Mathematics. — *A theorem concerning analytic continuation II.* By J. DE GROOT. (Communicated by Prof. J. A. SCHOUTEN.)

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In this paper we have two objects in view; in 1. we shall give a discussion of the results reached in a previous paper (A theorem concerning analytic continuation, Proc. Kon. Ned. Akad. v. Wet. **49** (1946), p. 213—222, denoted by [1]), in connection with a possible simplification of a certain proof, and the results already known about this subject. In 2. — 4. we shall give a generalisation of the theorem of continuation of [1], where the following question will be answered: *under which conditions does a one-valued function f, defined on an arbitrary closed point-set of the complex plane, present a one-valued and analytic function?*

1. In [1] the following problem (among others) was considered: in the complex plane be given a sequence of points $z_0, z_1, z_2, \ldots$, converging to z'. Each point z_i is given a function-value w_i. Under which conditions is it possible to find a one-valued and analytic function $f(z)$ defined in a region containing z' (and therefore also almost all points z_i) such that $f(z_i) = w_i$ for *nearly* all values of i? We found a necessary and adequate condition ([1], theorem I), which implied that for the nth difference-quotient $\triangle^n$ (defined for the points $z_0, z_1, z_2, \ldots, z_n$):

$$|\triangle^n| \leqq n!\, r^n \qquad (n = 1, 2, 3, \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where r is a suitably chosen positive constant.

The necessity of this condition was proved by means of an unequality ([1], theorem IV), which again was proved by [1], theorem III, i.e., the generalized mean value theorem for complex functions. It now appears that this unequality may be proved in a very simple way by well-known expedients [1]), while the necessity of (1) may also be proved immediately by use of the following well-known integral which represents $\triangle^n$ [2]) — Mr. POPKEN kindly drew my attention to this —

$$\triangle^n = \frac{n!}{2\pi i} \int\limits_C \frac{f(\zeta)\, d\zeta}{\prod\limits_{i=0}^{n} (\zeta - z_i)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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[1]) This unequality follows for instance from

$$\triangle^n = \lambda f^{(n)} (\lambda_0 z_0 + \ldots + \lambda_n z_n), \; (|\lambda| \leqq 1, \lambda_i \text{ real, with } \sum_0^n \lambda_i = 1),$$

comp. NÖRLUND, Differenzenrechnung.

[2]) Compare also NÖRLUND, op. cit.

Here $f(z)$ is an analytic function on and within the contour C, while the points $z_0, z_1, z_2, \ldots, z_n$ are within C. That (2) is true is immediately obvious if one applies the theorem of residues on the right side of (2). Thus we get

$$n! \sum_{i=0}^{n} \frac{f(z_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (z_i - z_j)}$$

and this is exactly $\triangle^n$. From (2) one now finds immediately a relation of the shape (1).

Further it may be remarked, that theorem V, following immediately from [1], theorem I, is known (comp. BENDIXSON, Acta Mathematica **9** (1887), p. 1—34). Summing up we may say that [1], theorem I is related to already known theorems and may, by means of simple expedients, be deduced from those theorems. Moreover [1], theorem III (mean value theorem for complex functions) is already known. This theorem was proved by MONTEL (comp. F. MONTEL, Sur quelques propriétés des différences divisées, Journal de Mathématiques (9me s.) **16** (1937), p. 222).

Mr. POPKEN informed me that [1], theorem I may be generalized in the following manner. While we considered a sequence of points, which converged to z', consisting of different points $z_0, z_1, z_2, \ldots$, POPKEN permits the coinciding of finitely many points z_i succeeding each other, for instance $z_0, z_1, z_1, z_1, z_2, z_2, z_3, z_4, \ldots$. To each point are given function-values w_i, to the sequence just mentioned for instance the values $w_0, w_1, w_1', w_1'', w_2, w_2', w_3, w_4, \ldots$, and we now look for a necessary and adequate condition that there exist a one-valued and analytic function, which in the points z_i has the values w_i, while moreover if a point z_i occurs n times at one stretch, the first until and including the $(n-1)th$ derivate of the function in z_i should be (respectively) $w', w'', \ldots, w^{(n-1)}$. Naturally the common shape of the difference-quotient is senseless for coinciding points z_i, but one may — in a well-known way — give a definition of the difference-quotient by means of determinants, which holds sense, when a number of points z_i coincide. As necessary and adequate condition one finds again (1).

2. The problem of continuation considered in [1] may be generalized as follows: there be given a (for the time being) bounded, closed but for the rest arbitrary set A in the complex plane. To each point $a \subset A$ is given a certain complex value $f(a)$, in other words, a one-valued function f has been defined on A. The problem now is to find a necessary and adequate condition to be imposed upon the values $f(a)$, in order that f be a one-valued analytic function on A.

A function f is, as known, analytic in a point or on a curve if it is possible to find a region containing that point or that curve — so this region is a neighbourhood of this point or curve — on which f is a one-valued analytic function. This definition may also be used when considering an arbitrary bounded closed set A instead of a point or a curve. In case A is not connected, in other words, is not a continuum, we shall ask less; then we shall only ask that there may be found finitely many disjunct regions containing A (i.e., which together form a neighbourhood of A), such that f is a one-valued and analytic function on each of those regions. We do not ask, for instance, that the function-values which f takes in one region be found by analytic continuation of the function-values of another region. We shall, however, call our f — unlike the custom — in this case, for the sake of simplicity, *analytic on that neighbourhood of* A, consisting of a finite number of regions. The problems rising by this definition we hope to discuss later. In case A *is a continuum,* however, *only one region is formed and our definition coincides with the known one.*

We consider the n^{th} *divided difference* $\Phi^n = \Phi^n\,[z_0 z_1 \ldots z_n]$ of $f(A)$ corresponding with $(n+1)$ arbitrary points $z_0, z_1, \ldots, z_n$ of A for every natural number n. Φ^n is defined by

$$\Phi^n = \frac{\Delta^n}{n!} = \sum_{k=0}^{n} \frac{f(z_k)}{\prod\limits_{\substack{j=0\\ j\neq k}}^{n} (z_k - z_j)}.$$

Now we demand that there may be found a fixed positive number r such that

$$\sqrt[n]{|\Phi^n|} \leqq r \qquad (n = 1, 2, \ldots) \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (3)$$

We contend that condition (3), composed for every finite combination of different points of A, is a necessary and adequate condition for the given f being analytic on A (i.e., on a neighbourhood of A consisting of a finite number of regions).

Theorem I'. *A one-valued function f defined on a bounded closed set A is analytic on A only if the set of all values*

$$\sqrt[n]{|\Phi^n|} \qquad\qquad (n = 1, 2, 3, \ldots),$$

composed for every combination of n points of A, is bounded.

P r o o f. *The condition is adequate.*

We may suppose that A is an infinite point-set (in the opposite case the following proof is greatly simplified). It is possible to construct a countable set Z, which is everywhere dense in A (i.e., A is the closure of Z: $\bar{Z} = A$). Be a an arbitrary, but not isolated point of A. Then there may be found a sequence $a_0, a_1, a_2, \ldots$, consisting of points belonging to Z, which converges to a. If we take all points a_i $(i = 0, 1, 2, \ldots)$ out of Z there remains a countable set $b_0, b_1, b_2, \ldots$. We may suppose there are infinitely many points b_i. We now enumerate Z by taking alternately points a_i and b_i: $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$ and shall note this sequence in the following by $z_0, z_1, z_2, \ldots$.

We define the function $g(z)$ by

$$g(z) = f(z_0) + \sum_{n=0}^{\infty} \Phi^{n+1} \cdot (z-z_0)(z-z_1)\dots(z-z_n) \quad . \quad . \quad . \quad (4)$$

where $\Phi^n = \Phi^n \ [z_0 z_1 \dots z_n]$. We shall prove first that $g(z)$ is a one-valued and analytic function in a certain neighbourhood O of a and that in the points of the intersection $O \cdot A$ this function takes precisely the given f-values. In the first place it is obvious — according to the interpolation-formula of NEWTON — that in the points z_i $(i = 0, 1, 2, \dots)$ $g(z)$ takes exactly the prescribed values $f(z_i)$.

Further be $\frac{1}{2}^d$ the diameter of A, then for the points z of a certain adequately small neighbourhood of a

$$|z-z_j| < d \qquad (j = 0, 1, 2, \dots).$$

Since $z_{2j} = a_j$ and the points a_i are converging to a there may be found an index m for every arbitrarily small $\delta > 0$, such that

$$|z-z_{2j}| < \delta \text{ for } j > m.$$

From these two unequalities one may deduce, in connection with (3) and (4), in a simple way that there may be found an index k and two positive constants C_1 and C_2 such that for the values z which are within a δ-neighbourhood of a

$$|g(z)| < C_1 + C_2 \cdot \sum_{n=k}^{\infty} r^n \cdot \delta^{1/2 n} \cdot d^{1/2 n} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now we choose $\delta < \dfrac{1}{dr^2}$, from which follows that the series of (5) converges, so that the series of (4) is uniformly converging in the δ-neighbourhood of a. As the terms of the series of (4) are all analytic functions in that neighbourhood, $g(z)$ is regular within that neighbourhood.

Further we must prove that in the points a' of A which belong to the mentioned δ-neighbourhood, $g(z)$ takes exactly the given values $f(a')$. Be a' such a point; in the first place we suppose a' to be a limit-point of A and consider the sequence $a', z'_0, z'_1, z'_2, \dots$, where z'_i $(i = 0, 1, 2, \dots)$ runs through the countable set Z, while z'_{2i} is a sequence converging to a'. We consider the function

$$g_{a'}(z) = f(a') + \Phi^1_{a'} \cdot (z-a') + \sum_{n=0}^{\infty} \Phi^{n+2}_{a'} \cdot (z-a')(z-z'_0)\dots(z-z'_n)$$

with

$$\Phi^n_{a'} = \Phi^n \, [a' \ z'_0 \, z'_1 \, \dots \, z'_{n-1}] .$$

This series, like that of $g(z)$, is certainly uniformly converging in an adequately small neighbourhood of a' and is there a one-valued and analytic function. $g_{a'}(z)$ further takes the given values $f(a')$ and $f(z_i)$ in a' and $z'_i = z_j$ respectively; $g(z)$ and $g_{a'}(z)$ have the same value particularly in the sequence of points z'_{2i} $(i = 1, 2, \dots)$ of Z, converging to a', in other words, $g(z)$ and $g_{a'}(z)$ have the same value in infinitely many points which have a limit-point belonging to their intersection, i.e., they have the same value, according to a well-known theorem of analytic continuation, in their whole intersection; so especially in a' $g(z)$ takes exactly the value $g_{a'}(a') = f(a')$.

Secondly it is possible that a' is not a limit-point but an isolated point of A. In this case a' already appears in the mentioned Z and according to the above of course $g(a') = f(a')$.

For every not-isolated point a of A it is possible, according to (4), to find a function $g(z) = g_a(z)$, which is one-valued and analytic in a certain δ_a-neighbourhood of a and which in the points of A belonging to that neighbourhood has the same values as the given function f. Apart from the open δ_a-neighbourhoods we also consider the closed $\frac{1}{2}\delta_a$-neighbourhoods (the closures of the circle-regions with center a and radius $\frac{1}{2}\delta_a$). Since A is bounded and closed it is — excepting a finite number of isolated points of A, which, for the time being, we remove out of A, while we call the remaining set again A — according to the covering-theorem of HEINE–BOREL, already covered by a finite number of such closed $\frac{1}{2}\delta_a$-neighbourhoods. Take two arbitrary closed neighbourhoods of that kind, denoted by $O = O(\frac{1}{2}\delta_{a'})$ and $O' = O'(\frac{1}{2}\delta_{a''})$. If the intersection $O \cdot O'$ of these two does contain only isolated points of A or no points of A at all, then there are none or finitely many, for the intersection is also closed. $O \cdot O'$ in this case has a certain distance from the set $A - A \cdot O \cdot O'$; $O \cdot O'$ then has an adequately small open neighbourhood $U(O \cdot O')$ which has no common point with $A - A \cdot O \cdot O'$. We now remove from O and O' the intersections of these both with $U(O \cdot O')$. Then we have left two closed sets O_1 and O'_1, while at most finitely many isolated points (viz. the set $A \cdot O \cdot O'$) have been removed from A. This new set $A - A \cdot O \cdot O'$ is again denoted by A. The closed sets O_1 and O'_1 have a vacuous intersection and therefore have a positive distance.

The second possibility is that O and O' have one or more limit-points of A for intersection. In these cases nothing is changed. By doing this for every pair of $\frac{1}{2}\delta_a$-neighbourhoods an at most finite number of isolated points is removed from A; the new set is again denoted by A. Thus we have achieved that the new set A is covered by a finite number of closed O_1-sets. Such a set of the kind O_1 which is finally formed, is not necessarily connected, but consists of an at most finite number of components. From each O_1-set we finally remove those components which contain only isolated points of A. The remaining set we again call A, and this set is finally covered by a finite number of closed sets $A_1, A_2, A_3, \ldots, A_s$ which originated from the sets of the kind O_1. These closed sets are divided into a finite number of systems, where every system will consist of those sets A_i which together form a component (i.e., a maximally connected set). Two sets A_j and A_k of such a component, having no vacuous intersection, must according to the above have a limit-point of A in this intersection.

Be these components $C_1, C_2, \ldots, C_r$. These components are, being closed sets, all at a certain positive distance from each other.

We now consider open neighbourhoods $O(A_1), O(A_2), \ldots, O(A_s)$ of the closed sets $A_1, A_2, \ldots, A_s$, satisfying the following conditions:

1°. A_i is formed from a certain corresponding closed $\frac{1}{2}\,\delta_a$-neighbourhood; $O(A_i)$ will belong to the corresponding open δ_a-neighbourhood.

2°. $O(A_i)$ will not contain any of the finitely many isolated points which have been removed from A.

3°. $O(A_i)$ will have a vacuous intersection with $O(A_j)$ if $A_i \cdot A_j = 0$.

It is easy to see that these conditions may be satisfied. The sum of the $O(A_i)$ $(i = 1, 2, \ldots, s)$ is divided into the components $O(C_1), O(C_2), \ldots, O(C_r)$ corresponding with $C_1, C_2, \ldots, C_r$ respectively. Apparently on every $O(A_i)$ a certain one-valued and analytic function $g(z)$ had been defined. We shall prove that therefore also on an arbitrary $O(C_k)$ a one-valued and analytic function is defined. For: if two sets $O(A_i)$ and $O(A_j)$ belonging to $O(C_k)$, have a not-vacuous intersection, $A_i \cdot A_j \neq 0$ and then according to the above there even is a limit-point $a' \subset A$ belonging to $A_i \cdot A_j$. Since Z is dense in A, a sequence of points belonging to Z converges to a', in other words, the functions $g(z)$ defined on $O(A_i)$ and $O(A_j)$ are taking the same values in a sequence of points having a limit-point a' which belongs to the intersection, and therefore in those points denote the same one-valued and analytic function. On every $O(C_i)$ there has in this way been defined a one-valued and analytic function. There still remain finitely many isolated points $a_1, a_2, \ldots, a_v$ of A; for every point a_i of this sequence we consider an open neighbourhood $O(a_i)$, which has no points in common either with $A - a_i$ or with one of the sets $O(C_i)$; to all points of $O(a_i)$ we are giving the constant given function-value $f(a_i)$. In this manner A is finally covered by a finite number of disjunct regions, on each of which a one-valued and analytic function has been defined, which in the points of A coincides with the given f. These functions together produce the required analytic function, defined on a neighbourhood of A.

The condition is necessary.

We must start from an arbitrary bounded closed set A and a neighbourhood O' of A. Within O' is a neighbourhood O of A consisting of a finite number of disjunct regions $G_1, G_2, \ldots, G_r$; the easy proof of this contention we shall not give. On O', and therefore on O, an arbitrary one-valued and analytic function f has been defined, i.e., on every G_i has been defined a one-valued and analytic function $f_i(z)$. We must prove that condition (3) is satisfied. Since A is closed it may be covered by a finite number of open circle-regions, which including their boundaries belong to O and the sum of which is divided into a finite number of disjunct regions $C_1, C_2, \ldots, C_s$ $(s \geq r)$. Every C_i must be an n_i-fold connected region, since C_i is composed from a finite number of open circle-regions.

The boundary R_i of C_i apparently consists of n_i curves $k_1, k_2, \dots, k_{n_i}$ belonging to O (every k_j consists of a finite number of circular arcs). The integral of $f(z)$ along the curves $k_1, k_2, \dots, k_{n_i}$, so along R_i, all described in positive direction, vanishes according to a well-known extension of the integral-theorem of CAUCHY, since $f(z)$ is analytic on the region G_i. The set A has a certain distance 2ε from the closed sum $\sum_{i=1}^{s} R_i$. Further we consider $(n+1)$ arbitrary points $z_0, z_1, \dots, z_n$ of A. Then apparently (one applies the theorem of residues a finite number of times):

$$\Phi^n = \sum_{i=0}^{n} \frac{f(z_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (z_i - z_j)} = \frac{1}{2\pi i} \sum_{k=1}^{s} \int_{\dot{R}_k} \frac{f(\zeta)}{\prod\limits_{j=0}^{n} (\zeta - z_j)} \, d\zeta \quad . \quad . \quad (6)$$

Further

$$\left| \int_{\dot{R}_k} \frac{f(z)}{\prod\limits_{j=0}^{n} (\zeta - z_j)} \, d\zeta \right| \leqq \frac{M_k \, . \, L_k}{\varepsilon^{n+1}},$$

where M_k is the maximum of $|f(z)|$ on R_k and L_k is the length of R_k. From this follows immediately

$$|\Phi^n| \leqq \frac{1}{2\pi\varepsilon^{n+1}} \, . \sum_{k=0}^{s} M_k \, . \, L_k \leqq r^n,$$

where r is a suitably chosen positive number independent of n. Since this holds true for every natural number n the proof is hereby finished.

3. *Other expressions for the necessary and adequate condition.* The necessary and adequate condition (3) may be varied in different ways. In (3) there are more than countably many conditions for the case that A is a not-countable set. This number of conditions may be made countable, however, in the following way: be Z an arbitrary countable set, everywhere dense in A. A necessary and adequate condition for f to be analytic is *the demand that* (3) *holds true for every finite combination of points of* Z *and that* f *is continuous on* A. The number of conditions (3) is now indeed at most countably infinite; but, on the other hand, we must now ask the continuity of f, which was not the case in theorem I′. But if A itself is countable this demand is not necessary of course, since we then may take $Z = A$.

Condition (3) may also be weakened in the following way: for every enumeration (α) of Z, say $z_1^\alpha, z_2^\alpha, \dots$, there is a positive number r_α, an index i_α and a natural number n_α such that for every $n \geqq n_\alpha$

$$\sqrt[n]{|\Phi_\alpha [z_i \, z_{i+1} \dots z_{i+n}]|} \leqq r_\alpha .$$

The numbers r_α may be different for different enumerations (α) and e.g. need not have an upper boundary. This condition may be weakened still more. The conditions we mentioned are conditions for entire A or Z. One may replace these by conditions in the points themselves; for instance: for every point of A a neighbourhood O may be found such that on $O \cdot Z$ condition (3) holds true.

4. *Further generalization of theorem I'.*

Of more importance is the question to what extent the conditions of boundedness and closedness, imposed upon A, are essential.

The condition of boundedness may be left out provided one only asks that condition (3) holds true for every bounded closed subset D of A (for different subsets D the value of r may be different); one may also impose the somewhat weaker condition that for the part of A lying within every closed circle-region with the origin for center and radius R, condition (3) holds true, while for different values R one may have different r's. The necessity and adequacy of this condition are proved by means of the method developed in 2. There is, however, one difference with the result reached in theorem I': for bounded closed A the given f may be continued analytically on a neighbourhood of A which consisted of a finite number of disjunct regions; in case boundedness is no longer conditioned the number of regions on each of which the continued f is one-valued and analytic may be infinitely great. In every bounded set of the complex plane there are, on the other hand, only finitely many regions of that kind; the regions accumulate only in the infinite.

In case one also takes into consideration the point in infinity, in other words, in case one considers a closed set on the complex sphere, one may find again a necessary and adequate condition perfectly analogous with (3). In that case one studies $f(z)$ in a neighbourhood of the point in infinity by examining the conduct of $f(1/z')$ in a neighbourhood of the point $z' = 0$, i.e., in a suitably chosen neighbourhood of $z' = 0$ condition (3) should hold true for $f(1/z')$.

The condition that A be closed plays a more essential part. If A is not closed and f is a one-valued function on A one may impose different conditions concerning the required analytic continuation, as we already explained in [1], 2. The problem is, whether one requires an analytic function which does or does not contain the limit-points of A, not belonging to A, in its region of regularity. In case one does not demand all limit-points of A to belong to the region of regularity, weaker conditions than (3) will often suffice. This problem will not be discussed here further; we only remark that for certain categories of not-closed sets one may find a necessary and adequate condition analogous with (3); for an arbitrary region e.g. by taking this as the sum of countably many closed sets. If one demands, however, continuation on a neighbourhood of the closure $\overline{A}$ of A (consisting of a finite number of regions) one apparently can use condition (3) again. Summarizing we may give the following theorem.

Theorem I''. *A one-valued function $f(A)$, defined on an arbitrary set A of the complex plane, may be continued to a one-valued and analytic function $g(z)$, defined on a certain neighbourhood $O(\overline{A})$ of the closure $\overline{A}$ of A (such that therefore $f(a) = g(a)$ for all points $a \subset A$), only if for every bounded subset A' of A there may be found a number $r_{A'}$ such that for the n^{th} divided difference Φ^n of f, composed for every combination of n points of A',*

$$\sqrt[n]{|\Phi^n|} \leqq r_{A'} \qquad (n = 1, 2, \ldots).$$

Here we may manage that the at most countably many regions which together form $O(\overline{A})$ do not accumulate anywhere in the finite (on each of those regions $g(z)$ is one-valued and analytic, but the values of $g(z)$ in one region need not necessarily form an analytic continuation of the values of $g(z)$ in another region).

As a special case we mention further:

Corollary. If A is a bounded closed point-set in the complex plane, consisting of a continuum and an arbitrary number of isolated points, and if on A is defined a one-valued function f, then f is analytic on A (except in a finite number of isolated points of A) and may be continued analytically on a *region* containing A (excepting again an at most finite number of isolated points of A), only if $f(A)$ satisfies condition (3).

Finally we remark that the generalisation of POPKEN, mentioned in 1., may be combined with our generalisation, which combination brings us to new theorems that are obvious without further discussion.