h\, dy^i\, dy^j, \qquad (h, i, \ldots = 1, 2) \quad . \; . \; . \; . \; . \quad (1)$$

which is supposed to have in each point of a certain region three different real null directions. The purpose of this paper is to derive the conditions which has to be satisfied by the $p_{hij}$ $(y^1, y^2)$ in order that this 3-web be hexagonal.

We shall arrive at the following result:

The (symmetrical) tensor $p_{hij}$ determines a tensor $g^{hi}$ up to a multiplicative factor by the equation

$$p_{hij}\, g^{hi} = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

We choose one of the solutions of this equation (different from the zero solution), which has always the rank two. Therefore there exists a covariant tensor $g_{hi}$ defined by

$$g_{ij}\, g^{jh} = A_i^h \begin{cases} = 0, & h \neq i \\ = 1, & h = i \end{cases} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

[1]) Springer, Berlin 1938, par. 17.

The $g^{hi}$ and $g_{ih}$ are used for the raising and lowering of indices. Then a covariant derivative can be defined by the equation

$$\nabla_j g_{hi} = p_{jhi} \quad \ldots \ldots \ldots \quad (4)$$

If the curvature affinor of this covariant derivative is denoted by $N_{kji}^{\cdot\cdot\cdot h}$, we may write

$$N_{ji} = N_{hji}^{\cdot\cdot\cdot h} = n_h p_{ji}^h + n\, g_{ji}, \quad \ldots \ldots \quad (5)$$

which equation determines the coefficients $n_h$ and $n$ uniquely.

*A hexagonal 3-web is characterised by the relation*

$$\partial_1 n_2 - \partial_2 n_1 = 0 \quad \ldots \ldots \ldots \quad (6)$$

*which means that $n_i$ has to be a gradient.*

In order to prove this theorem we choose the coordinate system so that the curves $y^1$ = constant and $y^2$ = constant form two of the three systems. Then the directions (1.0) and (0.1) satisfy the equation

$$p_{hij}\, dy^h\, dy^i\, dy^j = 0, \quad \ldots \ldots \ldots \quad (7)$$

from which it follows $p_{111} = p_{222} = 0$. As $p_{112} \neq 0$, because the three null directions are supposed to be different, and the 3-web only depends on the ratio of the components $p_{hij}$, we may take

$$p_{111} = p_{222} = 0 \qquad p_{112} = 1 \qquad p_{221} = -\beta \neq 0. \quad \ldots \quad (8)$$

From this we see that the vector $(\beta, 1)$ is a tangent vector of the curve of the third system. Now a 3-web is hexagonal if the third system has an equation of the form

$$\varphi(y^1) = \psi(y^2) + C, \quad \ldots \ldots \ldots \quad (9)$$

where C is a parameter of the system [2]). If $(\beta, 1)$ is the tangent vector of this system we have

$$\beta = \frac{\psi'(y^2)}{\varphi'(y^1)} \quad \ldots \ldots \ldots \quad (10)$$

In this case $\beta$ is the product of a function of $y^1$ and a function of $y^2$ and therefore

$$\partial_1 \partial_2 \log \beta = 0. \quad \ldots \ldots \ldots \quad (11)$$

Conversely if $\beta$ satisfies the equation (11) it can be written in the form (10) from which (9) follows by integration. Thus (11) is a necessary and sufficient condition that the 3-web be hexagonal.

From (2) and (8) we obtain the ratio of the components of the tensor $g^{hi}$. We choose

$$g^{11} = 2, \qquad g^{12} = \frac{1}{\beta}, \qquad g^{22} = \frac{2}{\beta^2}, \quad \ldots \ldots \quad (12)$$

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[2]) See f.i. W. BLASCHKE und G. BOL, l.c. pag. 17.

from which it follows

$$G_{11}=\frac{2}{3},\ g_{12}=-\frac{1}{3}\beta,\ g_{22}=\frac{2}{3}\beta^2. \quad \ldots\ldots \quad (13)$$

The parameters $\Pi^h_{ji}$ of the covariant derivative are on account of (4)

$$\Pi^h_{ji}=\left\{ {h \atop j\,i} \right\}-\frac{1}{2}p^{\ h}_{ji}, \quad \ldots\ldots \quad (14)$$

where $\left\{ {h \atop j\,i} \right\}$ are the CHRISTOFFEL symbols belonging to the tensor $g_{hi}$ and

$$p^h_{ji}=g^{hl}\,p_{jil}. \quad \ldots\ldots \quad (15)$$

With respect to the chosen coordinate system $p^h_{ji}$ has the following components

$$p^1_{11}=\frac{1}{\beta},\ p^1_{12}=-1,\ p^1_{22}=-2\beta,\ p^2_{11}=\frac{2}{\beta^2},\ p^2_{12}=-\frac{1}{\beta},\ p^2_{22}=-1, \quad (16)$$

whereas the components of $p^{hi}_j$ are

$$p^{11}_1=\frac{3}{\beta},\ p^{12}_1=\frac{3}{\beta^2},\ p^{22}_1=0,\ p^{11}_2=0,\ p^{12}_2=-\frac{3}{\beta},\ p^{22}_2=-\frac{3}{\beta^2} \quad . \quad (17)$$

The $\Pi^h_{ji}$ computed from (14) are

$$\left.\begin{aligned}\Pi^1_{11}&=-\frac{1}{3}\frac{\beta_1}{\beta}-\frac{1}{2\beta},\quad \Pi^1_{12}=\frac{2}{3}\beta_1-\frac{1}{2},\quad \Pi^1_{22}=-\frac{4}{3}\beta_1\beta+\beta,\ (\beta_i=\partial_i\beta)\\ \Pi^2_{11}&=-\frac{2}{3}\frac{\beta_1}{\beta^2}-\frac{1}{\beta^2},\quad \Pi^2_{12}=\frac{4}{3}\frac{\beta_1}{\beta}+\frac{1}{2\beta},\quad \Pi^2_{22}=\frac{\beta_2}{\beta}-\frac{2}{3}\beta_1+\frac{1}{2}\end{aligned}\right\}(18)$$

Then the $N_{ji}$ can be found from

$$N_{ji}\equiv\partial_h\Pi^h_{ji}-\partial_j\Pi^h_{hi}+\Pi^h_{hl}\Pi^l_{ji}-\Pi^h_{jl}\Pi^l_{hi} \quad \ldots \quad (19)$$

This leads to

$$\left.\begin{aligned}N_{11}&=-\frac{4}{3}\frac{\beta_{11}}{\beta}-\frac{2}{3}\frac{\beta_{12}}{\beta^2}+\frac{2}{3}\frac{\beta_1\beta_2}{\beta^3}+\frac{\beta_2}{\beta^3}-\frac{\beta_1}{\beta^2}-\frac{3}{2}\frac{1}{\beta^2}\\ N_{12}&=\frac{2}{3}\beta_{11}+\frac{1}{3}\frac{\beta_{12}}{\beta}-\frac{1}{3}\frac{\beta_1\beta_2}{\beta^2}-\frac{1}{2}\frac{\beta_2}{\beta^2}-\frac{\beta_1}{\beta}+\frac{3}{4}\frac{1}{\beta}=N_{21}\\ N_{22}&=-\frac{4}{3}\beta_{11}\beta-\frac{2}{3}\beta_{12}+\frac{2}{3}\frac{\beta_1\beta_2}{\beta}-\frac{1}{2}\frac{\beta_2}{\beta}+2\beta_1-\frac{3}{2}.\end{aligned}\right\}. \quad (20)$$

From

$$p^{ji}_l\,p^h_{ji}=I\,A^h_l \quad ; \quad I=\frac{9}{\beta^2} \quad \ldots\ldots \quad (21)$$

and (5) it follows

$$n_i = \frac{1}{I} N_{hj}\, p_i^{\;hj} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

This equation enables us to compute the components of $n_i$. They are

$$n_1 = - \partial_1 \log \beta \; ; \; n_2 = \frac{1}{2} \partial_2 \log \beta \quad . \; . \; . \; . \; . \quad (23)$$

from which it follows

$$\theta \equiv \partial_1 n_2 - \partial_2 n_1 = \frac{3}{2} \partial_1 \partial_2 \log \beta . \quad . \; . \; . \; . \; . \quad (24)$$

And as we have seen is a hexagonal 3-web exactly characterised by the vanishing of this expression. But with these considerations the proof is not yet complete. It is true that the result is independent of the special choice of the coordinate system, the left hand side of the equation (24) being an affinor, but we still have to show that $\theta$ remains unaltered when $p_{hij}$ and $g_{ij}$ are multiplied by arbitrary factors, for only the ratio of the components of these tensors are determined by the 3-web.

If $g_{ij}$ is multiplied by $\sigma^2$ and $p_{ij}^{h}$ by $\lambda$, the $\Pi_{ji}^{h}$ will change into

$$\Pi_{ji}^{h} \rightarrow \Pi_{ji}^{h} + T_{ji}^{h} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (25)$$

with

$$T_{ji}^{h} = A_j^h \sigma_i + A_i^h \sigma_j - g_{ij}\, \sigma^h - \frac{1}{2}(\lambda - 1)\, p_{ji}^{h} , \; (\partial_i = \partial_i \log \sigma) \, . \quad (26)$$

As a consequence we have

$$N_{ij} \rightarrow N_{ij} + \nabla_h T_{ji}^{h} - \nabla_j T_{hi}^{h} + T_{hl}^{h} T_{ji}^{l} - T_{jl}^{h} T_{hi}^{l} \quad . \; . \quad (27)$$

For the calculation of this expression the following relations are useful:

$$\nabla_j g^{hi} = - p_j^{\;hi} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

$$\left.\begin{aligned} \nabla_{[k} p_{j]i}^{h} &= g^{hm} \nabla_{[k} p_{j]im} - p_{[k}^{\;lm} p_{j]im} = g^{hm} \nabla_{[k} \nabla_{j]} g_{im} - \\ &- p_{[k}^{\;km} p_{j]im} = - \frac{1}{2} g^{hm} (N_{kjim} + N_{kjmi}) - p_{[k}^{\;km} p_{j]im} \end{aligned}\right\} . \quad (29)$$

from which

$$\nabla_h p_{ji}^{h} = - 2 N_{ji} + c\, g_{ij} \quad . \; . \; . \; . \; . \; . \quad (30)$$

where the coefficient $c$ need not be indicated more precisely. It turns out that

$$N_{ji} \rightarrow \lambda N_{ji} + p_{ji}^{h} \left(- \lambda \sigma_h - \frac{1}{2} \partial_h \lambda\right) + C g_{ji} \quad . \; . \; . \quad (31)$$

Consequently

$$n_i \rightarrow n_i - \sigma_i - \frac{1}{2}\partial_h \log \lambda, \quad \ldots\ldots \quad (32)$$

which shows that the new $n_i$ is obtained from the first one by addition of a gradient. So the expression

$$\theta = \partial_1 n_2 - \partial n_1 \quad \ldots\ldots\ldots\ldots \quad (33)$$

is invariant and our special choice of $g_{hi}$ and $p_{hij}$ has had no influence upon the result:

$\theta = 0$ *characterises a hexagonal 3-web.*