Applied Mechanics. — *Large distortions of circular rings and straight rods.* II. By A. VAN WIJNGAARDEN. (Nationaal Luchtvaartlaboratorium.) (Communicated by Prof. C. B. BIEZENO.)

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6. *The closed circular ring, loaded by two diametral forces.* We shall apply our theory first to the case of a closed circular ring, loaded diametrically by two radial forces P, the first example, treated by BIEZENO and KOCH in their second paper [5]). We attribute the positive sign to P if it is a compressive force and accordingly we attribute the positive sign to the radial deflection u, if it is directed inwards. We write for shortness $\lambda = u/r$. The forces P may be applied at the points $\sigma = 0$ and $\sigma = \pi$ respectively. From reasons of symmetry it follows immediately, that for these points $\varphi = 0$ and $\varphi = \pi$ respectively, and that to the point $\sigma = \pi/2$ corresponds $\varphi = \pi/2$. Further it is evident that $D_0 = P_0/2$, and $N_0 = 0$. So $\nu = 0$ and $\beta = \pi/2$.

If we substitute $\varphi = \pi/2$ in the eqs. (10), they become:

$$\left.\begin{aligned} \sqrt{\alpha} &= \frac{2\sqrt{2}}{\pi} f_1(k,\varphi)\Big|_{-\pi/4}^{0} \\ \lambda(0) &= 1 - \sqrt{\frac{2}{\alpha}}\, f_2(k,\varphi)\Big|_{-\pi/4}^{0} \\ \lambda\left(\frac{\pi}{2}\right) &= 1 - \sqrt{\frac{2}{\alpha}}\, f_3(k,\varphi)\Big|_{-\pi/4}^{0} \end{aligned}\right\} \quad \ldots\ldots \quad (30)$$

Assuming a number of values of k, we calculate from the first of the equations (30) the corresponding values of α, and then from the second and third equation the corresponding of $\lambda(0)$ and $\lambda\left(\frac{\pi}{2}\right)$. For small positive values of α, γ is large and $k < 1$. For $\gamma = 1$ we have $k = 1$, and for smaller values of γ, i.e. larger values of α, evidently k is > 1, so that eqs. (13) must be used to calculate the elliptic integrals. To $\gamma = 0$ corresponds $k = \frac{1}{2}\sqrt{2}$ and consequently $\varphi_{\min} = 0$. A lower inflexion occurs, and we use the eqs. (22), (23) and (24) for case II. For negative values of α, γ is < -1, and k is imaginary; so we use eqs. (15). A survey of the values of k, k^*, k^{**} and $\varphi_{\min}$ as functions of $\lambda(0)$, which itself varies within the limits $1 - \pi/2$

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[5]) Compare also R. SONNTAG, Die Kreisringfeder, Ing. Arch., **13**, 380—397 (1943). The author solves the same problem, at least, as far as no inflexions occur. His method is exact but somewhat akward and unnecessarily intricate.

and $1 + \pi/2$, is given in fig. 2. The dotted part of the graph for φ_{min} is that one, where φ_{min}, though real, has no physical sense.

For very small positive or negative values of a, we use the expansions of number **5.** As $\nu = 0$, it follows from (28) that $\nu_0 = \nu_1 = \nu_2 = \ldots = 0$. Furthermore $\sigma\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$; so from the first of the eqs. (29) it follows that $\delta_0 = 1$, $\delta_1 = 1/\pi$ and $\delta_2 = 3/\pi^2 - 3/16$, and further:

$$\left.\begin{aligned} \lambda(0) &= \left(\frac{\pi}{8} - \frac{1}{\pi}\right) a + \left(\frac{5}{16} - \frac{3}{\pi^2}\right) a^2 \ldots = 0{,}074389\, a + 0{,}008536\, a^2 \ldots \\ \lambda\left(\frac{\pi}{2}\right) &= -\left(\frac{1}{\pi} - \frac{1}{4}\right) a - \left(-\frac{1}{16} - \frac{3}{4\pi} + \frac{3}{\pi^2}\right) a^2 \ldots = -0{,}068310\, a - 0{,}002731\, a^2 \ldots \end{aligned}\right\} \quad (31)$$

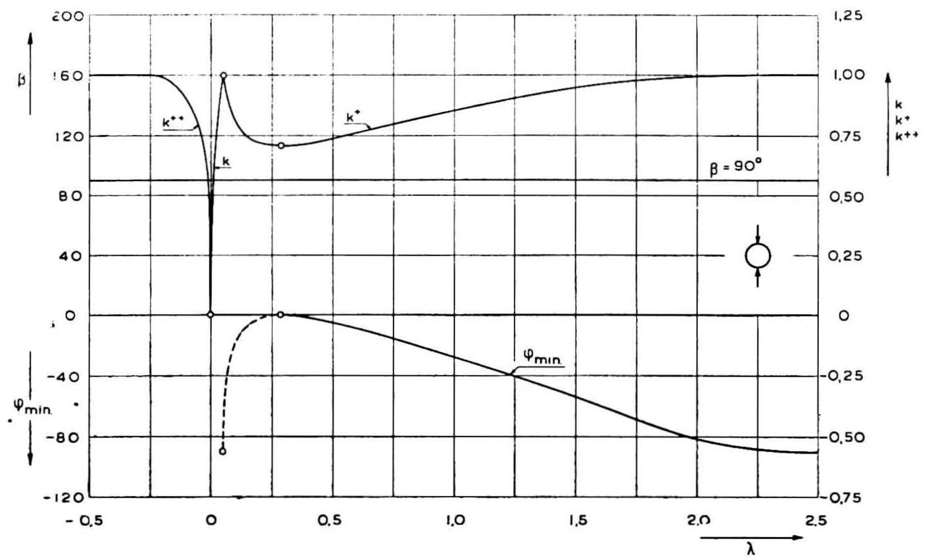

Fig. 2.

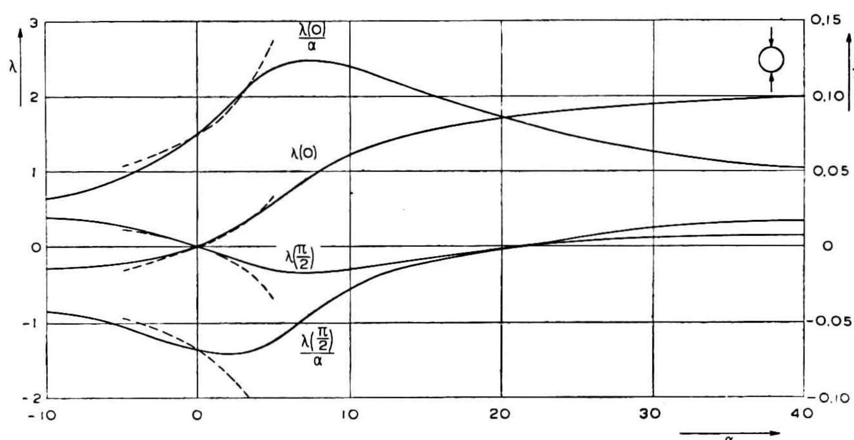

Fig. 3.

These exact developments can be compared with those, derived from the corresponding expressions of BIEZENO and KOCH. The coefficients of α are of course the same, as this term is the "elementary" one; as a matter of fact the authors did not calculate this term separately, but took it from the elementary theory. For the coefficients of α^2 they find 0,006289 and — 0,006425 respectively.

In fig. 3, $\lambda(0)$, $\lambda(\pi/2)$, $\lambda(0)/\alpha$ and $\lambda(\pi/2)/\alpha$ are given as functions of α. The drawn lines give the exact values. The dotted ones represent the approximate values of BIEZENO and KOCH.

7. *The semicircular ring, loaded by a radial force.* We consider the problem of a semicircular ring, clamped at both ends, and loaded in its middle by a radial force P. We attribute signs as in number **6**. Now the shearing force D_0 for $\sigma = 0$ ($\varphi = 0$) is again $P/2$, but the normal force N_0 is not zero, and, moreover, unknown. On the other hand, apart from the condition $\sigma = \pi/2$ for $\varphi = \pi/2$, also the condition $\xi = 1$ for $\varphi = \pi/2$ has to be fulfilled. So, the eqs (10) take the form:

$$\left.\begin{aligned}
\sqrt{\frac{\alpha}{2\sin\beta}} &= \frac{2}{\pi} f_1(k,\varphi)\Bigg|_{-\beta/2}^{\pi/4-\beta/2} \\
\sqrt{\frac{\alpha}{2\sin\beta}} &= \{\cos\beta\, f_2(k,\varphi) + \sin\beta\, f_3(k,\varphi)\}\Bigg|_{-\beta/2}^{\pi/4-\beta/2} \\
\lambda(0) &= 1 - \sqrt{\frac{2\sin\beta}{2}}\{\sin\beta\, f_2(k,\varphi) - \cos\beta\, f_3(k,\varphi)\}\Bigg|_{-\beta/2}^{\pi/4-\beta/2}
\end{aligned}\right\} . \qquad (32)$$

In this case we first take a certain value of k and together with this value, some different values of β. Then we calculate the difference of the expressions at the right hand side of the first and the second equation (32). By interpolation we determine that value of β, for which this difference is zero, (as it should be, because both expressions must be equal to the same quantity at the left hand side). If in this way we have found a pair k, β, we calculate α, and from the third equation $\lambda(0)$. The physical interpretation of this proces is, that the ring of section **6** is considered, but now loaded by a second pair of diametral forces at right angles to the first pair, the magnitude of which is determined in such a way, that the displacement of the points $\sigma = \pm\pi/2$ is zero.

To start with the process, we solve first the problem for small α, by means of the results of number **5**. If we substitute in the first of the eqs. (29): $\sigma(\pi/2) = \pi/2$, and in the second one: $\xi(\pi/2) = 1$, the values of δ_k and ν_k can be calculated. The first values are:

$$\left.\begin{aligned}
\delta_0 &= 1 \\
\delta_1 &= \frac{\pi-2}{\pi^2-8} = 0{,}610606 \\
\delta_2 &= \frac{-3\pi^6+124\pi^4-256\pi^3-640\pi^2+1792\pi-512}{16(\pi^2-8)^3} = 0{,}556041 \\
\nu_0 &= \frac{8-2\pi}{\pi^2-8} = 0{,}918277 \\
\nu_1 &= \frac{-\pi^5-8\pi^4+76\pi^3-64\pi^2-448\pi+768}{2(\pi^2-8)^3} = 0{,}007376
\end{aligned}\right\} . \quad (33)$$

With these values, we find from the third of the eqs. (29):

$$\left.\begin{aligned}
\lambda(0) &= \frac{\pi^3-20\pi+32}{8(\pi^2-8)}\,\alpha + \\
&+ \frac{-5\pi^6+6\pi^5+204\pi^4-680\pi^3-576\pi^2+4608\pi-4608}{16(\pi^2-8)^3}\,\alpha^2 - \ldots = \\
&\qquad 0{,}0116618\,\alpha + 0{,}0007136\,\alpha^2 - \ldots
\end{aligned}\right\} . \quad (34)$$

This can be compared with the result of BIEZENO and KOCH, where in stead of the coefficient 0,0007136 appears 0,00047.

The value of β for $\alpha = 0$ proves to be 47,4397°, so that we have a preliminary idea, what the value of β amounts to for small values of α. The numerical work is much more complicated than in the case of the foregoing section. The elastic behaviour of the ring appears however to be highly interesting. For $\lambda(0) = 0{,}0974$ and $\alpha = 5{,}5206$ a lower inflexion appears, followed (for $\lambda(0) = 0{,}1236$ and $\alpha = 7{,}1617$) by an upper in-

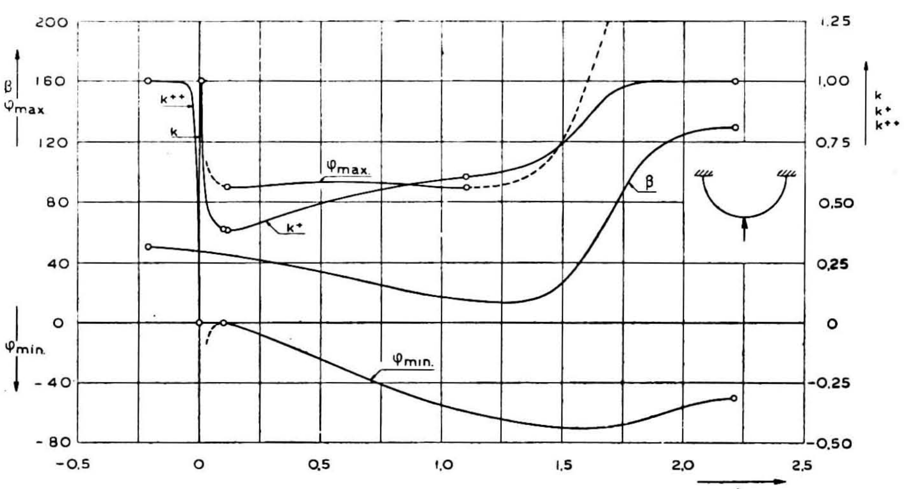

Fig. 4.

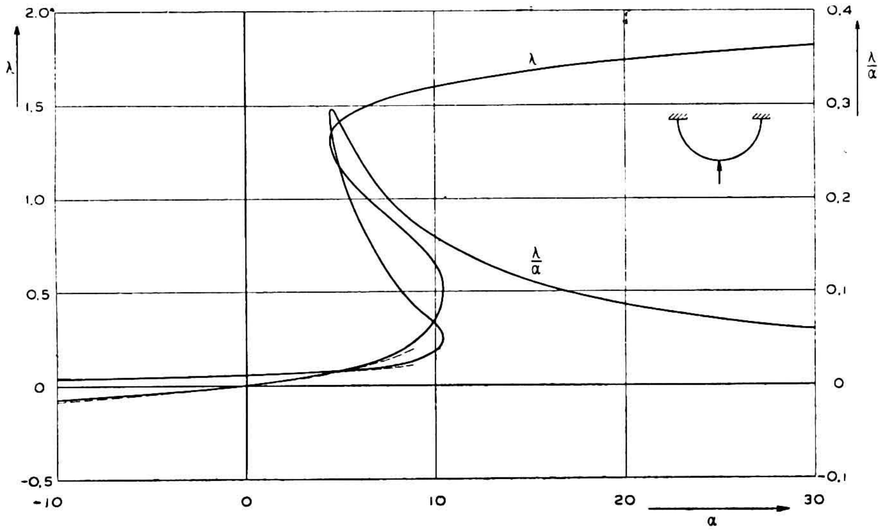
2.0
1.5
1.0
0.5
0
-0.5
λ
-10
0
10
20
30
α
0.4
0.3
0.2
0.1
0
-0.1
λ/α

Fig. 5.

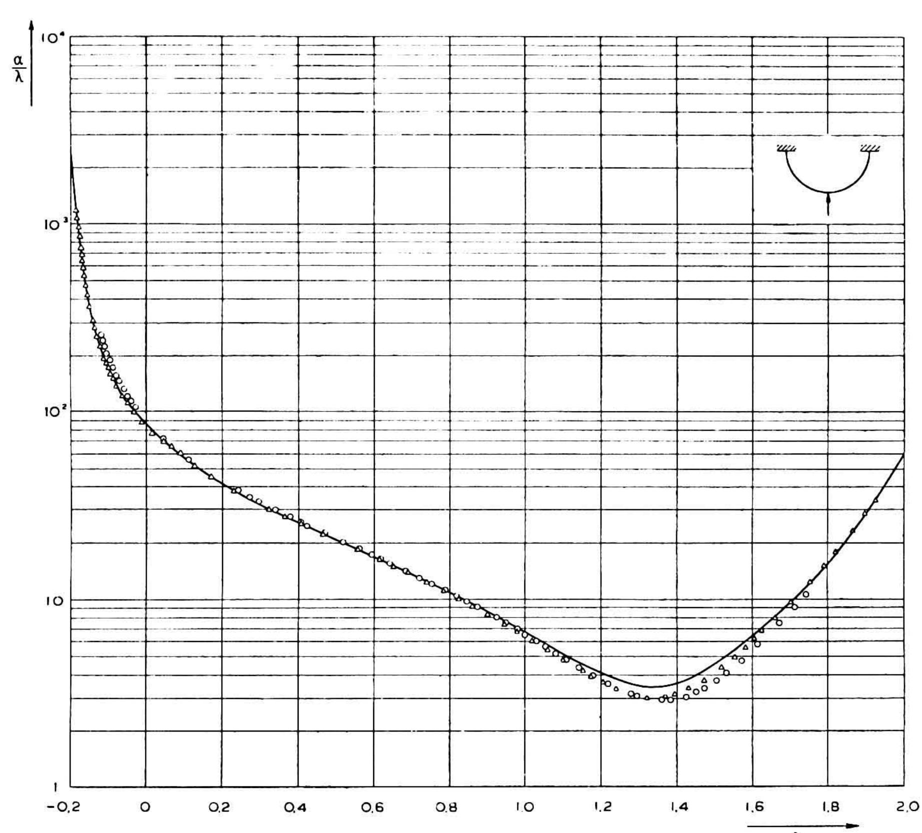
α/λ
10⁴
10³
10²
10
1
-0,2
0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0
λ

Fig. 6.

flexion. In the mean time the flexability, as defined by the ratio of displacement and loading force (λ/α) has increased extraordinarily. If $\lambda(0) = 0{,}48$, α assumes a maximum value 10,45. With increasing λ, α then decreases. This region therefore is an unstable one. For $\lambda(0) = 1{,}1000$ and $\alpha = 5{,}5637$ the upper inflexion disappears again. For $\lambda(0) = 1{,}27$, α reaches its minimum value 4,45 and up from here it increases rapidly, to become infinite for $\lambda(0) = 1 + \frac{1}{2}\sqrt{\pi^2 - 4}$. For negative values of $\lambda(0)$, $|\alpha|$ increases rapidly with $|\lambda|$, to become infinite for $\lambda(0) = 1 - \frac{1}{2}\sqrt{\pi^2 - 4}$.

In fig. 4, the values of β, k (or k^*, k^{**}), $\varphi_{\min}$ and $\varphi_{\max}$ are given as functions of $\lambda(0)$, which itself varies within the limits $1 - \frac{1}{2}\sqrt{\pi^2 - 4}$ and $1 + \frac{1}{2}\sqrt{\pi^2 - 4}$. The dotted parts of the lines for $\varphi_{\min}$ and $\varphi_{\max}$ represent again the regions, where these quantities have no physical sense.

In fig. 5 the quantities $\lambda(0)$ and $\lambda(0)/\alpha$ are given as functions of α. The drawn lines are the exact results from our theory; the dotted lines represent the approximate results of BIEZENO and KOCH.

Some experiments were made on two semicircular rings, made of steel strip, with a width of 30 mm and a thickness of 0,2 mm and 0,3 mm respectively. The radius of the ring was $r = 300$ mm. The results are plotted in fig. 6. The drawn line is the theoretical curve and the small triangles and circles represent the experimental values.

8. *The straight beam, supported in two points and loaded by a force in its middle.* We shall now give two other applications of our theory, dealing with a straight beam, supported in two points, and loaded by a normal force P in its middle.

As a first possibility, we assume, that the beam is free to move over its supports, which are supposed to remain at a fixed distance. The length of the deflected beam between the supports is then larger than this distance and increases with the deflection. In this simple example the load P again shows a maximum P_{cr} at a critical deflection u_{cr}, as was allready shown by SONNTAG.

The standard of length, which in the foregoing examples we choose equal to the radius r of the ring, is now taken equal to l, viz. the half distance of the supports (see fig. 7). These supports are supposed to exert on the

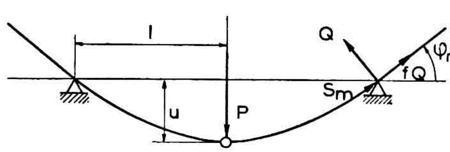

Fig. 7.

beam a normal reaction force Q and a tangential force, due to friction, fQ, where $f = tg\, \varphi_f$ is the coefficient of friction. The sign of f must be chosen in accordance with the fact that the load P is reached in an increasing or decreasing way. The slope of the beam at the supports may be φ_m. Then it follows from considerations of symmetry and equilibrium, that $D_0 = -P/2$ and $N_0 = \frac{1}{2}P \tan(\varphi_m - \varphi_f)$, so that $\beta = -\frac{1}{2}\pi + \varphi_m - \varphi_f$. As for $\varphi = \varphi_m$, we have $d\varphi/d\sigma = 0$, it follows from equation (7) that $\gamma = -\sin\varphi_f$ and consequently $k = 1/\sin(\frac{1}{4}\pi - \frac{1}{2}\varphi_f) \geqq 1$. We therefore have to use the transformation (13) with $k^* = \sin(\frac{1}{4}\pi - \frac{1}{2}\varphi_f)$.

We know, that for $\varphi = \varphi_m$, $\xi = 1$, and so we can find from the second equation (10) for a number of values of φ_m the corresponding α and from this we calculate the relative lengthening $\mu = \sigma_m - 1$, and the deflection $\lambda = u/l$ with the aid of the first and third equation (10).

The calculations have been actually performed for the case that no friction was present $\varphi_f = 0$) and no inflexions occurred in the elastic line. The results are plotted in fig. 8. The approximate results of SONNTAG and

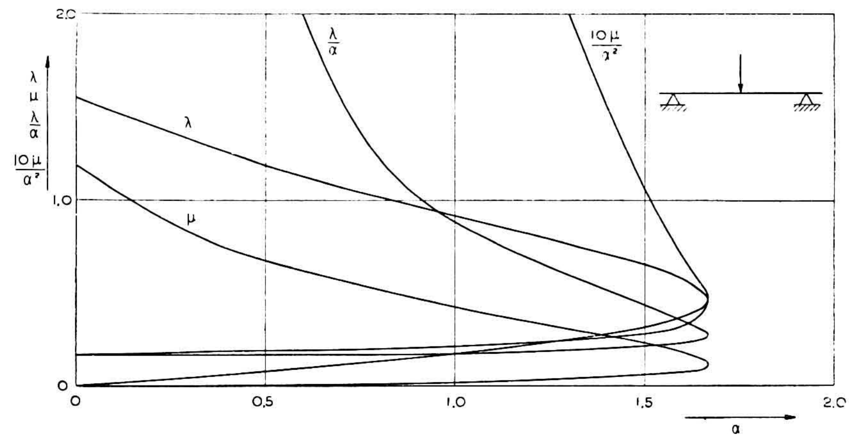

Fig. 8.

those of BIEZENO are very well confirmed. Of course we can give more accurate figures. So we find for the critical load $P_{cr} = 1{,}667\ EI/l^2$, and the corresponding values $\varphi_{m\,cr} = 38{,}30^\circ$, $u_{cr} = 0{,}4766\ l$, and $s_{m\,cr} = 1{,}125\ l$. SONNTAG's values are 1,70; 39°, and 0,48 respectively, the critical length being not explicitly given.

As a last example, we assume that the beam does not move over its supports, so that its length between the supports remains constant ($2l$), but that on the other hand, one of the supports is free to move towards the other (see fig. 9). We suppose this movement not to be influenced by

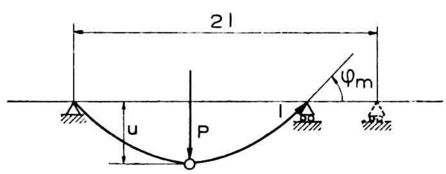

Fig. 9.

friction. The considerations are similar to those in the foregoing example. For $\varphi = \varphi_m$ we now have $\sigma = 1$. So we find α from the first equation (10) and calculate the deflection $\lambda = u/l$ and the contraction $\mu = 1 - \xi_m$ from the second and the third equation. The results are plotted in fig. 10.

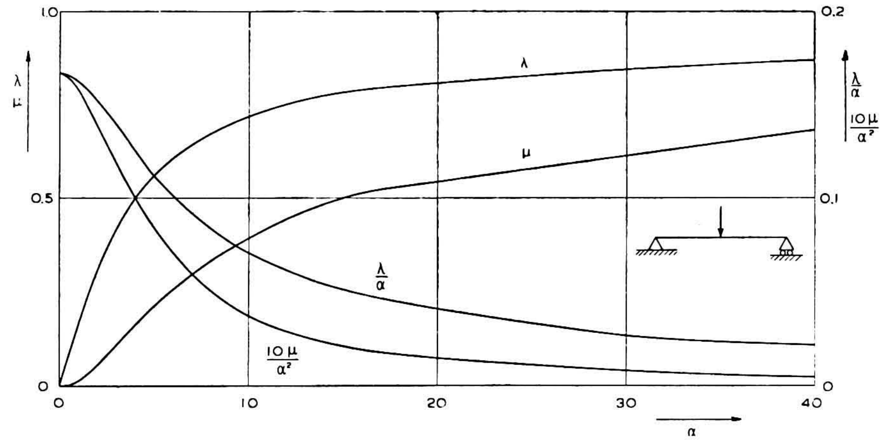

Fig. 10.