Mathematics. - On the theory of linear integral equations. V. By A. C. Zaanen. (Communicated by Prof. W. van der Woude.)
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## § 1. Introduction.

We suppose the reader to be acquainted with the contents of the papers I, II and IV, bearing the same title ${ }^{1}$ ). In this paper we shall consider linear integral equations

$$
\begin{equation*}
\int_{\Delta} K(x, y) f(y) d y-\lambda f(x)=g(x) \tag{1}
\end{equation*}
$$

in the space $L_{2}^{(m)}(\triangle)$ with kernel $K(x, y)=A(x) H(x, y)$, where $A(x)$ is a measurable, bounded and real function on the $m$-dimensional interval $\triangle$, and $H(x, y) \in L_{2}^{(2 m)}$ is a positive (positive means here: of positive type) Hermitean kernel. Equations with a kernel $K(x, y)$ of this category are sometimes called of the third kind, and, if $A(x)$ takes on only the values +1 and -1 , they are called of polar type (D. Hilbert). E. Garbe ${ }^{2}$ ) has discussed the equation of the third kind under the assumptions that $\triangle$ is a linear interval, $A(x)$ is continuous except for a finite number of jumps, $[A(x)]^{-1}$ is bounded ${ }^{3}$ ), and that the kernel $H(x, y)$ is continuous and general (that is, the functions $\int_{\Delta} H(x, y) f(y) d y$ are lying dense in the space $\left.L_{2}^{(m)}(\Delta)\right)$. Denoting the characteristic values $\neq 0$ of (1) by $\lambda_{i}(i=1,2, \ldots)$, and a corresponding $H$-orthonormal system of characteristic functions by $\psi_{i}(x)(i=1,2, \ldots)$, he obtained the following results:
$1^{\circ}$. If

$$
a_{i}=\int_{\Delta \dot{x} \Delta} H(x, y) \overline{\psi_{i}(x)} f(y) d x d y
$$

for an arbitrary $f(x) \in L_{2}$, then

$$
\begin{equation*}
\int_{\triangle} K(x, y) f(y) d y=\Sigma \lambda_{i} a_{i} \psi_{i}(x), \tag{2}
\end{equation*}
$$

uniformly in $x$,
$2^{\circ}$.

$$
H(x, y)=\Sigma \frac{\lambda_{i}^{2} \psi_{i}(x) \overline{\psi_{i}(y)}}{A(x) A(y)}
$$

[^0]uniformly in $x$ and $y$, or, writing
\[

$$
\begin{gather*}
\int_{\Delta} H(x, y) \psi_{i}(y) d y=\chi_{i}(x), \text { so that } \lambda_{i} \psi_{i}(x)=\int_{\Delta} K(x, y) \psi_{i}(y) d y=A(x) \chi_{i}(x) \\
H(x, y)=\Sigma \chi_{i} \overline{(x) \chi_{i}}(y), \quad . \quad . \quad . \tag{3}
\end{gather*}
$$
\]

uniformly in $x$ and $y$.
We shall show now, as a consequence of other more general results, that Garbe's Theorems are valid under less stringent conditions upon the function $A(x)$. The formula (2) holds if $A(x) \neq 0$ almost everywhere on $\triangle$, and also, in the case that $\triangle$ is a linear interval, if $A(x)$ is, for every $x \in \triangle$, continuous to the left or to the right (in thise case it is therefore permitted that $A(x)=0$ on a set of positive measure), and for the validity of (3) it is sufficient that $A(x) \neq 0$ almost everywhere in $\Delta$. We shall prove, moreover, that although the series $\Sigma \chi_{i}(x) \overline{\chi_{i}(y)}$ converges uniformly whenever $H(x, y)$ is continuous, its sum is not necessarily equal to $H(x, y)$ when $A(x)=0$ in a subinterval of $\triangle$, even in the case that $H(x, y)$ is general.
§ 2. The equation of the third kind.
Let $K(x, y)=A(x) H(x, y)$, where $H(x, y) \in L_{2}^{(2 m)}(\triangle)$ is a positive Hermitean kernel, and $A(x)$ is on $\triangle$ measurable, bounded and real. Then the linear transformation $H$ in the space $L_{2}^{(m)}(\Delta)$, defined by .

$$
H f=\int H(x, y) f(y) d y
$$

is completely continuous, self-adjoint and positive, while the linear transformation $A$, defined by

$$
A f=A(x) f(x)
$$

is bounded and self-adjoint. The completely continuous linear transformation $K=A H$ is then determined by

$$
K f=A H f=A(x) \int_{\triangle} H(x, y) f(y) d y=\int_{\Delta} K(x, y) f(y) d y
$$

As we know, the transformation $K$ is symmetrisable relative to $H$, and we observe that every $f(x) \in L_{2}$, satisfying $H f=0$, satisfies also $K f=0$. The kernel $K(x, y)$ is therefore what we have called in IV a Marty-kernel. Supposing that

$$
\|H(x, y)\|_{2 m}^{2}=\int_{\Delta x \Delta}|H(x, y)|^{2} d x d y \neq 0
$$

so that $H$ is not identical with the nulltransformation $O$, the theorems proved in IV may therefore be applied to the equation (1). We shall not
repeat them all here, and only pay attention to IV, Theorems 4,6 and 11 , since these may be replaced by stronger theorems. Instead of IV, Theorem 4 we have

Theorem 1. (Expansion Theorem.) Writing

$$
a_{i}=\left(f, \chi_{i}\right)=\int_{\Delta} f(x) \overline{\chi_{i}(x)} d x
$$

for an arbitrary $f(x) \in L_{2}$, we have

$$
\begin{gathered}
\int_{\Delta} K(x, y) f(y) d y \sim \Sigma \lambda_{i} a_{i} \psi_{i}(x)+p(x) \\
\int_{\Delta} K_{n}(x, y) f(y) d y \sim \sum_{i} \lambda_{i}^{n} a_{i} \psi_{i}(x) \quad(n \geqslant 2),
\end{gathered}
$$

where the function $p(x)$ satisfies the relation

$$
H p=\int_{\triangle} H(x, y) p(y) d y=0
$$

for almost every $x \in \triangle$.
Proof. Follows from I, Theorem 15.
Instead of IV, Theorem 6 we have
Theorem 2. Let $\lambda \neq 0$, and let $g(x) \in L_{2}$ be $H$-orthogonal to all characteristic functions of (1) belonging to the characteristic value $\lambda$ (If $\lambda$ is no characteristic value, $g(x)$ is therefore arbitrary). Then every solution of (1) satisfies a relation of the form

$$
f(x) \sim-\frac{g(x)}{\lambda}-\Sigma^{\prime} \frac{\lambda_{i}}{\lambda\left(\lambda-\lambda_{i}\right)} a_{i} \psi_{i}(x)+q(x) .
$$

where

$$
\mathbf{a}_{i}=\int_{\Delta} g(x) \overline{\chi_{i}(x)} d x \text { for } \lambda_{i} \neq \lambda, \int_{\Delta} H(x, y) q(y) d y=0
$$

for almost every $x \in \triangle$, and where $\Sigma^{\prime}$ denotes that for those values of $i$ for which $\lambda_{i}=\lambda$ the coefficient of $\psi_{i}(x)$ has the value $\int_{\Delta} f(x) \overline{\chi_{i}(x)} d x$. For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of (1).

Proof. Follows from I, Theorem 17.
Instead of IV, Theorem 11, we have

Theorem 3. (Expansion Theorem). We have
where

$$
\begin{equation*}
K_{2}(x, y)-p_{2}(x, y) \backsim \Sigma \lambda_{i}^{2} \psi_{i}(x) \overline{\chi_{i}(y)}, . \tag{4}
\end{equation*}
$$

$$
\int_{\triangle} H(x, z) p_{2}(z, y) d z=0
$$

almost everywhere in $\Delta \times \Delta$;

$$
\begin{align*}
K_{n}(x, y) \propto \sum_{i} \lambda_{i}^{n} \psi_{i}(x) \overline{\chi_{i}(y)} & (n \geqslant 3) ;  \tag{5}\\
\int_{\Delta} K_{n}(x, x) d x=\sum_{i} \lambda_{i}^{n} & (n \geqslant 3) \tag{6}
\end{align*}
$$

Proof. The formulae (4) and (5), and also the formula (6) for $n \geq 4$ have already been proved in IV, Theorem 11. The only thing that remains to be proved is

$$
\int_{\Delta} K_{3}(x, x) d x=\Sigma \lambda_{i}^{3}
$$

Now, in the proof of IV, Theorem 7 we have obtained the formula (4), stating that

$$
\int_{\Delta} P(z, y) K(y, z) d y=\Sigma \lambda_{i}^{2}\left|\chi_{i}(z)\right|^{2}
$$

for almost every $z \in \Delta$, where

$$
P(z, y)=\int_{\Delta} H(z, x) K(x, y) d x
$$

Hence

$$
\begin{aligned}
\int_{\Delta} H(z, x) K_{2}(x, z) d x=\int_{\Delta x \Delta} H(z, x) K(x, y) K(y, z) d x d y & = \\
& \int_{\Delta} P(z, y) K(y, z) d y=\Sigma \lambda_{i}^{2}\left|\chi_{i}(z)\right|^{2}
\end{aligned}
$$

or, observing that $A(z) \chi_{i}(z)=\lambda_{i} \psi_{i}(z)$,

$$
\begin{aligned}
K_{3}(z, z)= & \int_{\Delta} K(z, x) K_{2}(x, z) d x= \\
& A(z) \int_{\Delta} H(z, x) K(x, z) d x= \\
& \Sigma \lambda_{i}^{2} A(z) \chi_{i}(z) \overline{\chi_{i}(z)}=\Sigma \lambda_{i}^{3} \psi_{i}(z) \overline{\chi_{i}(z)}
\end{aligned}
$$

for almost every $z \in \Delta$. This shows that

$$
\int_{\Delta} K_{3}(z, z) d z=\Sigma \lambda_{i}^{3}
$$

Theorem 4. (Expansion Theorem). If $\Sigma \mu_{i}$ converges, where

$$
\mu_{i}(i=1,2, \ldots)
$$

is the sequence of characteristic values of the kernel $H(x, y)$, we have

$$
\begin{equation*}
K(x, y)-p(x, y) \sim \Sigma \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)} \tag{7}
\end{equation*}
$$

where

$$
\int_{\Delta} H(x, z) p(z, y) d z=0
$$

almost everywhere in $\Delta \times \Delta$;

$$
\begin{gather*}
K_{n}(x, y) \sim \sum_{i} \lambda_{i}^{n} \psi_{i}(x) \overline{\chi_{i}}(y) \quad(n \geqslant 2)  \tag{8}\\
\int_{\Delta} K_{n}(x, x) d x=\sum_{i} \lambda_{i}^{n} \quad(n \geqslant 2) \tag{9}
\end{gather*}
$$

Proof. If $\Sigma \mu_{i}$ converges, the uniquely determined, positive self-adjoint transformation $H^{1 / 3}$ is of the form

$$
H^{1 / 2} f=\int_{\Delta} H_{1 / 2}(x, y) f(y) d y
$$

where $H_{1_{2}}(x, y) \in L_{2}^{(2 m)}$, so that the transformation $\mathrm{Q}=A H^{1_{2}}$ is expressible as

$$
Q f=\int_{\Delta} \mathrm{Q}(x, y) f(y) d y
$$

where $Q(x, y)=A(x) H_{1_{2}}(x, y) \in L_{2}^{(2 m)}$. The theorem to be proved is therefore a consequence of IV, Theorem 12.
§ 3. The case that $H(x, y)$ is continuous.
Theorem 5. (Expansion Theorem for the kernel). If $H(x, y)$ is continuous, then

$$
\begin{equation*}
K(x, y)-p(x, y)=\Sigma \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)} \tag{10}
\end{equation*}
$$

uniformly in $\Delta \times \Delta$, where the function $p(x, y)$ satisfies the relation

$$
q(x, y)=\int_{\Delta} H(x, z) p(z, y) d z=0
$$

for every point $(x, y) \in \triangle X \triangle$.
Proof. Since $H(x, y)$ is continuous, the transformation $H^{1 / s}$ is, on account of II, Theorem 10, given by

$$
H^{1_{3}} f=\int_{\Delta} H_{1_{9}}(x, y) f(y) d y
$$

where $\int_{\triangle}\left|H_{1_{1}}(x, y)\right|^{2} d y$ is bounded. For almost every $x_{1} \in \Delta$ and almost every $x_{2} \in \triangle$ we have now

$$
\begin{aligned}
& \int_{\Delta}\left|H_{1 / 2}\left(x_{2}, y\right)-H_{1 / 2}\left(x_{1}, y\right)\right|^{2} d y= \\
& \int_{\Delta}\left\{H_{1_{2}}\left(x_{2}, y\right)-H_{1 / 2}\left(x_{1}, y\right)\right\} \overline{\left\{H_{1 / 2}\left(x_{2}, y\right)-H_{1 / 2}\left(x_{1}, y\right)\right\} d y}= \\
& \int_{\Delta}\left\{H_{1 / 2}\left(x_{2}, y\right)-H_{1 / 2}\left(x_{1}, y\right)\right\}\left\{H_{1 / 2}\left(y, x_{2}\right)-H_{1 / 2}\left(y, x_{1}\right)\right\} d y= \\
& H\left(x_{2}, x_{2}\right)-H\left(x_{2}, x_{1}\right)-H\left(x_{1}, x_{2}\right)+H\left(x_{1}, x_{1}\right) ;
\end{aligned}
$$

consequently, since $H(x, y)$ is continuous in $\Delta \times \Delta$, there exists for any $\varepsilon>0$ a number $\delta(\varepsilon)>0$ such that

$$
\begin{equation*}
\int_{\Delta}\left|H_{1 / 2}\left(x_{2}, y\right)-H_{1 / 2}\left(x_{1}, y\right)\right|^{2} d y<\varepsilon_{1} \tag{11}
\end{equation*}
$$

if only the distance

$$
\varrho\left(x_{1}, x_{2}\right)=\left(\sum_{i=1}^{m}\left|x_{1}^{(i)}-x_{2}^{(i)}\right|^{2}\right)^{1 / 2}
$$

of the points $x_{1}$ and $x_{2}$ satisfies the relation $\varrho\left(x_{1}, x_{2}\right)<\delta$, and $x_{k}(k=1,2)$ does not belong to a set $E_{k} \in \triangle(k=1,2)$ of measure 0 . Further, on account of

$$
\chi_{i}(x)=H \psi_{i}=H^{1 / 2} H^{1_{2}} \psi_{i}=H^{1_{2}} \Psi_{i}=\int_{\Delta} H_{1_{2}}(x, y) \Psi_{i}(y) d y
$$

holding for almost every $x \in \Delta$, we have

$$
\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)=\int_{\Delta}\left\{H_{1 / 2}\left(x_{2}, y\right)-H_{1 / 2}\left(x_{1}, y\right)\right\} \Psi_{i}(y) d y
$$

for almost every $x_{1} \in \triangle$ and almost every $x_{2} \in \triangle$; hence, in virtue of Bessel's inequality (the system $\Psi_{i}(x)$ is orthonormal),

$$
\begin{equation*}
\sum_{i=1}^{p}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2} \leqslant \int_{\Delta}\left|H_{1_{2}}\left(x_{2}, y\right)-H_{1_{1 / 2}}\left(x_{1}, y\right)\right|^{2} d y \tag{12}
\end{equation*}
$$

for these values of $x_{1}, x_{2}$ and for arbitrary $p$.
From (11) and (12) we deduce that

$$
\sum_{i=1}^{p}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2}<\varepsilon
$$

for almost every $x_{1} \in \Delta$ and almost every $x_{2} \in \Delta$, if only $\varrho\left(x_{1}, x_{2}\right)<\delta$. Since however the functions $\chi_{i}(x)=\int_{\Delta} H(x, y) \psi_{i}(y) d y$ are continuous in $\triangle$, the function $\sum_{i=1}^{p}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2}$ is continuous in $x_{1}$ and in $x_{2}$, so that the relation

$$
\sum_{i=1}^{p}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2} \leqslant \varepsilon
$$

holds evidently for every pair of points $x_{1}, x_{2} \in \triangle$, if only $\varrho\left(x_{1}, x_{2}\right)<\delta$. Observing that $p$ is arbitrary, we obtain finally

$$
\sum_{i}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2} \leqslant \varepsilon \text { for } \varrho\left(x_{1}, x_{2}\right)<\delta(\varepsilon) ;
$$

in other words

$$
\lim _{x_{2} \rightarrow x_{1}} \sum_{i}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2}=0
$$

By Minkowski's inequality we see now that

$$
\left|\left(\sum_{i}\left|\chi_{i}\left(x_{2}\right)\right|^{2}\right)^{1 / 2}-\left(\sum_{i}\left|\chi_{i}\left(x_{1}\right)\right|^{2}\right)^{1 / 2}\right| \leqslant\left(\sum_{i}\left|\chi_{i}\left(x_{2}\right)-\chi_{i}\left(x_{1}\right)\right|^{2}\right)^{1 / 2} ;
$$

the sumfunction of the series $\Sigma\left|\chi_{i}(x)\right|^{2}$ is therefore a continuous function. Hence, on account of Dini's well-known Theorem, since the functions $\left|\chi_{i}(x)\right|^{2}$ are non-negative and continuous, the uniform convergence of $\Sigma\left|\chi_{i}(x)\right|^{2}$. The inequality

$$
\Sigma\left|\chi_{i}(x) \overline{\chi_{i}(y)}\right| \leqslant\left(\Sigma\left|\chi_{i}(x)\right|^{2}\right)^{1 / 2} \cdot\left(\Sigma\left|\chi_{i}(y)\right|^{2}\right)^{1 / 2}
$$

shows then that $\Sigma_{\chi_{i}}(x) \overline{\chi_{i}(y)}$ converges uniformly in $\Delta \times \Delta$. Writing

$$
\begin{equation*}
H(x, y)-p_{1}(x, y)=\Sigma \chi_{i}(x) \overline{\chi_{i}(y)} \tag{13}
\end{equation*}
$$

we see, since both $H(x, y)$ and $\Sigma \chi_{i}(x) \overline{\chi_{i}(y)}$ are continuous in $\triangle \times \Delta$, that $p_{1}(x, y)$ is continuous in $\triangle \times \triangle$. Multiplying the relation (13) with $A(x)$ and writing $A(x) p_{1}(x, y)=p(x, y)$, we find

$$
K(x, y)-p(x, y)=\Sigma \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)}
$$

uniformly in $\Delta \times \Delta$, and we know already (cf. Theorem 4) that

$$
q(x, y)=\int_{\Delta} H(x, z) p(z, y) d z=0
$$

for almost every point $(x, y) \in \Delta \times \Delta$. We have still to prove that $q(x, y)=0$ everywhere in $\Delta \times \Delta$. For this purpose we observe that, since $H(x, z)$ is continuous, $q(x, y)$ is continuous in $x$ for a fixed value of $y$, and, since

$$
q(x, y)=\int_{\Delta} H(x, z) A(z) p_{1}(z, y) d z
$$

where $p_{1}(z, y)$ is continuous, $q(x, y)$ is continuous in $y$ for a fixed value of $x$. Starting now with the fact that, for almost every $y \in \Delta, q(x, y)=0$ for almost every $x \in \Delta$, we find on account of the continuity in $x$ that, for almost every $y \in \Delta, q(x, y)=0$ for every $x \in \Delta$; in other words, for every $x \in \Delta$ we have $q(x, y)=0$ for almost every $y \in \triangle$. On account of the continuity in $y$ we have therefore $q(x, y)=0$ for every point $(x, y) \in \Delta X \Delta$. This completes the proof.

Theorem 6. (Expansion Theorem for the iterated kernels). If $H(x, y)$ is continuous, then

$$
\begin{equation*}
K_{n}(x, y)=\sum_{i} \lambda_{i}^{n} \psi_{i}(x) \overline{\chi_{i}(y)} \quad(n \geqslant 2) \tag{14}
\end{equation*}
$$

uniformly in $\Delta \times \Delta$.
Proof. The result for $n=2$ follows immediately from the preceding theorem, since

$$
\int_{\Delta} K(x, z) p(z, y) d z=A(x) \int_{\Delta} H(x, z) p(z, y) d z=0
$$

in $\Delta \times \Delta$. The relation

$$
K_{n}(x, y)=\sum_{i} \lambda_{i}^{n} \psi_{i}(x) \overline{\chi_{i}(y)} \quad(n>2)
$$

follows easily by induction.
Theorem 7. (Expansion Theorem). If $H(x, y)$ is continuous, and

$$
a_{i}=\left(f, \chi_{i}\right)=\int_{\Delta} f(x) \overline{\chi_{i}(x)} d x \text { for an arbitrary } f(x) \in L_{2}
$$

we have

$$
\begin{gathered}
\int_{\Delta} K(x, y) f(y) d y-p(x)=\Sigma \lambda_{i} a_{i} \psi_{i}(x) \\
\int_{\Delta}^{\infty} K_{n}(x, y) f(y) d y=\sum_{i} \lambda_{i}^{n} a_{i} \psi_{i}(x) \quad(n \geqslant 2)
\end{gathered}
$$

uniformly in $\triangle$, where $p(x)$ satisfies the relation

$$
\int_{\triangle} H(x, y) p(y) d y=0
$$

for every $x \in \triangle$.
Proof. The formula (10) implies

$$
\int_{\triangle} K(x, y) f(y) d y-p(x)=\Sigma \lambda_{i} a_{i} \psi_{i}(x)
$$

uniformly in $\Delta$, where

$$
p(x)=\int_{\Delta} p(x, y) f(y) d y
$$

Hence

$$
\int_{\triangle} H(x, y) p(y) d y=\int_{\Delta x \Delta} H(x, y) p(y, z) f(z) d y d z=0
$$

for every $x \in \triangle$.
The formula (14) implies

$$
\int_{\Delta} K_{n}(x, y) f(y) d y=\sum_{i} \lambda_{i}^{n} a_{i} \psi_{i}(x) \quad(n \geqslant 2)
$$

uniformly in $\triangle$.
A Hermitean kernel $A(x, y) \in L_{2}^{(2 m)}$ is called a general kernel (D. HilBERT), when the set of all functions $A g=\int_{\Delta} A(x, y) g(y) d y$ is lying dense in the space $L_{2}$; in other words, when, given $f(x) \in L_{2}$ and the number $\varepsilon>0$, there exists a function $g(x) \in L_{2}$ such that

$$
\int_{\Delta}\left|f(x)-\int_{\triangle} A(x, y) g(y) d y\right|^{2} d x<\varepsilon
$$

Theorem 8. In order that the Hermitean kernel $A(x, y) \in L_{2}^{(2 m)}$ be general, it is necessary and sufficient that $A f=0$ should imply $f=0$.

Proof. Denoting by $\mu_{i}(i=1,2, \ldots)$ the sequence of all characteristic values $\neq 0$ of $A(x, y)$, and by $\varphi_{i}(x)$ a corresponding orthonormal system of characteristic functions, it is not difficult to prove that the condition that $A f=0$ should imply $f=0$ is equivalent with the condition that the system $\varphi_{i}(x)$ is complete (that is, the finite linear combinations $\Sigma a_{i} \varphi_{i}(x)$ are lying dense in $L_{2}$ ).

Let now $A f=0$ imply $f=0$, and let $f(x) \in L_{2}$ and $\varepsilon>0$ be given. Since the system $\varphi_{i}(x)$ is orthonormal and complete, we have $f=\Sigma \mathrm{a}_{i} \varphi_{i}$ with $a_{i}=\left(f, \varphi_{i}\right)$. Taking the index $N$ such that $\left\|f-\sum_{i=1}^{N} a_{i} \varphi_{i}\right\|^{2}<\varepsilon$, and writing $a_{i}=\mu_{i} b_{i}$, we have for $g=\sum_{i=1}^{N} b_{i} \varphi_{i}$ the relation

$$
A g=\sum_{i=1}^{N} b_{i} A \varphi_{i}=\sum_{i=1}^{N} \mu_{i} b_{i} \varphi_{i}=\sum_{i=1}^{N} a_{i} \varphi_{i}
$$

hence

$$
\int_{\triangle}\left|f(x)-\int_{\triangle} A(x, y) g(y) d y\right|^{2} d x=\|f-A g\|^{2}=\left\|f-\sum_{i=1}^{N} a_{i} \varphi_{i}\right\|^{2}<\varepsilon
$$

The kernel $A(x, y)$ is therefore general.

Conversely, if $A(x, y)$ is general, the elements $A g=\Sigma \mu_{i}\left(g, \varphi_{i}\right) \varphi_{i}$ are lying dense in $L_{2}$, so that the system $\varphi_{i}(x)$ is complete. Then however, as we have seen, $A f=0$ implies $f=0$.

Let $f(x)$ be a measurable function on the interval $\triangle, E_{1}$ the set where $f(x)=0, E_{2}=\Delta-E_{1}$ the set where $f(x) \neq 0$.

Definition. We shall say that $f(x)$ possesses the property $(G)$ when every measurable set $E_{3} \subset E_{2}$, for which meas. $\left(E_{2}-E_{3}\right)=0$, is lying dense in $E_{2}$, in other words, when $E_{2}$ is contained in the closure $\bar{E}_{3}$ of $E_{3}$.

Theorem 9. If $f(x) \neq 0$ almost everywhere in $\triangle, f(x)$ possesses the property (G).

Proof. Since now meas. $E_{2}=$ meas. $\triangle$, we have for every $E_{3}$ for which meas. $\left(E_{2}-E_{3}\right)=0$, also meas. $E_{3}=$ meas. $\Delta$. This implies $E_{2} \subset \bar{E}_{3}=\triangle$, so that $f(x)$ possesses the property ( $G$ ).

Theorem 10. If $f(x)$ is continuous in $\triangle, f(x)$ possesses the property $(G)$. In the case that $\Delta$ is a linear interval, it is even sufficient to suppose that $f(x)$ is, for every $x \in \triangle$, continuous to the left or to the right.

Proof. Let $f(x)$ be continuous in $\Delta$. Then the set $E_{1}$ is closed, so that $E_{2}$ is open (relative to $\triangle$ ); in other words, $E_{2}$ contains only internal points. Given now the set $E_{3} \subset E_{2}$ such that meas. $\left(E_{2}-E_{3}\right)=0$, every neighbourhood of a point $x \in E_{2}$ must contain points of $E_{3}$; hence $E_{2} \subset \bar{E}_{3}$. Every continuous function possesses therefore the property $(G)$. In the case that $\Delta$ is a linear interval, the same proof holds if only $f(x)$ is, for every $x \in \Delta$, continuous to the left or to the right.

Theorem 11. (Expansion Theorem). If $H(x, y)$ is continuous and general, and if $A(x)$ possesses the property ( $G$ ), then

$$
\begin{equation*}
K(x, y)=\Sigma \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)}, \tag{15}
\end{equation*}
$$

uniformly in $\Delta \times \Delta$, and, writing $a_{i}=\left(f, \chi_{i}\right)=\int_{\Delta} f(x) \overline{\chi_{i}(x)} d x$ for an arbitrary $f(x) \in L_{2}$,

$$
\begin{equation*}
\int_{\Delta} K(x, y) f(y) d y=\Sigma \lambda_{i} a_{i} \psi_{i}(x), \quad . \quad . \tag{16}
\end{equation*}
$$

uniformly in $\triangle$. Furthermore

$$
\begin{equation*}
\int_{\Delta} K(x, x) d x=\Sigma \lambda_{i} . \quad . \quad . \quad . \quad . \tag{17}
\end{equation*}
$$

Proof. By Theorem 5 we have

$$
K(x, y)-p(x, y)=\Sigma \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)}
$$

uniformly in $\Delta \times \Delta$, where $p(x, y)=A(x) p_{1}(x, y)$, the function $p_{1}(x, y)$ is continuous in $\Delta \times \Delta$, and $\int H(x, z) p(z, y) d z=0$ in $\Delta \times \Delta$. Since $H(x, y)$ is a general kernel, $\stackrel{\Delta}{H} p=0$ implies $p=0$, so that, for every $y \in \Delta, p(x, y)=0$ for almost every $x \in \Delta$. Denoting by $E_{1} \subset \triangle$ the set where $A(x)=0$, and by $E_{2}$ the complementary set where $A(x) \neq 0$, we see that $p(x, y)=A(x) p_{1}(x, y)=0$ for $x \in E_{1}$. Furthermore

$$
p_{1}(x, y)=p(x, y) / A(x)=0
$$

almost everywhere in $E_{2}$; hence, in a set $E_{3} \subset E_{2}$ for which meas. $\left(E_{2}-E_{3}\right)=0$. In virtue of the continuity of $p_{1}(x, y)$ the relation $p_{1}(x, y)=0$ holds also for $x \in \bar{E}_{3}$. But, $A(x)$ possessing the property $(G)$, we have $E_{2} \subset \bar{E}_{3}$, so that $p_{1}(x, y)=0$, and therefore also $p(x, y)=0$, for $x \in E_{2}$ Having established thus that, for every $y \in \Delta, p(x, y)=0$ for $x \in E_{1}$ and $x \in E_{2}$, we see that $p(x, y)=0$ in $\Delta \times \triangle$, hence

$$
\begin{equation*}
K(x, y)=\sum \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)} \tag{15}
\end{equation*}
$$

uniformly in $\triangle \times \triangle$.
The formulae (16) and (17) follow immediately from (15).
Theorem 12. Let $H(x, y)$ be continuous and general, and $A(x)$ possess the property (G). Let furthermore $\lambda \neq 0$, and $g(x) \in L_{2}$ be H-orthogonal to all characteristic functions of $K(x, y)$, belonging to the characteristic value. (If $\lambda$ is no characteristic value, $g(x)$ may be any function belonging to $L_{2}$ ). Then the solution of the equation (1) is given by

$$
f(x)=-\frac{g(x)}{\lambda}-\Sigma^{\prime} \frac{\lambda_{i}}{\lambda\left(\lambda-\lambda_{i}\right)} a_{i} \psi_{i}(x)
$$

where $a_{i}=\int_{\Delta} g(x) \overline{\chi_{i}(x)} d x$ for $\lambda_{i} \neq \lambda, \Sigma^{\prime}$ denotes that for those values of $i$ for which $\lambda_{i}=\lambda$ the coefficient of $\psi_{i}(x)$ is arbitrary, and the series

$$
\Sigma^{\prime} \frac{\lambda_{i}}{\lambda\left(\lambda-\lambda_{i}\right)} a_{i} \psi_{i}(x)
$$

converges uniformly in $\triangle$.
Proof. By the preceding theorem we have

$$
g(x)+\lambda f(x)=\int_{\Delta} K(x, y) f(y) d y=\Sigma \lambda_{i} b_{i} \psi_{i}(x)
$$

uniformly in $\triangle$, where $b_{i}=\left(f, \chi_{i}\right)$. Since

$$
\begin{aligned}
& \lambda_{i} b_{i}=\lambda_{i}\left(f, \chi_{i}\right)=\lambda_{i}\left(f, H \psi_{i}\right)=\left(f, H K \psi_{i}\right)=\left(H K f, \psi_{i}\right)= \\
& \left(K f, H \psi_{i}\right)=\left(g+\lambda f, \chi_{i}\right)=a_{i}+\lambda b_{i}
\end{aligned}
$$

we find $b_{i}=-a_{i} /\left(\lambda-\lambda_{i}\right)$ for $\lambda_{i} \neq \lambda$. Furthermore, the solution $f(x)$ being determined except for a characteristic function of $K(x, y)$, belonging to the characteristic value $\lambda$, the coefficients $b_{i}$ may be taken arbitrary for those values of $i$ for whih $\lambda_{i}=\lambda$. Hence

$$
f(x)=-\frac{g(x)}{\lambda}-\Sigma^{\prime} \frac{\lambda_{i}}{\lambda\left(\lambda-\lambda_{i}\right)} a_{i} \psi_{i}(x),
$$

uniformly in $\triangle$.
Theorem 13. (Garbe's Theorem). If $H(x, y)$ is continuous and general, and if moreover $A(x) \neq 0$ for almost every $x \in \Delta$, then

$$
\begin{equation*}
H(x, y)=\Sigma \chi_{i}(x) \overline{\chi_{i}(y)} \tag{18}
\end{equation*}
$$

uniformly in $\triangle \times \triangle$.
Proof. In the proof of Theorem 5 we have seen that $\Sigma_{\chi_{i}}(x) \overline{\chi_{i}(y)}$ converges uniformly. Furthermore, since $A(x)$ possesses the property (G), we have by Theorem 11

$$
K(x, y)=A(x) H(x, y)=\Sigma \lambda_{i} \psi_{i}(x) \overline{\chi_{i}(y)}=A(x) \Sigma \chi_{i}(x) \overline{\chi_{i}(y)}
$$

for every point $(x, y) \in \triangle \times \triangle$. Hence, for every $y \in \triangle$,

$$
\begin{equation*}
H(x, y)=\Sigma \chi_{i}(x) \overline{\chi_{i}(y)} \tag{18}
\end{equation*}
$$

for those values of $x$ for which $A(x) \neq 0$, that is, for almost every $x \in \Delta$. Since however both $H(x, y)$ and $\Sigma \chi_{i}(x) \overline{\chi_{i}(y)}$ are continuous in $\triangle \times \triangle$, the relation (18) holds for every point $(x, y) \in \Delta \times \triangle$.

Remark. It is not difficult to show that $A(x) \neq 0$ almost everywhere in $\Delta$ is the necessary and sufficient condition that $A f=A(x) f(x)=0$ should imply $f=0$ in the space $L_{2}$. This condition therefore is, for a measurable, bounded and real function $A(x)$, the analogue of the condition to be general for a Hermitean kernel $A(x, y) \in L_{2}^{(2 m)}$.

It may be asked whether, in the case that $A(x)=0$ in a set of positive measure, the relation

$$
H(x, y)=\Sigma \chi_{i}(x) \overline{\chi_{i}(y)}
$$

remains valid. We shall show that this is not necessarily true. Let, for this purpose, $\triangle$ be the linear interval $0 \leq x \leq 2 \pi$, and $t_{i}(x)$ the orthonormal trigonometrical system, hence
$t_{1}(x)=(2 \pi)^{-1 / 2}, t_{2 n}(x)=\pi^{-1 / 2} \cos n x(n \geqslant 1), t_{2 n+1}(x)=\pi^{-1 / 2} \sin n x(n \geqslant 1)$.
It is well-known that the system $t_{i}(x)$ is complete in the space $L_{2}(0,2 \pi)$ of all functions $f(x)$ for which $|f(x)|^{2}$ is summable over $\triangle$.

Let now the continuous, general, positive Hermitean kernel $H(x, y)$ be defined by

$$
H(x, y)=\sum_{i} i^{-4} t_{i}(x) t_{i}(y)
$$

it has the characteristic values $i^{-4}(i=1,2, \ldots)$ with the corresponding characteristic functions $t_{i}(x)$. The transformation $H^{1 / 2}$ corresponds then with the continuous, general, positive Hermitean kernel

$$
H_{1_{2}}(x, y)=\sum_{i} i^{-2} t_{i}(x) t_{i}(y),
$$

having the characteristic values $i^{-2}$ with the characteristic functions $t_{i}(x)$. Furthermore we define the bounded self-adjoint transformation $A$ by $A f=A(x) f(x)$, where

$$
A(x)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant x \leqslant \pi \\
1 \text { for } \pi<x \leqslant 2 \pi .
\end{array}\right.
$$

Let us consider now the periodic function $f(x)$, defined by

$$
f(x)=\left\{\begin{array}{cl}
x^{5}(\pi-x)^{5} & \text { for } 0 \leqslant x \leqslant \pi \\
0 & \text { for } \pi \leqslant x \leqslant 2 \pi
\end{array}\right.
$$

The fourth derivative of $f(x)$ is periodic and continuous; if therefore the Fourier series of $f(x)$ is $\Sigma\left(f, t_{i}\right) t_{i}(x)=\Sigma b_{i} t_{i}(x)$, we find without any difficulty by partial integration that there exists a constant $M$ such that $\left|b_{i}\right| \leq M i^{-4}(i=1,2, \ldots)$, so that $f(x)=\Sigma b_{i} t_{i}(x)$, uniformly in $\triangle$. Defining now $a_{i}=i^{2} b_{i}(i=1,2, \ldots)$, we have $\left|a_{i}\right| \leq M i^{-2}$, which shows that $\sum a_{i} t_{i}(x)$ also converges uniformly in $\triangle$. Writing $\Psi(x)=\Sigma a_{i} t_{i}(x)$, we find

$$
H^{1 / 2} \Psi=\Sigma a_{i} H^{1 / 2} t_{i}=\sum a_{i} i^{-2} t_{i}=\Sigma b_{i} t_{i}=f
$$

while from the definitions of $A(x)$ and $f(x)$ follows immediately $A(x) f(x)=0$; hence

$$
\begin{equation*}
H^{1 / 2} A H^{1 / 2} \Psi=H^{1 / 2} A f=0 \tag{19}
\end{equation*}
$$

Denoting the $H$-orthonormal characteristic functions with characteristic values $\neq 0$ of the kernel $K(x, y)=A(x) H(x, y)$ by $\psi_{i}(x)$, we know by I, Theorem 18 that the functions $\Psi_{i}=H^{1 / 2} \psi_{i}$ are the orthonormal characteristic functions with characteristic values $\neq 0$ of the self-adjoint transformation $H^{11_{2}} A H^{1_{2}}$. From (19) follows therefore that $\Psi(x)$ is orthogonal to all functions $\Psi_{i}(x)$. Writing $\widetilde{\Psi}(x)=\Psi(x) /\|\Psi\|$, the system $\left\{\widetilde{\Psi}(x), \Psi_{i}(x)\right\}$ is orthonormal, and, since for every $x \in \triangle$

$$
\begin{gathered}
\chi_{i}(x)=H \psi_{i}=H^{1 / 2} \Psi_{i}=\int_{\Delta} H_{1_{2}}(x, y) \psi_{i}(y) d y \\
f(x) /\|\Psi\|=H^{1 / 2} \widetilde{\Psi}=\int_{\Delta} H_{1_{2}}(x, y) \widetilde{\Psi}(y) d y
\end{gathered}
$$

we find in virtue of Bessel's inequality

$$
\Sigma\left|\chi_{i}(x)\right|^{2}+|f(x)|^{2} /\|\Psi\|^{2} \leqslant \int_{\Delta}\left|H_{y_{2}}(x, y)\right|^{2} d y=H(x, x)
$$

so that, on account of $f(x) \neq 0$ for $0<x<\pi$,

$$
\Sigma \chi_{i}(x) \overline{\chi_{i}(x)}<H(x, x)
$$

for these values of $x$. The functions $H(x, y)$ and $\Sigma \chi_{i}(x) \overline{\chi_{i}(y)}$ being continuous in $\Delta \times \Delta$, there exists consequently for every point $\left(x_{0}, x_{0}\right) \in \Delta X \Delta$, subject to $0<x_{0}<\pi$, a two-dimensional neighbourhood $E\left(x_{0}\right) \subset \Delta \times \Delta$ such that

$$
H(x, y) \neq \Sigma \chi_{i}(x) \overline{\chi_{i}(y)}
$$

for $(x, y) \in E\left(x_{0}\right)$.

## § 4. An example.

We shall illustrate the theorems, proved in the preceding paragraph, by an example showing that the functions $p(x)$ and $p(x, y)$, occurring in the Theorems 7 and 5, need not vanish identically.

Let $\Delta$ be the linear interval $[0,2 \pi]$, and $\Delta_{1}, \Delta_{2}, \Delta_{3}$ the subintervals $[0, \pi / 2],(\pi / 2, \pi),[\pi, 2 \pi]$. The orthonormal system of functions $\varphi_{1}(x)$, $\phi_{2}(x), \varphi_{3}(x)$ is defined by

$$
\begin{aligned}
& \varphi_{1}(x)=\left\{\begin{array}{cl}
(2 / \pi)^{1 / 2}|\sin 2 x| & \text { for } x \in \triangle_{1}+\triangle_{2}, \\
0 & \text { for } x \in \triangle_{3}
\end{array},\right. \\
& \varphi_{2}(x)=\left\{\begin{array}{cl}
-(2 / \pi)^{1 / 2} \sin 2 x & \text { for } x \in \triangle_{1}+\triangle_{2}, \\
0 & \text { for } x \in \triangle_{3} \\
0 & \text { for } x \in \triangle_{1}+\triangle_{2},
\end{array}\right. \\
& \varphi_{3}(x)=\left\{\begin{array}{cl}
0 & \text { for } x \in \triangle_{3} \\
-(2 / \pi)^{1 / 2} \sin x
\end{array}\right.
\end{aligned}
$$

The positive, self-adjoint transformation $H$ is then defined by

$$
H f=\int_{\Delta} H(x, y) f(y) d y
$$

where

$$
H(x, y)=\varphi_{1}(x) \varphi_{1}(y)+\varphi_{3}(x) \varphi_{3}(y) .
$$

The kernel $H(x, y)$ is evidently continuous, and

$$
H \varphi_{1}=\varphi_{1}, H \varphi_{2}=0, H \varphi_{3}=\varphi_{3}
$$

Let now the bounded, self-adjoint transformation $A$ be given by $A f=A(x) f(x)$, where

$$
A(x)=\left\{\begin{aligned}
-1 & \text { for } x \in \triangle_{1} \\
1 & \text { for } x \in \triangle_{2}+\triangle_{3} .
\end{aligned}\right.
$$

Then

$$
A \varphi_{1}=\varphi_{2}, A \varphi_{2}=\varphi_{1}, A \varphi_{3}=\varphi_{3}
$$

hence

$$
K(x, y)=A(x) H(x, y)=\varphi_{2}(x) \varphi_{1}(y)+\varphi_{3}(x) \varphi_{3}(y) .
$$

Since evidently $H^{1_{2}}=H$, the self-adjoint transformation $\widetilde{K}=H^{1_{2}} A H^{1_{3}}$ is corresponding with the kernel

$$
\widetilde{K}(x, y)=\int_{\Delta} H(x, z) K(z, y) d z=\varphi_{3}(x) \varphi_{3}(y)
$$

As we know, $K(x, y)$ and $\widetilde{K}(x, y)$ have the same characteristic values $\neq 0$; hence, since $\widetilde{K}(x, y)$ has evidently only the characteristic value $\lambda=1$, different from 0 , with the characteristic function $\varphi_{3}(x)$, the kernel $K(x, y)$ has also $\lambda_{1}=1$ as the only characteristic value $\neq 0$ with the $H$-normal characteristic function $\psi_{1}(x)=\varphi_{3}(x)$. Observing that $\chi_{1}(x)=H \psi_{1}=$ $=H \varphi_{3}=\varphi_{3}(x)$, so that $\lambda_{1} \psi_{1}(x) \overline{\chi_{1}(y)}=\varphi_{3}(x) \varphi_{3}(y)$, we have therefore

$$
K(x, y)-p(x, y)=\lambda_{1} \psi_{1}(x) \overline{\chi_{1}(y)},
$$

where $p(x, y)=\varphi_{2}(x) \varphi_{1}(y) \neq 0$ for $x$ and $y$ in the interior of $\Delta_{1}$ or $\Delta_{2}$; and $\int H(x, z) p(z, y) d z=0$, as required by Theorem 5 .

Furthermore, by Theorem 7,

$$
\int_{\triangle} K(x, y) f(y) d y=\lambda_{1} a_{1} \psi_{1}(x)+p(x)
$$

where

$$
a_{1}=\int_{\Delta} f(x) \overline{\chi_{1}(x)} d x \text { and } H p=\int_{\Delta} H(x, y) p(y) d y=0 .
$$

Taking $f(x)=p_{1}(x)$, we have

$$
\int_{\Delta} K(x, y) f(y) d y=K \varphi_{1}=A H \varphi_{1}=\varphi_{2}(x) \text { and } a_{1}=\left(\varphi_{1}, \varphi_{3}\right)=0 ;
$$

hence

$$
\varphi_{2}(x)=p(x)
$$

which shows that $p(x) \neq 0$ for $x$ in the interior of $\triangle_{1}$ and $\triangle_{2}$. Evidently $H p=H \varphi_{2}=0$, as required.


[^0]:    ${ }^{1)}$ Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 194-204, 205-212 and 409-416 (1946).
    ${ }^{2}$ ) E. Garbe, Zur Theorie der Integralgleichung dritter Art, Math. Annalen, 76, 527-547 (1915).
    ${ }^{3}$ ) Without stating it very clearly, Garbe uses the boundedness of $[A(x)]^{-1}$ in his formulae (10) and (34).

