Aerodynamics. — *Some problems of the motion of interstellar gas clouds. II* *). By J. M. BURGERS. (Mededeling No. 48 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

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6. *Quasi-stationary solution with "barometric" pressure gradient.* — Our starting point again will be eqs. (1) — (4). In order to be able to take a formal account of radiation loss, of ψ erg per cm³ and per sec, we multiply eq. (3) by $R/(k-1)$ and to the right hand side add the term $-\psi/\varrho$, so that it takes the form:

$$\frac{R}{k-1}\frac{DT}{Dt}=\frac{R}{k-1}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)=-RT\frac{\partial u}{\partial x}-\frac{\psi}{\varrho}\quad . \; . \; . \quad (3a)$$

The most convenient way for constructing an appropriate solution is to start from the *streamlines*, for which tentatively the following equation is assumed:

$$x=\varphi_0(t)-s\,\varphi_1(t)\quad . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

A streamline is defined by $s=$ constant. We take $s=0$ as the front of the moving gas layer and restrict s to positive values; further at $t=0$ (present epoch) we take: $\varphi_1(0)=1$. Hence the motion of the front is defined by: $x=\varphi_0(t)$, and s measures the distance behind the front at $t=0$. With constant t we have for any function of x and t:

$$\partial/\partial x=-(1/\varphi_1)\,.\,\partial/\partial s.\quad . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

From (17) we obtain for the velocity:

$$u=(dx/dt)_{\text{for constant } s}=\dot\varphi_0-s\,\dot\varphi_1\quad . \; . \; . \; . \; . \quad (19)$$

(using dots to denote derivatives with respect to t, while an accent will be used for derivatives of functions of s). This gives: $\partial u/\partial x=\dot\varphi_1/\varphi_1$.

For the density the following function is taken:

$$\varrho=\varrho^*(s)/\varphi_1(t)\quad . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

which satisfies the equation of continuity. Making use of (4), eq. (1) then gives:

$$\ddot\varphi_0-s\,\ddot\varphi_1=\frac{\varrho^{*\prime}}{\varrho^*}\frac{RT}{\varphi_1}+\frac{R}{\varphi_1}\frac{\partial T}{\partial s}\quad . \; . \; . \; . \; . \; . \quad (21)$$

Tentatively for ϱ^* we assume the formula:

$$\varrho^*=\varrho_m\,e^{-\beta s}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

*) Part I has appeared in these Proceedings **49** (1946), p. 589.

which gives the desired exponential decrease of the density with s. The coefficient ϱ_m determines the maximum value (ϱ_m/φ_1) of the density at the front of the advancing layer. Equation (21) now can be considered as a differential equation for T as a function of s; its integral is [10]):

$$RT = -\ddot{\varphi}_0 \varphi_1/\beta + (1+\beta s)\ddot{\varphi}_1 \varphi_1/\beta^2 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The pressure at the front of the advancing layer: $p_{fr} = R\,T_{fr}\,\varrho_{fr} = -\varrho_m(\ddot{\varphi}_0/\beta - \ddot{\varphi}_1/\beta^2)$, must be equal to the pressure generated in the interstellar gas into which our layer is penetrating. We will not enter into a detailed investigation of the shock waves and other phenomena set up in this gas, but assume that the "impact pressure" is given by $\frac{1}{2}(k+1)\,\varrho_0\,u_{fr}^2$, where ϱ_0 is the original density of the interstellar gas, while u_{fr}, the front velocity of the advancing layer, is equal to $\dot{\varphi}_0$. Hence we obtain:

$$\ddot{\varphi}_0/\beta - \ddot{\varphi}_1/\beta^2 = -\tfrac{1}{2}(k+1)(\varrho_0/\varrho_m)\,\dot{\varphi}_0^2 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Introduction of (23) into (3a) finally gives, after multiplication by $-(k-1)$:

$$\frac{\ddot{\varphi}_0 \varphi_1}{\beta}\left(\frac{\dddot{\varphi}_0}{\ddot{\varphi}_0} + k\frac{\dot{\varphi}_1}{\varphi_1}\right) - \frac{(1+\beta s)\ddot{\varphi}_1\varphi_1}{\beta^2}\left(\frac{\dddot{\varphi}_1}{\ddot{\varphi}_1} + k\frac{\dot{\varphi}_1}{\varphi_1}\right) = (k-1)\frac{\psi}{\varrho} \quad . \quad (25)$$

In general ψ/ϱ will be a complicated function of s, and it will not be possible to satisfy (25) in an exact way, which shows that the assumption (22) apparently was too restricted. We therefore start by investigating the case $\psi = 0$, and provisionally assume $\ddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$. Then to a first approximation (24) gives:

$$\ddot{\varphi}_0/\beta \cong -\tfrac{1}{2}(k+1)(\varrho_0/\varrho_m)\,\dot{\varphi}_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

from which:

$$u_{fr} = \dot{\varphi}_0 = \frac{2\varrho_m}{(k+1)\varrho_0}\frac{1}{\beta(t+t_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

t_0 being an integration constant. At the same time eq. (25) reduces to:

$$k\,\dot{\varphi}_1/\varphi_1 \cong -\dddot{\varphi}_0/\ddot{\varphi}_0 = 2/(t+t_0). \quad . \quad . \quad . \quad . \quad . \quad (25a)$$

from which, taking account of the condition $\varphi_1(0) = 1$:

$$\varphi_1 = \{(t+t_0)/t_0\}^{2/k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

With the same degree of approximation (23) becomes:

$$RT \cong -\frac{\ddot{\varphi}_0\varphi_1}{\beta} = \frac{2\varrho_m}{(k+1)\varrho_0}\frac{1}{\beta^2 t_0^2}\left(\frac{t_0}{t+t_0}\right)^{2-2/k} \quad . \quad . \quad . \quad . \quad (28)$$

[10]) A term proportional to $e^{\beta s}$ might be added to this solution, which term could be written $\varphi_1\varphi_2\,e^{\beta s}$, where φ_2 is an arbitrary function of t. This term, however, has been discarded. The discrepancy observed in eq. (25) cannot be removed by introducing such a term. — As will be seen below we suppose that the second term of (23) is of minor importance.

As the function φ_1 determines a gradual expansion of the moving gas layer, (28) expresses the adiabatic decrease of T consequent upon this expansion, which is obtained as radiation is neglected.

7. *Continuation. — Numerical data for a thin strip or sheet of cloud in NGC 6960. — Influence of radiation loss.* — From data supplied by OORT (in view of their vagueness slightly adjusted so as to obtain a satisfactory fit in calculating the temperature) we take: $u_{fr} = 7{,}0 \,.\, 10^6$ (70 km/sec) at the present epoch ($t = 0$); $\varrho_m = 5{,}2 \,.\, 10^{-23}$ (31,5 H-atoms per cm^3); $\varrho_0 = 10^{-24}$. Then (26) gives: $u_{fr} = 39/\beta(t + t_0)$, so that for $t = 0$: $\beta\, t_0 = 5{,}57 \,.\, 10^{-6}$. The timescale evidently is proportional to the linear scale, which according to (22) is fixed by $1/\beta$. Taking $1/\beta$ equal to $2{,}5 \,.\, 10^{16}$ cm (the estimated visual thickness of a sheet is about 5″), we find $\beta = 4{,}0 \,.\, 10^{-17}$, giving: $t_0 = 1{,}39 \,.\, 10^{11}$ sec $= 4410$ year.

From (26) and (27) we now calculate for $t = 0$: $\ddot{\varphi}_0 = -5{,}03 \,.\, 10^{-5}$; $\ddot{\varphi}_1 = +1{,}24 \,.\, 10^{-23}$; $\ddot{\varphi}_1/\beta = 3{,}1 \,.\, 10^{-7}$, so that the assumption $\ddot{\varphi}_1/\beta \langle\langle |\ddot{\varphi}_0|$ appears to be satisfied. The expansion determined by φ_1 is very gradual; with $t = 4410$ year we have $\varphi_1 = 2{,}3$.

Further from (28) we deduce, again for $t = 0 : T = 15100°$, which is in accordance with the astronomical estimate.

We now turn back to the radiation loss. From data supplied by OORT it follows that in the present problem the radiation loss cannot be neglected. We might try to obtain a better approximation than that given by (25*a*) by taking for ψ/ϱ a function of the time alone, calculated with an average value of the density, replacing (25) by:

$$k\,\frac{\dot{\varphi}_1}{\varphi_1} = -\frac{\dddot{\varphi}_0}{\ddot{\varphi}_0} + \frac{(k-1)\,\beta}{\ddot{\varphi}_0\,\varphi_1}\,\frac{\psi}{\varrho}. \qquad (25b)$$

Without integrating this equation it will be evident that the radiation loss tends to decrease the rate of expansion and even may turn it into a contraction. The assumption $\ddot{\varphi}_1/\beta \langle\langle |\ddot{\varphi}_0|$ does not seem to be impaired by the correction. As the temperature is given by $R\,T \cong -\ddot{\varphi}_0\varphi_1/\beta$, the decrease of T with time becomes faster than that given previously by (28).

Although the solution is incomplete, the following conclusions seem possible. The fact that the astronomical data for u_{fr}, ϱ_m, ϱ_0 lead to the correct value for T, can be taken as an indication that the assumption concerning the "impact pressure" experienced in moving through the interstellar gas is not far from the truth. The result concerning φ_1 gives an acceptable explanation of the mechanism by which the very thin sheets of cloud for long times can retain their elegant appearance. The estimate for the timescale depends directly upon the assumption concerning the impact pressure. It appears rather short when considered in relation to the distance between the two nebulae NGC 6960 and 6992 ($155' = 48{,}7 \,.\, 10^{18}$ cm), which are believed to have originated from a single explosion and thus should have travelled each about $24{,}4 \,.\, 10^{18}$ cm. As formula (26) cannot be applied to velocities approaching that of light, we may ask what would be the distance covered in the interval of time evolved since u_{fr} decreased from 10.000 km/sec (10^9 cm/sec; $t + t_0 = 30{,}9$ year) until its present value of 70 km/sec ($7 \cdot 10^6$ cm/sec; $t + t_0 = 4410$ year). This distance amounts to $4{,}83 \cdot 10^{18}$ cm, which is less than $^1/_5$ of the total amount travelled since the explosion (to cover the remaining $19{,}6 \cdot 10^{18}$ cm at 10^9 cm/sec would require ca. 620 year). Evidently eq. (26) must be considered as an approximation valid for the present epoch only, and the decrease of velocity with time originally must have taken place at a slower rate [11]).

8. *Possibility of surface waves on interstellar clouds.* — Thus far we have restricted to the consideration of motion in one dimension only. It is evident that the real motions and currents in interstellar space will be of a much more complicated character. A comparison with atmospheric movements in many cases will have impressed itself on the mind of the observer. In particular the question has been brought forward whether certain types of structure, resembling cirrus clouds in the earth's atmosphere, may indicate the presence of waves or perhaps of vortices, such as can arise owing to the instability of the motion of fluid layers sliding over each other with different velocities. As an instance of such structures the clouds around the Pleiades must be mentioned; in particular the cloud around the star Merope shows a marked periodic structure. In certain parts of these clouds, *e.g.* in those around Alcyone and between Maia and Merope, several systems of periodic structure seem to be present, crossing each other at rather large angles. It is probable that the illuminated patches all belong to one single cloud, extending at least over about $1^\circ = 5{,}4 \cdot 10^{18}$ cm, and that the periodic structures in the various patches are connected with each other.

When it is attempted to explain such structures by assuming a wave

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[11]) The relations from which the timescale has been deduced, can be put in the form:

$$p_{fr} = R\,T_{fr}\,\varrho_{fr} = \tfrac{1}{2}(k+1)\,\varrho_0\,u_{fr}^2 = -\,M_1\,(d\,u_{fr}/dt),$$

where $M_1 = \varrho_{fr} \cdot L$ (M_1 being the mass of the whole layer per unit area and L its effective thickness). This equation gives: $d\,u_{fr}/dt = -R\,T_{fr}/L$, and thus fixes the present timescale from very simple data.

With the formulae of the text: $\varrho_{fr} = \varrho_m/\varphi_1$; $L = \varphi_1/\beta$. The assumption of a smaller value of β (leading to thinner sheets of cloud) reduces both the timescale and the distance covered. Larger values of β are in contradiction with observation. The assumption of a substantially smaller value of ϱ_0, which would lead to a greater value of $\beta\,t_0$, would change the value of the temperature in the same ratio, and thus seems unlikely. It may be thought that perhaps originally the clouds have moved through a portion of the interstellar gas of smaller density, but the problem then presents itself whether both nebulae have suffered the same adventure.

No account has been taken of the possibility of lateral motions (a divergent motion in the plane of the sheet might produce a continuous reduction of thickness), but in view of the elementary character of the relation between temperature, density and impact pressure, such motions are not likely to lead to an appreciably different timescale.

formation of similar nature as observed with certain atmospheric clouds. several difficulties present themselves. The wave motion in the atmosphere, arising between layers of air sliding over each other, is controlled by gravity. Furthermore in atmospheric clouds the wave pattern becomes visible in consequence of condensation of water vapour in those elements of volume which have suffered a reduction in temperature through their upward displacement. In the case of the interstellar gas an increased condensation of dust particles perhaps may be connected with a local increase in density and thus could make visible periodic changes of the density.

As to gravity, local gravitational fields due to separate clouds, as have been considered in section 1, may bring with them values of g of the order of $1{,}67 \,.\, 10^{-9}$. An estimate of the general gravitational field of the Galaxy can be obtained by taking the mass inside a sphere described with the distance from the centre of the Galaxy to the sun as radius (ca. $3 \,.\, 10^{22}$ cm) equal to $2{,}4 \,.\, 10^{11}$ times the mass of the sun, giving $M_{\mathrm{Gal}} = 4{,}7 \,.\, 10^{44}$ gr, which makes the value of g in our neighbourhood $3{,}5 \,.\, 10^{-8}$. The general gravitational field, however, will be compensated by the centripetal acceleration of the clouds, so long as these move freely. Residual effects might be found when clouds collide with each other and produce abnormal accelerations in their surface layers. These accelerations even may exceed considerably the value of g calculated from the gravitational attraction: in the example treated in the preceding section the acceleration in the thin sheet is $du/dt = -u/(t + t_0) = -5 \,.\, 10^{-5}$ at the present epoch. It is not easy to decide whether fields produced in this way can regulate the appearance of wave motions, supposing that the colliding masses at the same time should have a tangential velocity with respect to each other. As one of the two colliding masses will suffer a deceleration and the other an acceleration, the equivalent g-vectors will have opposite directions, pointing towards the surface of separation. This produces a situation different from that found in ordinary wave motion, which appears in stratified material where gravity everywhere acts in the same direction, pointing from the region of lower density to that of higher density.

A comparison between interstellar and atmospheric motions might be made by applying the theory of similarity, according to which similar motions can exist when in the first place there is geometrical similarity in pattern of motion and in density distribution, while in the second place REYNOLDS' number $Re = \varrho U \lambda / \mu$, FROUDE's number $Fr = U^2 / g\lambda$ and MACH's number $Ma = U/c$ must have the same values in the cases to be compared (U: a characteristic value of the velocities to be considered; λ: wavelength; ϱ: density, for the interstellar gas approx. 10^{-22}, for air at the height of say 4 km 0,0008; μ: viscosity, for the interstellar gas approx. 0,0023, for air 0,00017; c: velocity of sound). All ordinary cases of wave motion are such in which Ma is far below unity and then is of no importance. We therefore take FROUDE's number first, and consider a case of stationary wave motion along the surface of separation between two layers of material, having the densities ϱ, ϱ', the surface of separation being perpendicular to the local gravity vector with density increasing in the direction to which the vector points, the layers sliding over each other with velocities U, U' (measured with respect to the wave pattern). Then the wavelength must satisfy the relation [12]):

$$(\varrho - \varrho')\, g\, \lambda/2\pi = \varrho\, U^2 + \varrho'\, U'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

provided λ is small compared with the thickness of the layers. In this case $Fr = (\varrho - \varrho')/2\pi\varrho - \varrho' U'^2/\varrho g \lambda$, from which appears $Fr < 1/2\pi$. Hence in order to obtain a wavelength of $3 \, . 10^{16}$ cm as found in the Merope cloud, we must have $U^2/g < 5 \, . \, 10^{15}$, which would require either rather small values of the velocity (*e.g.*: $U = 2 \, . \, 10^3$ with g of the order 10^{-9}) or very high values of g, of the order 10^{-7} and more, provided of course we have to do with stationary (stable) wave motion.

With larger values of U^2/g the wave motion is unstable and in course of time increases in amplitude. This in itself is not impossible, but when U^2/g considerably exceeds the value mentioned the observed wavelength must be determined by some other cause; one might think of the thickness of some intermediate layer. Problems referring to wave motion in stratified systems have been treated by RAYLEIGH, TAYLOR, and others [13]). But as there is no direct clue to the case which should be chosen as a basis for comparison and as we miss any trustworthy datum about the value of g, no promising way for the construction of an appropriate solution as yet is to be seen.

It might be that we should discard any reference to the influence of gravity, and should look exclusively to features of the velocity distribution in order to find an explanation for the observed value of λ. The question arises if viscosity can play a part. If we assume a velocity U of $5 \, . \, 10^5$ cm/sec, the value of REYNOLDS' number $\varrho U \lambda/\mu$ comes out as 650. This appears rather small when compared with the results of some theoretical investigations on the stability of laminar motion when viscosity is effective, which would make us to expect a value above 15000 [14]). The supposition of a much higher value of U, however, would bring us to velocities above

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12) See H. LAMB, Hydrodynamics (Cambridge 1932), p. 377 (Art. 234, form. 5).

13) RAYLEIGH, Theory of Sound (London 1945), Vol. II, Ch. XXI (p. 376 seqq.). — G. I. TAYLOR, Effect of variation in density on the stability of superposed streams of fluid, Proc. Roy. Soc. (London) A **132**, p. 499, 1931. — See also: V. BJERKNES, J. BJERKNES, H. SOLBERG und T. BERGERON, Physikalische Hydrodynamik (Berlin 1933), Kap. VIII, IX u. X (pp. 305—421).

14) See H. SCHLICHTING, Zur Entstehung der Turbulenz bei der Plattenströmung. Göttinger Nachrichten Math.-physik. Kl., 1933 (II, no. 38), p. 181. From fig. 3 (p. 197) the minimum value of $U_m \delta^*/\nu$ is read off as 575 (comp. p. 202; $\nu = \mu/\varrho$) with $\alpha\delta^* = 0{,}23$, where $\alpha = 2\pi/\lambda$; this gives $U_m \lambda/\nu = 15700$. The maximum value of $\alpha\delta^*$ is about 0,28 with $U_m \delta^*/\nu =$ ca. 860; $U_m \lambda/\nu = 19300$. In both cases the waves are just on the limit between stability and instability; experimental observations point to the appearance of waves well inside the domain of instability, with still greater values of $U_m \lambda/\nu$.

the speed of sound, in which case the problem takes a wholly different character. Not much is known concerning the formation of waves in this case [15]. Further investigations are necessary before the question raised in this section can be settled.

9. *Considerations on the motion of the interstellar gas in the Galaxy as a whole.* — OORT has raised the problem whether the erratic motions and forms of the interstellar clouds may be due to effects depending upon the variation of galactic rotation with the distance from the centre of the Galaxy. It may be supposed that rotational motion with a regular velocity distribution ("laminar" rotation) would prove to be unstable, and that a form of turbulence should set in. In this connection one may first make an estimate of the magnitude of REYNOLDS' number for the rotation of the gas in the Galaxy. Assuming a radius of 50000 light years $= 5 \cdot 10^{22}$ cm, a circumferential velocity of 260 km/sec $= 2{,}6 \cdot 10^{7}$ cm/sec, $\varrho = 3 \cdot 10^{-24}$ and $\mu = 0{,}0023$, we obtain: $Re = 1{,}7 \cdot 10^{9}$. This value is high, though not excessive. In experimental situations (rotating basin filled with water) values of $8 \cdot 10^{6}$ probably can be reached; in atmospheric cyclones with a radius of (often far) over 100 km values of the order 10^{9} certainly occur. The boundary conditions in the latter cases, however, are wholly different from those existing in the Galaxy, and this prevents a direct comparison.

It is extremely difficult to investigate the stability of laminar rotation, in particular as account must be taken of changes in density and temperature, while even the changes in the gravitational field may be of some importance. According to JEANS gravitational instability in a homogeneous gas may arise for disturbances of a period λ_g exceeding a certain limit of the order $c\sqrt{\pi/\gamma\varrho}$ (c: velocity of sound; γ: gravitation constant $= 6{,}67 \cdot 10^{-8}$; ϱ_0: original density) [16]. With $T = 10.000°$, $c = 1{,}18 \cdot 10^{6}$, $\varrho_0 = 3 \cdot 10^{-24}$ this limit becomes $4{,}7 \cdot 10^{21}$ cm. In view of the dimensions of the Galaxy such a form of instability consequently may play its part. Its timescale is of the order $\lambda_g/c = 4{,}0 \cdot 10^{15}$ sec $= 1{,}3 \cdot 10^{8}$ year. As the distances between the interstellar clouds and their linear dimensions are much smaller than the value of λ_g just mentioned, other causes must be operative as well, and these evidently should be found in the rotation.

We cannot attack this problem at present, but a few remarks may be added. A laminar rotation of the interstellar gas as a whole would require a nearly complete balance between gravitational attraction and centrifugal force, as otherwise extremely large pressure gradients would be necessary. Estimating the value of g in our neighbourhood as $\gamma M_{\mathrm{Gal}}/r^2 = 3{,}13 \cdot 10^{37}/r^2$, we find $U = 5{,}6 \cdot 10^{18}/\sqrt{r}$. Viscous friction in

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[15]) See for some provisional results (which rather would point to a greater stability of surfaces of separation with velocities exceeding that of sound): J. ACKERET, Ueber Luftkräfte bei sehr grossen Geschwindigkeiten, Helv. Phys. Acta **1**, p. 301, 1928.

[16]) J. H. JEANS, Astronomy and Cosmogony (Cambridge, 1928), p. 340.

this case will produce a resultant force per unit volume of magnitude: $-0{,}75\,\mu U/r^2$, which at the sun's distance from the centre of the Galaxy ($r = 3\,.\,10^{22}$) amounts to $1{,}92\,.\,10^{-48}\,U$. With the density $3\,.\,10^{-24}$ this would lead to an acceleration of $-6{,}4\,.\,10^{-25}\,U$. The timescale thus arrived at is so enormous that we must conclude that viscous friction is inefficient to regularize the rotation.

A force will also be produced by the resistance of the stars. Taking as example the sun, and writing U_{rel} for its motion relatively to the gas, we find, curiously enough, that REYNOLDS' number for this relative motion $\varrho U_{rel}\,R_{sun}/\mu$ (with $R_{sun} = 7\,.\,10^{10}$ cm) even for $U_{rel} = 300$ km/sec $=$ $= 3\,.\,10^7$ cm/sec comes out as $2{,}74\,.\,10^{-3}$, i.e. far below unity. Hence we must apply STOKES' law of resistance. With the star density in the neighbourhood of the sun $N = 1/(3\,.\,10^{56})$ the force per unit volume becomes: $6\pi\mu R_s\,U_{rel}\,N = 10^{-47}\,U_{rel}$. Although comparable with the internal viscous friction, this force just as little can play an important part.