Mathematics. — *On the G-function.* IV. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 10. Second expansion formula.

Theorem 2. Assumptions: *k, l, m, n, p and q are integers with*

$$q \geqq 1, \quad 0 \leqq l-1 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq k \leqq q; \quad . \quad . \quad (111)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_k$ *fulfil the conditions* (1), (99) *and* (100); *r is an arbitrary integer which satisfies the inequality*

$$r \geqq \operatorname{Max}(0, k+l-m-n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = A^{m,n-l+1}_{\;\;k} \sum_{s=0}^{r-1} \Omega^{m,n-l+1}_{\;\;k}(s)\, G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2s-1)\pi i}) \\ + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{\;\;k}(t)\, G_{p,q}^{k,l,n}(z\,e^{(k+l-m-n-2r)\pi i} \,\|\, a_t). \end{aligned}\right\} (113)$$

Proof. We may distinguish three cases:

First case:

$$1 \leqq l \leqq n \leqq p \leqq q, \quad k+l-n \leqq m \leqq k \leqq q, \quad r \geqq 0.$$

Formula (113) can be established by induction. If $r=0$, then (113) reduces to (102) with $\lambda = 0$. We may therefore suppose $r \geqq 1$ and assume that (113) with $r-1$ instead of r has yet been proved.

Now we have by (58)

$$G_{p,q}^{k,l,n}(z\,e^{(k+l-m-n-2r+2)\pi i} \,\|\, a_t) = e^{2\pi i a_t} G_{p,q}^{k,l,n}(z\,e^{(k+l-m-n-2r)\pi i} \,\|\, a_t)$$
$$-2\pi i\, e^{\pi i a_t} G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2r+1)\pi i}).$$

If this is substituted on the right-hand side of (113) with $r-1$ instead of r, the sum $\sum_{t=1}^{n-l+1}$ not only gives the corresponding sum in (113) but also

$$-2\pi i\, G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2r+1)\pi i}) \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r-1)\pi i a_t} \triangle^{m,n-l+1}_{\;\;k}(t)$$

and this expression may by means of (59) be reduced to

$$A^{m,n-l+1}_{\;\;k}\, \Omega^{m,n-l+1}_{\;\;k}(r-1)\, G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2r+1)\pi i}),$$

since $\overline{\Omega}^{m,n-l+1}_{\;\;k}(k+l-m-n-r) = 0$, because of $k+l-m-n-r \leqq -1$.

It follows therefore that the sum $\sum_{s=0}^{r-2}$ on the right-hand side of (113) with $r-1$ instead of r reduces to the sum $\sum_{s=0}^{r-1}$ in (113). Thus the first case is finished.

Second case:

$$1 \leqq l \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq k+l-n,\ r \geqq k+l-m-n.$$

This case may also be proved by induction. Owing to the first case formula (113) is true if $m = k+l-n$. We may therefore suppose $0 \leqq m \leqq k+l-n-1$ and assume that (113) with $m+1$ instead of m has yet been proved.

Now it follows from (113) with $m+1$ instead of m, $z\,e^{-\pi i}$ instead of z, $r-1$ instead of r and s replaced by $s-1$

$$e^{\pi i b_{m+1}} G_{p,q}^{m+1,n}(z\,e^{-\pi i}) = e^{\pi i b_{m+1}} A^{m+1,n-l+1}_{k} \sum_{s=1}^{r-1} \Omega^{m+1,n-l+1}_{k}(s-1)\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2s-1)\pi i})$$
$$+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} e^{\pi i (b_{m+1}-a_t)} \triangle^{m+1,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z\,e^{(k+l-m-n-2r)\pi i} \,\|\, a_t).$$

We further have by (113) with $m+1$ instead of m and $z\,e^{\pi i}$ instead of z

$$e^{-\pi i b_{m+1}} G_{p,q}^{m+1,n}(z\,e^{\pi i}) = e^{-\pi i b_{m+1}} A^{m+1,n-l+1}_{k} \sum_{s=0}^{r-1} \Omega^{m+1,n-l+1}_{k}(s)\, G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2s-1)\pi i})$$
$$+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} e^{\pi i (a_t-b_{m+1})} \triangle^{m+1,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z\,e^{(k+l-m-n-2r)\pi i} \,\|\, a_t).$$

From these two relations and (55) it appears

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = &-\frac{e^{-\pi i b_{m+1}}}{2\pi i} A^{m+1,n-l+1}_{k}\, \Omega^{m+1,n-l+1}_{k}(0)\, G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-1)\pi i}) \\ &-\frac{e^{-\pi i b_{m+1}}}{2\pi i} A^{m+1,n-l+1}_{k} \sum_{s=1}^{r-1} \{\Omega^{m+1,n-l+1}_{k}(s) - e^{2\pi i b_{m+1}} \Omega^{m+1,n-l+1}_{k}(s-1)\} \times \\ &\qquad\times G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2s-1)\pi i}) \\ &+\frac{1}{\pi} \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \sin(b_{m+1}-a_t)\pi \,.\, \triangle^{m+1,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,\|\, a_t). \end{aligned}\right\} \quad (114)$$

Now it is obvious on account of the definition of the coefficients A

$$-\frac{e^{-\pi i b_{m+1}}}{2\pi i} A^{m+1,n-l+1}_{k} = A^{m,n-l+1}_{k}.$$

Moreover we find without difficulty in view of the definition of the coefficients Ω

$$\Omega^{m+1,n-l+1}_{k}(0) = \Omega^{m,n-l+1}_{k}(0)$$

and

$$\Omega^{m+1,n-l+1}_{k}(s) - e^{2\pi i b_{m+1}}\,\Omega^{m+1,n-l+1}_{k}(s-1) = \Omega^{m,n-l+1}_{k}(s).$$

Finally it follows from the definition of the coefficients $\triangle$ that

$$\frac{1}{\pi}\sin(b_{m+1}-a_t)\pi\,.\,\triangle^{m+1,n-l+1}_{k}(t) = \triangle^{m,n-l+1}_{k}(t).$$

Formula (114) is therefore equivalent to (113). So the second case is also finished.

Third case:

$$q \geqq 1,\ n = l-1,\quad 0 \leqq l-1 \leqq p \leqq q,\quad 0 \leqq m \leqq k \leqq q,\quad r \geqq 1+k-m.$$

From the definition of the function G we easily deduce

$$G^{m,l-1}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = G^{m,l}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}\alpha,a_1,\ldots,a_p\\ b_1,\ldots,b_k,\alpha,b_{k+1},\ldots,b_q\end{matrix}\right); \tag{115}$$

herein is α an arbitrary number.

To the function $G^{m,l}_{p+1,q+1}(z)$ on the right-hand side of this relation we may apply (113) with $n = l$, $k+1$ instead of k, $p+1$ instead of p and $q+1$ instead of q. Now it is clear, on account of the definitions of the coefficients A, Ω and $\triangle$ and the function G, that in the particular case under consideration ($\alpha, a_1, \ldots, a_p$ instead of $a_1, a_2, \ldots, a_{p+1}$ and $b_1, \ldots, b_k, \alpha, b_{k+1}, \ldots, b_q$ instead of $b_1, \ldots, b_{q+1}$)

$$A^{m,1}_{k+1} = A^{m,0}_{k}\,,\quad \Omega^{m,1}_{k+1}(s) = \Omega^{m,0}_{k}(s),$$
$$G^{k+1,l-1,l}_{p+1,q+1}(\zeta) = G^{k,l-1,l-1}_{p,q}(\zeta)$$

and

$$\triangle^{m,1}_{k+1}(1) = 0.$$

We therefore get (113) with $n = l-1$ when we apply (113) to the right-hand side of (115).

With this the theorem has been completely proved.

§ 11. Third expansion formula.

Theorem 3. Assumptions: *k, l, m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq l-1 \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq k \leqq q \text{ and } m+n \leqq k+l;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions (1), (99) and (100); r is an arbitrary integer which satisfies the inequality

$$0 \leqq r \leqq k+l-m-n.$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = {} & A^{m,n-l+1}_{\ \ k} \sum_{s=0}^{r-1} \Omega^{m,n-l+1}_{\ \ k}(s)\, G_{p,q}^{k,l-1,n}\left(z\, e^{(k+l-m-n-2s-1)\pi i}\right) \\ & + \overline{A}^{m,n-l+1}_{\ \ k} \sum_{\tau=0}^{k+l-m-n-r-1} \overline{\Omega}^{m,n-l+1}_{\ \ k}(\tau)\, G_{p,q}^{k,l-1,n}\left(z\, e^{(m+n-k-l+2\tau+1)\pi i}\right) \\ & + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{\ \ k}(t)\, G_{p,q}^{k,l,n}\left(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right). \end{aligned}\right\} \quad (116)$$

Proof. The theorem can be established by induction. The formula is true if $r = k + l - m - n$, since (116) with $r = k + l - m - n$ is equivalent to (113) with $r = k + l - m - n$. We may therefore suppose $0 \leqq r \leqq k + l - m - n - 1$ and assume that (116) with $r + 1$ instead of r has yet been proved. Now it follows from (57), if $n \geqq l$,

$$G_{p,q}^{k,l,n}\left(z\, e^{(k+l-m-n-2r-2)\pi i} \,\|\, a_t\right) = e^{-2\pi i a_t} G_{p,q}^{k,l,n}\left(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right) + 2\pi i\, e^{-\pi i a_t} G_{p,q}^{k,l-1,n}\left(z\, e^{(k+l-m-n-2r-1)\pi i}\right).$$

If this is substituted on the right-hand side of (116) with $r + 1$ instead of r, the sum $\sum_{t=1}^{n-l+1}$ not only yields the sum $\sum_{t=1}^{n-l+1}$ in (116) but besides

$$2\pi i\, G_{p,q}^{k,l-1,n}\left(z\, e^{(k+l-m-n-2r-1)\pi i}\right) \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r+1)\pi i a_t} \triangle^{m,n-l+1}_{\ \ k}(t) \text{ [36]}$$

and this expression is by (59) equal to

$$-G_{p,q}^{k,l-1,n}\left(z\, e^{(k+l-m-n-2r-1)\pi i}\right) \left\{ A^{m,n-l+1}_{\ \ k} \Omega^{m,n-l+1}_{\ \ k}(r) - \overline{A}^{m,n-l+1}_{\ \ k} \overline{\Omega}^{m,n-l+1}_{\ \ k}(k + l - m - n - r - 1) \right\}.$$

The sums $\sum_{s=0}^{r}$ and $\sum_{\tau=0}^{k+l-m-n-r-2}$ on the right-hand side of (116) with $r + 1$ instead of r reduce therefore to the sums $\sum_{s=0}^{r-1}$, respect. $\sum_{\tau=0}^{k+l-m-n-r-1}$ in (116). So the theorem is established.

§ 12. Extension of theorem 3.

In the same manner as formula (113) we may prove the formula conjugate to (113)

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = {} & \overline{A}^{m,n-l+1}_{\ \ k} \sum_{s=0}^{r-1} \overline{\Omega}^{m,n-l+1}_{\ \ k}(s)\, G_{p,q}^{k,l-1,n}\left(z\, e^{(m+n-k-l+2s+1)\pi i}\right) \\ & + \sum_{t=1}^{n-l+1} e^{(k+l-m-n-2r)\pi i a_t} \triangle^{m,n-l+1}_{\ \ k}(t)\, G_{p,q}^{k,l,n}\left(z\, e^{(m+n-k-l+2r)\pi i} \,\|\, a_t\right). \end{aligned}\right\} \quad (117)$$

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[36]) This is still true if $n = l - 1$, since the sums $\sum_{t=1}^{n-l+1}$ then vanish.

This relation holds, provided that the conditions (111), (112), (1), (99) and (100) are satisfied.

We now replace r by $k+l-m-n-r$ and s by τ. Then formula (117) reduces to

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) &= \overline{A}^{m,n-l+1}_{k} \sum_{\tau=0}^{k+l-m-n-r-1} \overline{\Omega}^{m,n-l+1}_{k}(\tau)\, G_{p,q}^{k,l-1,n}\left(z e^{(m+n-k-l+2\tau+1)\pi i}\right) \\ &+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}\left(z e^{(k+l-m-n-2r)\pi i} \| a_t\right); \end{aligned}\right\} \quad (118)$$

herein is r an arbitrary integer which satisfies the inequality

$$r \leqq \operatorname{Min}(0, k+l-m-n).$$

We may now show that formula (116) holds under conditions which are much more general than those of theorem 3. Indeed, I will prove:

Theorem 4. *Suppose that k, l, m, n, p and q are integers which satisfy the conditions* (111); *further that the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions* (1), (99) *and* (100); *finally that r is an arbitrary integer (positive, negative or zero).*

Then formula (116) *is valid.*

P r o o f. Observing that $\Omega^{m,n-l+1}_{k}(s)$ and $\overline{\Omega}^{m,n-l+1}_{k}(s)$ vanish for $s=-1,-2,-3,\ldots$, we may distinguish six cases [37]):

F i r s t c a s e: $m+n \geqq k+l$, $r \geqq 0$. Formula (116) reduces to (113).

S e c o n d c a s e: $m+n \geqq k+l$, $k+l-m-n \leqq r \leqq 0$. Formula (116) reduces to (102) with $\lambda = -r$.

T h i r d c a s e: $m+n \geqq k+l$, $r \leqq k+l-m-n$. Formula (116) reduces to (118).

F o u r t h c a s e: $m+n \leqq k+l$, $r \geqq k+l-m-n$. Formula (116) reduces to (113).

F i f t h c a s e: $m+n \leqq k+l$, $0 \leqq r \leqq k+l-m-n$. This is the case of theorem 3.

S i x t h c a s e: $m+n \leqq k+l$, $r \leqq 0$. Formula (116) reduces to (118).

§ 13. Some more lemmas.

Lemma 19. *Suppose that k, l, p, q, $\varkappa$ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ \varkappa \geqq 1,\ 0 \leqq \nu \leqq k \leqq q \ \text{ and } \ l+\nu-1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions

$$a_j - b_h \neq 1, 2, 3, \ldots \ (j=\nu+1, \ldots, l+\nu-1;\ h=1, \ldots, k), \quad (119)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \ (j=1, \ldots, \nu;\ t=1, \ldots, \nu;\ j \neq t). \quad (120)$$

[37]) Comp. also definition 4.

Then the following formula holds [38]):

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) &= -\sum_{h=1}^{k-\nu+\varkappa-1} \Omega^{0,\nu}_{k}(h)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{-2h\pi i}) \\ &\quad - \frac{1}{A^{0,\nu}_{k}} \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-1)\pi i a_\sigma}\, \triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (121)$$

P r o o f. If we put $m = 0$, $n = l + \nu - 1$, $r = k - \nu + \varkappa$ and $z = \zeta\, e^{(\nu-k)\pi i}$ in (113) and suppose that $\nu \leq k$, then we find (121), because of (5) and (50).

Lemma 20. *Suppose that $k, l, p, q, \varkappa, \lambda$ and ν are integers with*

$$l \geqq 1,\; q \geqq 1,\; 1 \leqq \lambda \leqq \varkappa,\; 0 \leqq \nu \leqq k \leqq q \quad \text{and} \quad l + \nu - 1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) &= \sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi^{k,0}_{\nu,k}(h;\lambda)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i}) \\ &- \frac{1}{A^{0,\nu}_{k}} \sum_{\tau=0}^{\lambda-1} \Phi^{k,0}_{\nu,k}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\tau-1)\pi i a_\sigma}\, \triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (122)$$

P r o o f. From (74) (with $\lambda = 1$) and (73) (with $\lambda = 0$) it follows

$$\Phi^{k,0}_{\nu,k}(h;1) = -\,\Omega^{0,\nu}_{k}(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

We further have by (74) if $\lambda \neq 1 - h$

$$\Phi^{k,0}_{\nu,k}(h;\lambda) = -\sum_{\tau=0}^{\lambda-2} \Phi^{k,0}_{\nu,k}(1;\tau)\, \Omega^{0,\nu}_{k}(h+\lambda-\tau-1) - \Phi^{k,0}_{\nu,k}(1;\lambda-1)\, \Omega^{0,\nu}_{k}(h);$$

in view of (74) we find therefore if $\lambda \neq 1 - h$

$$\Phi^{k,0}_{\nu,k}(h;\lambda) = \Phi^{k,0}_{\nu,k}(h+1;\lambda-1) - \Phi^{k,0}_{\nu,k}(1;\lambda-1)\, \Omega^{0,\nu}_{k}(h). \quad . \quad (124)$$

From (123) and (73) (with $\lambda = 0$) it appears that (122) with $\lambda = 1$ reduces to (121). Hence we may suppose $2 \leq \lambda \leq \varkappa$ and assume that (122) with $\lambda - 1$ instead of λ has already been proved.

Now formula (122) with $\lambda - 1$ instead of λ may be written in the following way

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) &= \Phi^{k,0}_{\nu,k}(1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2\lambda+2)\pi i}) \\ &+ \sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi^{k,0}_{\nu,k}(h+1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i}) \\ &- \frac{1}{A^{0,\nu}_{k}} \sum_{\tau=0}^{\lambda-2} \Phi^{k,0}_{\nu,k}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\tau-1)\pi i a_\sigma}\, \triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (125)$$

[38]) The products $\triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w \,\|\, a_\sigma)$ on the right of (121) must be defined by a limiting process when $a_\sigma - b_h = 1, 2, 3, \ldots$ $(1 \leq h \leq k)$; comp. the Remark at the end of § 9.

The first term on the right-hand side of this relation is because of (121) with $\zeta e^{(-2\lambda+2)\pi i}$ instead of ζ and $\varkappa-\lambda+1$ instead of $\varkappa$ equal to

$$\Phi_{\nu,k}^{k,0}(1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta e^{(-2\lambda+2)\pi i})$$

$$=-\sum_{h=1}^{k-\nu+\varkappa-\lambda}\Phi_{\nu,k}^{k,0}(1;\lambda-1)\,\Omega^{0,\nu}_{k}(h)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta e^{(-2h-2\lambda+2)\pi i})$$

$$-\frac{\Phi_{\nu,k}^{k,0}(1;\lambda-1)}{A^{0,\nu}_{k}}\sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\lambda+1)\pi i a_\sigma}\,\Delta^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta e^{(2\nu-2k-2\varkappa+1)\pi i}\,||\,a_\sigma).$$

If this is substituted on the right-hand side of (125), then (125) reduces in virtue of (124) to (122), so that the lemma has been proved.

Lemma 21. *Suppose that k, l, p, q, r and ν are integers with*

$$l\geqq 1,\ q\geqq 1,\ r\geqq 1,\ 0\leqq \nu\leqq k\leqq q \text{ and } l+\nu-1\leqq p\leqq q;$$

suppose further that the numbers $a_1,\ldots,a_{l+\nu-1}$ and $b_1,\ldots,b_k$ satisfy the conditions (119) and (120).

Then the following formula holds:

$$\left.\begin{aligned}G_{p,q}^{k,l-1,l+\nu-1}(\zeta)&=\sum_{h=1}^{k-\nu}\Phi_{\nu,k}^{k,0}(h;r)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta e^{(-2h-2r+2)\pi i})\\ &-\frac{1}{A^{0,\nu}_{k}}\sum_{\sigma=1}^{\nu} e^{(k-\nu+1)\pi i a_\sigma}\,\Theta_{\nu}^{k,0}(\sigma;r-1)\,\Delta^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta e^{(2\nu-2k-2r+1)\pi i}\,||\,a_\sigma).\end{aligned}\right\}\quad(126)$$

Proof. From (80) it follows

$$\sum_{\tau=0}^{r-1} e^{(2r-2\tau-2)\pi i a_\sigma}\,\Phi_{\nu,k}^{k,0}(1;\tau)=\Theta_{\nu}^{k,0}(\sigma;r-1).$$

We therefore find (126) if we put $\varkappa=\lambda=r$ in (122).

Remark. Formula (122) is also valid if the following conditions are satisfied: $k, l, p, q, \varkappa, \lambda$ and ν are integers with

$$l\geqq 1,\ q\geqq 1,\ 0\leqq k\leqq q,\ \nu\geqq 0,\ l+\nu-1\leqq p\leqq q,\ \lambda\leqq 0 \text{ and } \varkappa\geqq 1+\nu-k;$$

the numbers $a_{\nu+1},\ldots,a_{l+\nu-1}$ and $b_1,\ldots,b_k$ fulfil the condition (119).

For, if $\lambda\leqq 0$, the sum $\sum_{h=1}^{k-\nu+\varkappa-\lambda}$ on the right-hand side of (122) is because of (73) and (75) equal to $G_{p,q}^{k,l-1,l+\nu-1}(\zeta)$; since $\Phi_{\nu,k}^{k,0}(1;\tau)=0$ for $\tau<0$, the sum $\sum_{\tau=0}^{\lambda-1}$ is zero for $\lambda\leqq 0$ (comp. definition 4).

Similarly formula (126) is also true under the following conditions: k, l, p, q, r and ν are integers with

$$l\geqq 1,\ 1\leqq 1+\nu\leqq k\leqq q,\ 1+\nu-k\leqq r\leqq 0 \text{ and } l+\nu-1\leqq p\leqq q;$$

the numbers $a_{\nu+1},\ldots,a_{l+\nu-1}$ and $b_1,\ldots,b_k$ fulfil the condition (119).

Lemma 22. *Suppose that k, l, m, n, p, q and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,\ 0 \leqq n-l+1 \leqq \nu \leqq k \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ *and* $b_1, \ldots, b_k$ *satisfy the conditions* (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} &\sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{k}(s)\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{-2s\pi i}) \\ &= \sum_{h=1}^{k-\nu} \{ \Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{k}(h+\lambda-1)\}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(-2h-2\lambda+2)\pi i}) \\ &- \frac{1}{A^{0,\nu}_{k}} \sum_{\tau=1}^{\nu} e^{(k-\nu+1)\pi i a_\tau}\, \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\, \Delta^{0,\nu}_{k}(\sigma)\, G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+1)\pi i} \,||\, a_\tau). \end{aligned}\right\} \quad (127)$$

P r o o f. We first suppose $\lambda \leqq 0$. Then the left-hand side of (127) vanishes since $\Omega^{m,n-l+1}_{k}(s) = 0$ for $s < 0$. Because of $\Phi^{m,n-l+1}_{\nu,k}(1;\tau) = 0$ if $\tau < 0$, it appears from (71)

$$\Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{k}(h+\lambda-1) = 0 \text{ for } \lambda \leqq 0.$$

We further have by (54)

$$\Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1) = 0 \text{ for } \lambda \leqq 0.$$

Hence formula (127) is certainly true if $\lambda \leqq 0$.

We now consider the case with $\lambda > 0$. Because of (77) we have

$$\sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{k}(s)\, \Phi^{k,0}_{\nu,k}(h;\lambda-s) = \Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{k}(h+\lambda-1);\ . \quad (128)$$

besides it follows from (79)

$$\sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{k}(s)\, \Theta^{k,0}_{\nu}(\sigma;\lambda-s-1) = \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1) \quad . \quad . \quad (129)$$

If we replace in (126) ζ by $w\, e^{-2s\pi i}$ and r by $\lambda - s$ and use (128) and (129), we easily find (127).

Lemma 23. *Suppose that k, l, m, n, p, q, μ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,$$
$$0 \leqq n-l+1 \leqq \nu,\ 0 \leqq \mu \leqq k-\nu \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ *and* $b_1, \ldots, b_k$ *satisfy the conditions* (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned}
&A^{m,n-l+1}_{k} \sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{k}(s)\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{-2s\pi i})\\
&= A^{m,n-l+1}_{k} \sum_{h=1}^{k-\nu-\mu} \{ \Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{k}(h+\lambda-1)\}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(-2h-2\lambda+2)\pi i})\\
&\quad - \overline{A}^{0,\nu}_{k} B^{m,n-l+1}_{\nu} \sum_{\varkappa=1}^{\mu} \Psi^{m,n-l+1}_{\nu,k}(\varkappa;\lambda)\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\varkappa)\pi i})\\
&- B^{m,n-l+1}_{\nu} \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu+1)\pi i a_\sigma}\, \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\, \triangle^{0,\nu}_{k}(\sigma)\, G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu+1)\pi i} \,\|\, a_\sigma).
\end{aligned}\right\} \quad (130)$$

P r o o f. From the definitions 5 and 6 it follows

$$\frac{A^{m,n-l+1}_{k}}{A^{0,\nu}_{k}} = B^{m,n-l+1}_{\nu} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (131)$$

Formula (130) with $\mu = 0$ is therefore equivalent to (127). Hence we may suppose $1 \leqq \mu \leqq k - \nu$ and assume that (130) with $\mu - 1$ instead of μ has already been proved.

Now it follows from (57), if $\nu \geqq 1$,

$$\left.\begin{aligned}
G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu-1)\pi i} \,\|\, a_\sigma) &= e^{-2\pi i a_\sigma}\, G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu+1)\pi i} \,\|\, a_\sigma)\\
&\quad + 2\pi i e^{-\pi i a_\sigma}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}).
\end{aligned}\right\} \quad (132)$$

If this is substituted on the right-hand side of (130) with $\mu - 1$ instead of μ, the expression $-B^{m,n-l+1}_{\nu} \sum_{\sigma=1}^{\nu}$ not only yields the corresponding expression in (130) but besides

$$\begin{aligned}
&-2\pi i\, B^{m,n-l+1}_{\nu}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times\\
&\qquad \times \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu+2)\pi i a_\sigma}\, \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\, \triangle^{0,\nu}_{k}(\sigma)
\end{aligned}$$ [39]

and this expression is by virtue of (80) equal to

$$\begin{aligned}
&-2\pi i\, B^{m,n-l+1}_{\nu}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times\\
&\qquad \times \sum_{\tau=0}^{\lambda-1} \Phi^{m,n-l+1}_{\nu,k}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\lambda-2\mu-2\tau)\pi i a_\sigma}\, \triangle^{0,\nu}_{k}(\sigma)\\
&\qquad = G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \sum_{\tau=0}^{\lambda-1} \Phi^{m,n-l+1}_{\nu,k}(1;\tau) \times\\
&\times \{ A^{0,\nu}_{k} B^{m,n-l+1}_{\nu}\, \Omega^{0,\nu}_{k}(k-\nu+\lambda-\mu-\tau) - \overline{A}^{0,\nu}_{k} B^{m,n-l+1}_{\nu}\, \overline{\Omega}^{0,\nu}_{k}(\mu+\tau-\lambda)\}
\end{aligned}$$

(after (59)).

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[39]) This is still true for $\nu = 0$, since the sums $\sum_{\sigma=1}^{\nu}$ then vanish.

The right-hand side of this relation is on account of (131), (71) (with $h = k - \nu - \mu + 1$) and (81) equal to

$$G_{p,q}^{k,l-1,l+\nu-1}\left(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}\right) \times$$
$$\times \Big[A^{m,n-l+1}_{k}\left\{\Omega^{m,n-l+1}_{k}(k-\nu+\lambda-\mu) - \Phi^{m,n-l+1}_{\nu,k}(k-\nu-\mu+1;\lambda)\right\}$$
$$- \overline{A}^{0,\nu}_{k} B^{m,n-l+1}_{\nu} \Psi^{m,n-l+1}_{\nu,k}(\mu;\lambda)\Big].$$

It appears therefore that the sums $\sum\limits_{h=1}^{k-\nu-\mu+1}$ and $\sum\limits_{\varkappa=1}^{\mu-1}$ on the right-hand side of (130) with $\mu - 1$ instead of μ reduce by the substitution (132) to the corresponding sums $\sum\limits_{h=1}^{k-\nu-\mu}$ and $\sum\limits_{\varkappa=1}^{\mu}$ in (130). This establishes the lemma.