Mathematics. — *Space groups and their axioms.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

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Notations. Small letters denote a point of a set or an element of a group, but also the set consisting of one point or one element.

Capitals denote sets, while 0 indicates the vacuous set.

$U(p)$ is a neighbourhood of p.

$A \subset B$ or $B \supset A$ means: all points of A belong to B.

$A \cup B$ is the sum of A and B.

$\bigcup\limits_{i=1}^{\infty} A_i$ is the sum of the sets $A_1, A_2, \ldots$, etc.

$A \cap B$ is the intersection of A and B.

$A - B$ is the difference of A and B. This is only defined for $A \subset B$.

$A \cdot B$ or AB is the set of all products ab (the product indicates the relation in a group) with $a \subset A$ and $b \subset B$.

The notions: group, continuous and topological mapping, open and closed sets are supposed to be known.

1. In topology a general neighbourhood-space may be defined as a (not-vacuous) set satisfying both (A) and (A'):

(A) To each point p of the set belongs at least one subset as a neighbourhood. $U(p)$ always contains p. The system of all neighbourhoods is denoted by $\{U\}$.

(A') The HAUSDORFF *criterion of equivalence.* If in a set are introduced two systems of neighbourhoods $\{U\}$ and $\{V\}$ the corresponding neighbourhood-spaces will be considered as topologically not-different (or equivalent) if: each $U(p)$ of an arbitrary point p contains a $V(p)$ and, conversely, each $V(p)$ contains a $U(p)$.

If the neighbourhood-space G is defined by the system of neighbourhoods $\{U\}$, then the sets out of equivalent systems $\{V\}, \{W\}, \ldots$ are also called neighbourhoods in G.

A neighbourhood of a subset $A \subset G$ is defined as the sum of certain neighbourhoods $U(a)$, where a runs through the set A.

Further usually the following axioms are, starting from a general neighbourhood-space, successively introduced:

(B) The intersection of two neighbourhoods of one point is another neighbourhood of that point.

(C) Among the equivalent systems of neighbourhoods there is at least one system consisting of open sets.

(D') Of every two different points one has a neighbourhood which does not contain the other point (and conversely).

(D) Every two different points have disjunct neighbourhoods.

(E) Two disjunct closed sets of which at least one consists of only one point have disjunct neighbourhoods.

(F) Two disjunct closed sets have disjunct neighbourhoods.

(G) There is a system of neighbourhoods consisting of enumerably many sets.

Often the following names are used:

if for a general neighbourhood-space hold true the axioms

(B), (C),	it is called a *topological space*;
(B), (C), (D'),	„ „ „ „ RIESZ *space*;
(B), (C), (D'), (D),	„ „ „ „ HAUSDORFF *space*;
(B), (C), (D'), (D), (E),	„ „ „ „ *regular space*;
(B), (C), (D'), (D), (E), (F),	„ „ „ „ *normal space*;
(B), (C), (D'), (D), (E), (F), (G),	„ „ „ „ *separable space*.

2. By a *space group* R we shall mean a set R which

1°. is an (abstract) *group*,

2°. is a (general) *neighbourhood-space*; so the axioms (A) and (A') are satisfied,

3°. satisfies both *axioms of continuity* (indicating the connection between group and space), i.e.

a. for every two elements a and b of R and for each neighbourhood $U(ab)$ there may be found a $U(a)$ and a $U(b)$ with

$$U(a) \cdot U(b) \subset U(ab);$$

b. for every neighbourhood $U(a^{-1})$ of an arbitrary element a^{-1} of R may be found a $V(a)$ with $V^{-1} \subset U$.

The axioms (B), (C), etc., which in 1. were introduced for neighbourhood-spaces, may also be introduced for space groups and to the so formed space groups we may give the corresponding names.

It now stands to reason to suppose that for space groups there is a closer connection between the topological axioms than for neighbourhood-spaces. In the following will be investigated in how far this is the case and in how far therefore simplifications offer themselves in comparison with the neighbourhood-spaces considered in a purely topological way. It will appear that the axioms (C) and (E) hold true in every space group (that in a topological group (E) holds true is already known from D. VAN DANTZIG's paper, Math. Ann. **107** (1933), p. 608). (B) and (D') may or may not hold true; (D) and (D') are equivalent in every space group. Further it will be proved, among other things, that an enumerable space group satisfying (B) and (D') is a normal space group.

3. In this number we repeat some properties, known for "topological groups", which hold true for all space groups.

If every element x of a general space group R is mapped on ax (where a is a fixed but arbitrarily chosen element of R), then in that way a topological mapping is determined of R onto itself. In the first place the mapping is one-to-one, as $ax = b$ for a given a and b has but one solution. Further the mapping is continuous in the direction $x \to ax$. For an arbitrary $U(ax)$ there may be found, according to the first axiom of continuity, a $U(a)$ and a $U(x)$ with $U(a) \cdot U(x) \subset U(ax)$, so that in any case $a \cdot U(x) \subset U(ax)$, i.e., $U(x)$ is mapped on a set belonging to $U(ax)$. In the same way it is proved that the inverse mapping $ax \to x$ or $x \to a^{-1}x$ is continuous.

Also the mappings $x \to xa$ and $x \to x^{-1}$ are topological. So, if V is a neighbourhood of the unity e, then V^{-1} also is a neighbourhood of e and Va and aV are neighbourhoods of a.

Every space group R is homogeneous, i.e., for every two elements a and b of R there exists a topological mapping of R onto itself, which maps a on b. The mapping $x \to xa^{-1}b$ fills this requirement. This homogeneity is an important property. For instance, if one wants to prove a certain (topological) property for a neighbourhood of a point p it is, because of the homogeneity, necessary and sufficient to prove this property for a neighbourhood of the unity e.

Suppose O is an open subset, G a closed subset, W an arbitrary set and p a point of a space group. Then Gp is a closed set, for the mapping $x \to xp$ is topological, so G is mapped onto a closed set Gp. In the same way one proves that Op is an open set. But then OW also is an open set, as the sum of an arbitrary number of open sets is also open. So in general: OW, O^{-1}, WO are open; G, G^{-1}, Gp and pG are closed.

4. **Theorem.** *In every space group axiom* (*C*) *holds true.*

Proof. Because of the homogeneity we only have to prove that within an arbitrary neighbourhood $U(e)$ of the unity e may be found an open set containing e. As $e \cdot e = e$, it is possible, according to the second axiom of continuity, to find for U two neighbourhoods $V_1^1(e)$ and $V_2^1(e)$ with

$$V_1^1 \cdot V_2^1 \subset U.$$

For the same reason there may be found for V_1^1 two neighbourhoods V_1^2 and V_2^2 with $V_1^2 \cdot V_2^2 \subset V_1^1$. Thus continuing we construct

$$V_1^{j+1} \cdot V_2^{j+1} \subset V_1^j$$

for $j = 1, 2, \ldots$.

Now consider the sum

$$V = \bigcup_{i=1}^{\infty} V_2^i \cdot V_2^{i-1} \cdot \ldots V_2^1.$$

For each term of V

$$V_2^i \cdot V_2^{i-1} \cdot \ldots V_2^1 \subset V_1^{i-1} \cdot V_2^{i-1} \cdot \ldots V_2^1 \subset V_1^{i-2} \cdot V_2^{i-2} \cdot \ldots V_2^1 \subset \ldots \subset$$
$$\subset V_1^1 \cdot V_2^1 \subset U.$$

So V belongs to U, but at the same time V is open, for an arbitrary element

$$p_2^i \cdot p_2^{i-1} \cdot \ldots p_2^1 \qquad (p_2^k \subset V_2^k)$$

of V has the neighbourhood (belonging to V)

$$V_2^{i+1} \cdot p_2^i \cdot p_2^{i-1} \cdot \ldots p_2^1.$$

Theorem. *In every space group R a point p and a closed set, disjunct with p, have disjunct neighbourhoods.*

From this follows in particular, that axiom (E) holds true in every space group.

Proof. Because of the homogeneity we may suppose that e is the point mentioned in the theorem; further let Q be a closed set which does not contain e.

Then the open set $R - Q$ is a $U(e)$. There may be found two neighbourhoods $V_1(e)$ and $V_2(e)$ with $V_1 \cdot V_2 \subset U$. V_1 is a neighbourhood of e and QV_2^{-1} is a neighbourhood of Q, as V_2^{-1} is a neighbourhood of e. V_1 and QV_2^{-1} are the required disjunct neighbourhoods. For, supposing this was not the case, there might be found in V_1, V_2^{-1} and Q the points v_1, v_2^{-1} and q respectively, with $v_1 = qv_2^{-1}$ or $q = v_1 v_2$, in contradiction with the fact that Q and $V_1 V_2$ are disjunct.

As to the validity of (B) and (D'), it will appear from the instances given in 5. that all four possibilities may offer themselves. From (D') always follows (D) and conversely.

Theorem. *In every space group (D') and (D) are equivalent.*

Proof. We only have to prove that, in case (D') holds true, e and a point $a \neq e$ have disjunct neighbourhoods. As (D') holds true there must be a $U(e)$ which does not contain a. For the rest the proof runs exactly as that of the theorem here above, if only one substitutes a for Q (we do not, however, contend that the point a — seen as a set — is closed).

It now stands to reason to introduce, starting from a general space group, the following axioms and denominations:

A space group, satisfying (B), is called a topological space group (and is indeed, considered as a space, a topological space).

A space group satisfying (B) and (D') is called a regular space group (and again, considered as an abstract space, is indeed a regular space).

A space group satisfying (B), (D') and (G) is called a separable space group (and again is a separable space).

Remark. Instances of regular space groups which are not normal are unknown to me.

The separable spaces, which are topologically speaking the most important (these are topologically identical with the subsets of the HILBERT space) are reached already by the axioms (A), (A'), (B), (C), (D'), (E) and (G). The separable space groups, which are again the most important space groups, are reached as follows from the theorems above by the axioms (A), (A'), (C), (D'), (G) and the group- and continuity-axioms.

5. In this number we investigate in particular the *space groups of finite order.*

If a space group of that kind satisfies (B) and (D') it is of course a space group consisting of isolated points, so there is no spacial relation whatever.

Each of the three remaining possibilities may offer itself however in a not trivial way, denoted by

1°. $(-B)$ [1], (D');
2°. (B), $(-D')$;
3°. $(-B)$, $(-D')$.

To prove this we use the following lemma.

Lemma. If in an abstract commutative group a system of neighbourhoods $\{ V \}$ of the unity e is defined such that the axioms of continuity are satisfied as far as e is concerned (the a and b mentioned in the definition of the axioms of continuity always are e), and if one defines a system of neighbourhoods of an arbitrary but fixed element a of the group by $\{ aV \}$, then the abstract group is hereby transformed into a space group.

Proof. We apparently only have to prove the axioms of continuity. Let abU be a neighbourhood of ab, then for $U = U(e)$ there may be found a $V_1(e)$ and a $V_2(e)$ with $V_1 \cdot V_2 \subset U$. Now $aV_1 \cdot bV_2 = abV_1V_2 \subset abU$. In the same simple way one proves the second axiom of continuity.

We shall now give the required instances the correctness of which is easy to see.

1°. $(-B)$, (D'). The abstract commutative group consists of the elements e, a, b, and ab with moreover the relations $a^2 = b^2 = e$. A system of neighbourhoods of e consists by definition of the sets $\{ e, a \}$ and $\{ e, b \}$.

2°. (B), $(-D')$. The abstract group as in 1°. A system of neighbourhoods of e consists by definition of the set $\{ e, a \}$.

3°. $(-B)$, $(-D')$. The abstract commutative group consists of the elements e, a, b, c, ab, ac, bc and abc with moreover the relations $a^2 = b^2 = c^2 = e$. A system of neighbourhoods of e consists by definition of the sets $\{ e, a, b, ab \}$ and $\{ e, a, c, ac \}$.

[1] $(-B)$ means, that axiom (B) does not hold true.

6. For a space group of enumerable order we have the following

Theorem. Every space group of enumerable order, satisfying (B), also satisfies (F). From this follows in particular that *every regular space group of enumerable order is normal.*

Proof. Let P and Q be two closed disjunct sets, consisting respectively of the points $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ (the case where at least one of the sets consists of a finite number of points is simpler and is left out of consideration). For e may be constructed a neighbourhood U so that $p_1 U$ (this is a neighbourhood of p_1) contains no point of Q and $q_1 U$ (this is a neighbourhood of q_1) contains no point of P. The possibility of this follows from the validity of (B) and of the first theorem in 4. Within U we may construct a neighbourhood $V_1(e)$ with $V_1 \cdot V_1^{-1} \subset U$. Therefore also $p_1 V_1 V_1^{-1} \cap Q = 0$ and $q_1 V_1 V_1^{-1} \cap P = 0$. Thus continuing one constructs

$$V_1 V_1^{-1} \supset V_1 \cap V_1^{-1} \supset V_2 V_2^{-1} \supset V_2 \cap V_2^{-1} \supset V_3 V_3^{-1} \supset \ldots$$

with

$$p_i V_i V_i^{-1} \cap Q = 0 \text{ and } q_i V_i V_i^{-1} \cap P = 0 \qquad (i = 1, 2, \ldots) \ . \quad (1)$$

Now

$$U(P) = \bigcup_{i=1}^{\infty} p_i V_i \text{ and } U(Q) = \bigcup_{i=1}^{\infty} q_i V_i$$

are the required disjunct neighbourhoods of P and Q: For suppose $U(P)$ and $U(Q)$ had a common point; then this would have the shape

$$p_k v_k = q_l v_l.$$

We may suppose $k \geqq 1$. Then we find

$$p_k = q_l v_l v_k^{-1} \subset q_l V_l V_k^{-1} \subset q_l V_l V_l^{-1}$$

in contradiction with (1).