Mathematics. — *On the G-function.* V. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Lemma 24. *Suppose that k, l, m, n, p, q, μ and ν are integers with $l \geqq 1$, $q \geqq 1$, $\mu \geqq 0$, $0 \leqq m \leqq k \leqq q$, $0 \leqq n-l+1 \leqq \nu \leqq k$ and $l+\nu-1 \leqq p \leqq q$; further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{n-l+1}$, $a_{\nu+1}, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ fulfil the conditions (100) and (119) and satisfy besides the inequality*

$$a_j - b_h \neq 1, 2, 3, \ldots \quad (j = 1, \ldots, n-l+1;\ h = 1, \ldots, m) \ . \tag{133}$$

Then the following formula holds [40]:

$$\left.\begin{aligned}
&\sum_{t=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu)\pi i a_t} \Delta^{m,\,n-l+1}_{k}(t)\, G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta \,\|\, a_t) \\
&= -A^{m,\,n-l+1}_{k} \sum_{h=k-\nu-\mu+1}^{k-\nu} \Omega^{m,\,n-l+1}_{k}(h+\lambda-1)\, G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2k-2\nu-2h+1)\pi i}) \\
&\quad + \overline{A}^{m,\,n-l+1}_{k} \sum_{\varkappa=1}^{\mu} \overline{\Omega}^{m,\,n-l+1}_{k}(l-m-n-\lambda+\nu+\varkappa-1)\, G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2\varkappa-1)\pi i}) \\
&\quad\quad + \sum_{\sigma=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu-2\mu)\pi i a_\sigma} \Delta^{m,\,n-l+1}_{k}(\sigma)\, G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta e^{2\mu\pi i} \,\|\, a_\sigma).
\end{aligned}\right\} \tag{134}$$

P r o o f. Formula (134) is obvious if $\mu = 0$. We may therefore suppose $\mu \geqq 1$ and assume that (134) with $\mu - 1$ instead of μ has already been proved.

Now it follows from (57), if $\nu \geqq 1$,

$$\begin{aligned}
G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta e^{(2\mu-2)\pi i} \,\|\, a_\sigma) = e^{-2\pi i a_\sigma} G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta e^{2\mu\pi i} \,\|\, a_\sigma) \\
+ 2\pi i e^{-\pi i a_\sigma} G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2\mu-1)\pi i}).
\end{aligned}$$

If we substitute this on the right-hand side of (134) with $\mu - 1$ instead of μ, the sum $\sum\limits_{\sigma=1}^{n-l+1}$ not only gives the corresponding sum in (134) but also

$$2\pi i\, G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2\mu-1)\pi i}) \sum_{\sigma=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu-2\mu+1)\pi i a_\sigma} \Delta^{m,\,n-l+1}_{k}(\sigma) \text{ [41]}$$

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[40]) The products $\Delta^{m,\,n-l+1}_{k}(t)\, G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta \,\|\, a_t)$ in (134) must be defined by a limiting process if $a_t - b_h = 1, 2, 3, \ldots$ $(m+1 \leqq h \leqq k)$.

[41]) This is still true if $\nu = 0$, since then $n - l + 1 = 0$, so that the sums $\sum\limits_{\sigma=1}^{n-l+1}$ vanish.

and this expression is by (59) equal to

$$-G_{p,q}^{k,l-1,l+\nu-1}(\zeta e^{(2\mu-1)\pi i})\{A^{m,n-l+1}_{k}\,\Omega^{m,n-l+1}_{k}(k+\lambda-\nu-\mu)$$
$$-\overline{A}^{m,n-l+1}_{k}\,\overline{\Omega}^{m,n-l+1}_{k}(l-m-n-\lambda+\nu+\mu-1)\}.$$

The sums $\sum_{h=k-\nu-\mu+2}^{k-\nu}$ and $\sum_{\varkappa=1}^{\mu-1}$ on the right-hand side of (134) with $\mu-1$ instead of μ reduce therefore to the corresponding sums $\sum_{h=k-\nu-\mu+1}^{k-\nu}$ and $\sum_{\varkappa=1}^{\mu}$ in (134). With this the lemma has been proved.

Lemma 25. *Suppose that k, l, m, n, p, q and ν are integers with $l \geqq 1$, $q \geqq 1$, $0 \leqq m \leqq k \leqq q$, $0 \leqq n-l+1 \leqq \nu$ and $l+\nu-1 \leqq p \leqq q$; further that r is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{n-l+1}$, $a_{\nu+1}, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (100), (119) and (133).*

Then the following formula holds:

$$\left.\begin{aligned}
&G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_{n-l+1},a_{\nu+1},\ldots,a_p,a_{n-l+2},\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right)\\
&=A^{m,n-l+1}_{k}\sum_{s=0}^{r-1}\Omega^{m,n-l+1}_{k}(s)\,G_{p,q}^{k,l-1,l+\nu-1}(z\,e^{(k+l-m-n-2s-1)\pi i})\\
&+\overline{A}^{m,n-l+1}_{k}\sum_{\tau=0}^{k+l-m-n-r-1}\overline{\Omega}^{m,n-l+1}_{k}(\tau)\,G_{p,q}^{k,l-1,l+\nu-1}(z\,e^{(m+n-k-l+2\tau+1)\pi i})\\
&+\sum_{t=1}^{n-l+1}e^{(m+n-k-l+2r)\pi i a_t}\,\Delta^{m,n-l+1}_{k}(t)\,G_{p,q}^{k,l,l+\nu-1}(z\,e^{(k+l-m-n-2r)\pi i}\,\|\,a_t).
\end{aligned}\right\}\quad(135)$$

Proof. We consider formula (116) that holds under the general conditions which have been stated in theorem 4. On the left-hand side we replace $G_{p,q}^{m,n}(z)$ by

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_{n-l+1},a_{\nu+1},\ldots,a_p,a_{n-l+2},\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right).$$

Observing that $G_{p,q}^{k,\lambda}(z)$ is a symmetric function of $a_{\lambda+1}, \ldots, a_p$, it follows easily from the definitions of $G_{p,q}^{k,l-1,n}(z)$ and $G_{p,q}^{k,l,n}(z\,\|\,a_t)$ (see the end of § 1) that the functions $G_{p,q}^{k,l-1,n}(\zeta)$ and $G_{p,q}^{k,l,n}(w\,\|\,a_t)$ on the right of (116) must then be replaced by

$$G_{p,q}^{k,l-1}\left(\zeta\,\middle|\,\begin{matrix}a_{\nu+1},\ldots,a_p,a_1,\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right),$$

respect.

$$G_{p,q}^{k,l}\left(w\,\middle|\,\begin{matrix}a_t,a_{\nu+1},\ldots,a_p,a_1,\ldots,a_{t-1},a_{t+1},\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right)$$

(where $1 \leqq t \leqq n-l+1 \leqq \nu$)

and these functions are according to the definitions of $G_{p,q}^{k,l-1,n}(z)$ and $G_{p,q}^{k,l,n}(z \| a_t)$ equal to $G_{p,q}^{k,l-1,l+\nu-1}(\zeta)$, respect. $G_{p,q}^{k,l,l+\nu-1}(w \| a_t)$.

The coefficients A, $\bar{A}$, Ω, $\bar{\Omega}$ and $\triangle$ take in the new formula the same values as in the original one and the system of conditions (1), (99) needs to be replaced by the system (119), (133). This proves the lemma.

In the same manner as the above lemmas we may prove the conjugate lemmas. Thus the conjugate of lemma 23 is

Lemma 26. *Suppose that k, l, m, n, p, q, δ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,$$
$$0 \leqq n-l+1 \leqq \nu,\ 0 \leqq \delta \leqq k-\nu \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that β is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned}
&A_k^{m,n-l+1} \sum_{\tau=0}^{\beta-1} \bar{\Omega}_k^{m,n-l+1}(\tau)\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{2\tau\pi i}) \\
&= -A_k^{0,\nu} \bar{B}_\nu^{m,n-l+1} \sum_{h=1}^{\delta} \bar{\Psi}_{\nu,k}^{m,n-l+1}(h;\beta)\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{(2k-2\nu-2h+2\beta)\pi i}) \\
&+ A_k^{m,n-l+1} \sum_{\varkappa=1}^{k-\nu-\delta} \{\bar{\Phi}_{\nu,k}^{m,n-l+1}(\varkappa;\beta) - \bar{\Omega}_k^{m,n-l+1}(\varkappa+\beta-1)\}\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{(2\varkappa+2\beta-2)\pi i}) \\
&- B_\nu^{m,n-l+1} \sum_{\sigma=1}^{\nu} e^{(\nu-k+2\delta-1)\pi i a_\sigma}\, \bar{\Theta}_\nu^{m,n-l+1}(\sigma;\beta-1)\, \triangle_k^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w\, e^{(2k-2\nu+2\beta-2\delta-1)\pi i} \| a_\sigma).
\end{aligned}\right\} \quad (136)$$

§ 14. Fourth expansion formula.

The most important theorem of the present paper is

Theorem 5. Assumptions: *k, l, m, n, p, q and ν are integers with*

$$l \geqq 1, q \geqq 1, 0 \leqq m \leqq k \leqq q, 0 \leqq n-l+1 \leqq \nu \leqq k \text{ and } l+\nu-1 \leqq p \leqq q;$$

the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ fulfil the conditions (119), (120) *and* (133); *μ is an arbitrary integer which satisfies the inequality*

$$0 \leqq \mu \leqq k-\nu;$$

λ is an arbitrary integer.

Assertion:

$$\left.\begin{aligned}
&G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_{n-l+1}, a_{\nu+1}, \ldots, a_p, a_{n-l+2}, \ldots, a_\nu \\ b_1, \ldots, b_q \end{matrix}\right) \\
&= \sum_{h=1}^{k-\nu-\mu} R_{\nu,k}^{m,n-l+1}(h;\lambda)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(k+l-m-n-2h-2\lambda+1)\pi i}) \\
&+ \sum_{\varkappa=1}^{\mu} \bar{R}_{\nu,k}^{m,n-l+1}(\varkappa; l-m-n-\lambda+\nu)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(l-k-m-n+2\varkappa-2\lambda+2\nu-1)\pi i}) \\
&+ \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu)\pi i a_\sigma}\, T_{\nu,k}^{m,n-l+1}(\sigma;\lambda)\, G_{p,q}^{k,l,l+\nu-1}(z\, e^{(l-k-m-n-2\lambda+2\mu+2\nu)\pi i} \| a_\sigma).
\end{aligned}\right\} \quad (137)$$

Proof. In formula (135) we replace r by $k+\lambda-\nu$. Then we obtain

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_{n-l+1},a_{\nu+1},\dots,a_p,a_{n-l+2},\dots,a_\nu\\ b_1,\dots,b_q\end{matrix}\right)$$

$$=A^{m,n-l+1}_{\;\;k}\sum_{s=0}^{\lambda-1}\Omega^{m,n-l+1}_{\;\;k}(s)\,G_{p,q}^{k,l-1,l+\nu-1}\left(z\,e^{(k+l-m-n-2s-1)\pi i}\right)$$

$$+A^{m,n-l+1}_{\;\;k}\sum_{h=1}^{k-\nu}\Omega^{m,n-l+1}_{\;\;k}(h+\lambda-1)\,G_{p,q}^{k,l-1,l+\nu-1}\left(z\,e^{(k+l-m-n-2h-2\lambda+1)\pi i}\right)$$

$$+\overline{A}^{m,n-l+1}_{\;\;k}\sum_{\tau=0}^{l-m-n-\lambda+\nu-1}\overline{\Omega}^{m,n-l+1}_{\;\;k}(\tau)\,G_{p,q}^{k,l-1,l+\nu-1}\left(z\,e^{(m+n-k-l+2\tau+1)\pi i}\right)$$

$$+\sum_{t=1}^{n-l+1}e^{(m+n+k-l+2\lambda-2\nu)\pi i a_t}\,\triangle^{m,n-l+1}_{\;\;k}(t)\,G_{p,q}^{k,l,l+\nu-1}\left(z\,e^{(l-k-m-n-2\lambda+2\nu)\pi i}\,\|\,a_t\right).$$

On the right-hand side of this relation we may reduce the sum $\sum_{s=0}^{\lambda-1}$ by means of (130) (with $w=z\,e^{(k+l-m-n-1)\pi i}$), further the sum $\sum_{\tau=0}^{l-m-n-\lambda+\nu-1}$ by means of (136) (with $\beta=l-m-n-\lambda+\nu$, $\delta=k-\nu-\mu$ and $w=z\,e^{(m+n-k-l+1)\pi i}$) and finally the sum $\sum_{t=1}^{n-l+1}$ by means of (134) (with $\zeta=z\,e^{(l-k-m-n-2\lambda+2\nu)\pi i}$). Then we find

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_{n-l+1},a_{\nu+1},\dots,a_p,a_{n-l+2},\dots,a_\nu\\ b_1,\dots,b_q\end{matrix}\right)$$

$$=\sum_{h=1}^{k-\nu-\mu}\left\{A^{m,n-l+1}_{\;\;k}\,\Phi^{m,n-l+1}_{\nu,k}(h;\lambda)-A^{0,\nu}_{\;\;k}\,\overline{B}^{m,n-l+1}_{\nu}\,\overline{\Psi}^{m,n-l+1}_{\nu,k}(h;l-m-n-\lambda+\nu)\right\}\times$$

$$\times\,G_{p,q}^{k,l-1,l+\nu-1}\left(z\,e^{(k+l-m-n-2h-2\lambda+1)\pi i}\right)$$

$$+\sum_{\varkappa=1}^{\mu}\left\{\overline{A}^{m,n-l+1}_{\;\;k}\,\overline{\Phi}^{m,n-l+1}_{\nu,k}(\varkappa;l-m-n-\lambda+\nu)-\overline{A}^{0,\nu}_{\;\;k}\,B^{m,n-l+1}_{\nu}\,\Psi^{m,n-l+1}_{\nu,k}(\varkappa;\lambda)\right\}\times$$

$$\times\,G_{p,q}^{k,l-1,l+\nu-1}\left(z\,e^{(l-k-m-n+2\varkappa-2\lambda+2\nu-1)\pi i}\right)$$

$$-\sum_{\sigma=1}^{\nu}\left\{e^{(k-\nu-2\mu+1)\pi i a_\sigma}\,B^{m,n-l+1}_{\nu}\,\Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\right.$$

$$\left.+\,e^{(k-\nu-2\mu-1)\pi i a_\sigma}\,\overline{B}^{m,n-l+1}_{\nu}\,\overline{\Theta}^{m,n-l+1}_{\nu}(\sigma;l-m-n-\lambda+\nu-1)\right\}\times$$

$$\times\,\triangle^{0,\nu}_{\;\;k}(\sigma)\,G_{p,q}^{k,l,l+\nu-1}\left(z\,e^{(l-k-m-n-2\lambda+2\mu+2\nu)\pi i}\,\|\,a_\sigma\right)$$

$$+\sum_{\sigma=1}^{n-l+1}e^{(m+n+k-l+2\lambda-2\nu-2\mu)\pi i a_\sigma}\,\triangle^{m,n-l+1}_{\;\;k}(\sigma)\,G_{p,q}^{k,l,l+\nu-1}\left(z\,e^{(l-k-m-n-2\lambda+2\mu+2\nu)\pi i}\,\|\,a_\sigma\right).$$

Because of (84), (85), (86) and (87) the coefficients in this relation are equal to those in (137). This establishes the theorem.

§ 15. On a special transformation formula.

It is sometimes desirable to express a function $M(z)$, that is many-valued with a branch-point at $z=0$, in the following way

$$M(z)=\sum_{h=1}^{k} r_h\, M(z\,e^{(\gamma-2h)\pi i}),$$

or, more general, to write a function $N(z)$ as follows

$$N(z)=\sum_{h=1}^{k} s_h\, M(z\,e^{(\delta-2h)\pi i}).$$

In this § I will establish such relations for the function G. I will prove:

Theorem 6. Assumptions: *k, m, n, p and q are integers with*

$$q\geqq 1,\ 0\leqq n\leqq p\leqq q \quad \text{and} \quad 0\leqq m\leqq k\leqq q;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ satisfy the condition

$$a_j-b_h\neq 1,2,3,\ldots \qquad (j=1,\ldots,n;\ h=1,\ldots,k);$$

λ is an arbitrary integer.

Assertion:

$$G_{p,q}^{m,n}(z)=\sum_{h=1}^{k} R_{0,k}^{m,0}(h;\lambda)\, G_{p,q}^{k,n}(z\,e^{(k-m-2h-2\lambda+2)\pi i}). \quad . \quad . \quad (138)$$

Proof. This theorem is a special case of theorem 5. Indeed, it is obvious on account of the definition of $G_{p,q}^{k,l-1,n}(z)$ that

$$G_{p,q}^{k,n,n}(z)=G_{p,q}^{k,n}(z).$$

Hence we find, if we take $l=n+1$ and $\nu=0$ in (137),

$$\left.\begin{aligned} G_{p,q}^{m,n}(z)=&\sum_{h=1}^{k-\mu} R_{0,k}^{m,0}(h;\lambda)\, G_{p,q}^{k,n}(z\,e^{(k-m-2h-2\lambda+2)\pi i})\\ &+\sum_{\varkappa=1}^{\mu} \overline{R}_{0,k}^{m,0}(\varkappa;1-m-\lambda)\, G_{p,q}^{k,n}(z\,e^{(-k-m+2\varkappa-2\lambda)\pi i}),\end{aligned}\right\} . \quad (139)$$

where $0\leqq\mu\leqq k$. Putting $\mu=0,1,\ldots,k$, it seems as if there are $k+1$ different relations, formula (138) being the special case with $\mu=0$. But this is only true in seeming. For (139) may also be written in the following way

$$\left.\begin{aligned} G_{p,q}^{m,n}(z)=&\sum_{h=1}^{k-\mu} R_{0,k}^{m,0}(h;\lambda)\, G_{p,q}^{k,n}(z\,e^{(k-m-2h-2\lambda+2)\pi i})\\ &+\sum_{h=k-\mu+1}^{k} \overline{R}_{0,k}^{m,0}(k-h+1;1-m-\lambda)\, G_{p,q}^{k,n}(z\,e^{(k-m-2h-2\lambda+2)\pi i}).\end{aligned}\right\} \quad (140)$$

Subtracting (140) with $\mu+1$ instead of μ from (140), we find for $0\leqq\mu\leqq k-1$

$$\overline{R}_{0,k}^{m,0}(\mu+1;1-m-\lambda)=R_{0,k}^{m,0}(k-\mu;\lambda),$$

hence for $1 \leqq h \leqq k$

$$\overline{R}^{m,0}_{0,k}(k-h+1;1-m-\lambda) = R^{m,0}_{0,k}(h;\lambda),$$

so that (140) is equivalent to (138).

I will now apply formula (138) to MACDONALD's BESSEL function with imaginary argument $K_\nu(z)$ [42]), to the BESSEL function of the second kind $Y_\nu(z)$ and to WHITTAKER's function $W_{k,m}(z)$ [43]). These functions may in the following way be expressed in terms of the G-function [44])

$$K_\nu(z) = \tfrac{1}{2} G^{2,0}_{0,2}(\tfrac{1}{4}z^2 \mid \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu), \quad . \quad . \quad . \quad . \quad . \quad (141)$$

$$Y_\nu(z) = (-1)^t G^{2,0}_{1,3}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} -\tfrac{1}{2}\nu - t - \tfrac{1}{2} \\ \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu, -\tfrac{1}{2}\nu - t - \tfrac{1}{2} \end{matrix}\right) \quad (t=0, \pm 1, \pm 2, \ldots), \; (142)$$

$$e^{-\frac{1}{2}z} W_{k,m}(z) = z^{\frac{1}{2}} G^{2,0}_{1,2}\left(z \,\middle|\, \begin{matrix} \tfrac{1}{2}-k \\ m, -m \end{matrix}\right) \quad . \quad . \quad . \quad . \quad (143)$$

Now we have by lemma 16, if $b_1 = -b_2 = \frac{1}{2}\nu$,

$$R^{2,0}_{0,2}(1;\lambda) = \frac{\sin(\lambda+1)\nu\pi}{\sin\nu\pi} \quad \text{and} \quad R^{2,0}_{0,2}(2;\lambda) = -\frac{\sin\lambda\nu\pi}{\sin\nu\pi}.$$

Hence we obtain, if we apply (138) with $m = k = 2$ to (141),

$$K_\nu(z) = \frac{\sin(\lambda+1)\nu\pi}{\sin\nu\pi} K_\nu(z e^{-\lambda\pi i}) - \frac{\sin\lambda\nu\pi}{\sin\nu\pi} K_\nu(z e^{-(\lambda+1)\pi i});$$

this formula, wherein λ is an arbitrary integer, occurs in a somewhat other form by WATSON [45]); it may be used to obtain asymptotic expansions for $K_\nu(z)$ for large values of $|z|$ with $|\arg z| \geqq \frac{3}{2}\pi$ [46]).

Similarly we find, if we apply (138) to (142),

$$Y_\nu(z) = \frac{\sin(\lambda+1)\nu\pi}{\sin\nu\pi} Y_\nu(z e^{-\lambda\pi i}) - \frac{\sin\lambda\nu\pi}{\sin\nu\pi} Y_\nu(z e^{-(\lambda+1)\pi i}).$$

The corresponding relation for $W_{k,m}(z)$ follows from (138) and (143) [47])

$$W_{k,m}(z) = (-1)^\lambda \left\{ \frac{\sin 2(\lambda+1)m\pi}{\sin 2m\pi} W_{k,m}(z e^{-2\lambda\pi i}) + \frac{\sin 2\lambda m\pi}{\sin 2m\pi} W_{k,m}(z e^{-2(\lambda+1)\pi i}) \right\}. \; (144)$$

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[42]) WATSON, [**31**], 78. The function $K_\nu(z)$ is equal to $\frac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(z e^{\frac{1}{2}\pi i})$, where $H_\nu^{(1)}(z)$ is the first HANKEL function.

[43]) WHITTAKER and WATSON, [**32**], chapter XVI.

[44]) Comp. [**21**], 205—206 and [**22**], 187.

[45]) WATSON, [**31**], 75, formula (5); comp. also DEBYE, [**7**], 6.

[46]) The well-known asymptotic expansion

$$K_\nu(z) \backsim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} {}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; -\frac{1}{2z}\right)$$

is valid for $|\arg z| < \frac{3}{2}\pi$ (comp. WATSON, [**31**], 202—203).

[47]) The numbers k and m in (143) are, of course, not the same as those in (138)

§ 16. The particular cases with $l = 1$, $k = q$ of the expansion formulae (102), (113), (116) and (137).

The most important particular cases of the expansion formulae (102), (113), (116) and (137) are those with $l = 1$, $k = q$. The right-hand sides may then because of (10) and (11) be written in a somewhat simpler form. Theorem 1 gives theorem 7; theorem 2 gives theorem 8 A. Theorem 8 B is conjugate to theorem 8 A. Theorem 9 is furnished by theorem 3. Theorem 10 is a particular case of theorem 5 $(l = 1,\ k = q,\ \nu = p)$.

Theorem 7. Assumptions: *m, n, p and q are integers with*

$$1 \leqq n \leqq p \leqq q,\ 1 \leqq m \leqq q \text{ and } m + n \geqq q + 1;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ fulfil the conditions

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, n;\ h = 1, \ldots, m), \quad \ldots \quad (1)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, n;\ t = 1, \ldots, n;\ j \neq t); \quad (20)$$

λ is an arbitrary integer which satisfies the inequality

$$0 \leqq \lambda \leqq m + n - q - 1.$$

Assertion:

$$G_{p,q}^{m,n}(z) = \sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \Delta^{m,n}_{q}(t)\, G_{p,q}^{q,1}(z\, e^{(q-m-n+2\lambda+1)\pi i} \,\|\, a_t). \quad (145)$$

Theorem 8 A. Assumptions: *m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq q; \quad \ldots \quad (146)$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ fulfil the conditions (1) and (20); r is an arbitrary integer which satisfies the inequality

$$r \geqq \mathrm{Max}\,(0, q - m - n + 1). \quad \ldots \quad (147)$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) &= A^{m,n}_{q} \sum_{s=0}^{r-1} \Omega^{m,n}_{q}(s)\, G_{p,q}^{q,0}(z\, e^{(q-m-n-2s)\pi i}) \\ &+ \sum_{t=1}^{n} e^{(m+n-q+2r-1)\pi i a_t} \Delta^{m,n}_{q}(t)\, G_{p,q}^{q,1}(z\, e^{(q-m-n-2r+1)\pi i} \,\|\, a_t). \end{aligned}\right\} \quad (148)$$

Theorem 8 B. Assumptions: *m, n, p and q are integers which satisfy the conditions (146);*
the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ fulfil the conditions (1) and (20); r is an arbitrary integer which satisfies the inequality (147).

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) &= \overline{A}^{m,n}_{q} \sum_{s=0}^{r-1} \overline{\Omega}^{m,n}_{q}(s)\, G_{p,q}^{q,0}(z\, e^{(m+n-q+2s)\pi i}) \\ &+ \sum_{t=1}^{n} e^{-(m+n-q+2r-1)\pi i a_t} \Delta^{m,n}_{q}(t)\, G_{p,q}^{q,1}(z\, e^{(m+n-q+2r-1)\pi i} \,\|\, a_t). \end{aligned}\right\} \quad (149)$$

Theorem 9. Assumptions: *m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq q \text{ and } m + n \leqq q + 1;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequality*

$$0 \leqq r \leqq q - m - n + 1.$$

Assertion:

$$\left.\begin{aligned} G^{m,n}_{p,q}(z) = A^{m,n}_{q} \sum_{s=0}^{r-1} \Omega^{m,n}_{q}(s)\, G^{q,0}_{p,q}\left(z\, e^{(q-m-n-2s)\pi i}\right) \\ + \bar{A}^{m,n}_{q} \sum_{\tau=0}^{q-m-n-r} \bar{\Omega}^{m,n}_{q}(\tau)\, G^{q,0}_{p,q}\left(z\, e^{(m+n-q+2\tau)\pi i}\right) \\ + \sum_{t=1}^{n} e^{(m+n-q+2r-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, G^{q,1}_{p,q}\left(z\, e^{(q-m-n-2r+1)\pi i} \,\|\, a_t\right). \end{aligned}\right\} \quad (150)$$

Theorem 10. Assumptions: *m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq q;$$

the numbers $a_1, \ldots, a_p$ *and* $b_1, \ldots, b_m$ *fulfil the conditions*

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, n;\ h = 1, \ldots, m). \quad . \quad . \quad (1)$$

$$a_j - a_h \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, p;\ h = 1, \ldots, p;\ j \neq h); \quad (38)$$

μ is an arbitrary integer which satisfies the inequality

$$0 \leqq \mu \leqq q - p; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

λ is an arbitrary integer.

Assertion:

$$\left.\begin{aligned} G^{m,n}_{p,q}(z) = \sum_{h=1}^{q-p-\mu} R^{m,n}_{p,q}(h;\lambda)\, G^{q,0}_{p,q}\left(z\, e^{(q-m-n-2h-2\lambda+2)\pi i}\right) \\ + \sum_{\varkappa=1}^{\mu} \bar{R}^{m,n}_{p,q}(\varkappa;\, p-m-n-\lambda+1)\, G^{q,0}_{p,q}\left(z\, e^{(2p-q-m-n+2\varkappa-2\lambda)\pi i}\right) \\ + \sum_{\sigma=1}^{p} e^{(q-p-2\mu)\pi i a_\sigma} T^{m,n}_{p,q}(\sigma;\lambda)\, G^{q,1}_{p,q}\left(z\, e^{(2p-q-m-n-2\lambda+2\mu+1)\pi i} \,\|\, a_\sigma\right). \end{aligned}\right\} \quad (152)$$