Mathematics. — *On the fundamental theorem of algebra.* (Second communication) [1]. By J. G. VAN DER CORPUT.

(Communicated at the meeting of September 21, 1946.)

§ 2. *Proof of the transitivity.*

Definition. A polynomial is called simple if it is relatively prime to its derivative.

Lemma 1. *Every divisor $D(X)$ of a simple polynomial $F(X)$ is simple,* for a common divisor of $D(X)$ and $\frac{dD}{dX}$ would occur both in $F(X)=D(X)\,U(X)$ and in $\frac{dF}{dX}=U\frac{dD}{dX}+D\frac{dU}{dX}$, which is impossible.

Lemma 2. *The product of simple relatively prime polynomials is simple.*

It is sufficient to give a proof for the product of two polynomials. Be $F(X)=U(X)\,V(X)$. Then $\frac{dF}{dX}=U\frac{dV}{dX}+V\frac{dU}{dX}$ is relatively prime both to $U(X)$ and $V(X)$, hence also to $U(X)\,V(X)$.

Lemma 3. *Two polynomials, the product of which is simple, are simple and relatively prime.*

In fact, a common divisor of $U(X)$ and $\frac{dU}{dX}$ (or $V(x)$) occurs both in $F(X)=U(X)\,V(X)$ and in $\frac{dF}{dX}=U\frac{dV}{dX}+V\frac{dU}{dX}$, which is impossible, if $U(X)\,V(X)$ is simple.

Lemma 4. *Any polynomial $F(X)$ may be written in the form*

$$F(X)=F_1^{\alpha_1}\ldots F_\lambda^{\alpha_\lambda},$$

where the exponents are positive and $F_1(X),\ldots,F_\lambda(X)$ are simple, relatively prime polynomials.

Proof. The theorem is obvious if $F(X)$ is simple. If not, put $F(X)=D(X)\,F^*(X)$, where D is the greatest common divisor of $F(X)$ and $\frac{dF}{dX}$. In this case the degree of both $D(X)$ and $F^*(X)$ is less than the degree of $F(X)$. We may suppose, that for these polynomials the

---

1) Compare Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 722—732 (1946) and Indagationes Mathematicae **4** (1946).

proof is already given. Consequently we may write

$$F = F_1^{\alpha_1} \dots F_\lambda^{\alpha_\lambda}, \quad \dots \dots \dots \quad (6)$$

where the exponents α_i are natural numbers and $F_1 \dots F_\lambda$ are simple polynomials. We may even assume that these polynomials are relatively prime. For, if for instance $F_1 = UV$ and $F_2 = UW$, then U, V and W, being divisors of simple polynomials, are simple and

$$F = U^{\alpha_1 + \alpha_2} V^{\alpha_1} W^{\alpha_2} F_3^{\alpha_3} \dots F_\lambda^{\alpha_\lambda}.$$

Continuing in this manner we finally find $F(X)$ written in a form, similar to (6) with simple, relatively prime polynomials.

Lemma 5. *If a polynomial F is written in the form* (6), *where $F_1, \dots, F_\lambda$ are simple and relatively prime and the exponents are positive, then $F_1(X) \dots F_\lambda(X)$ is the characteristic divisor of $F(X)$.*

Proof. $\dfrac{dF}{dX}$ is the sum of λ terms; all but one of these terms are divisible by $F_1^{\alpha_1}$. The exceptional term possesses the form $\alpha_1 \dfrac{F}{F_1}$ and is divisible by $F_1^{\alpha_1 - 1}$ but has further no common factor with F_1. The derivative $\dfrac{dF}{dX}$ is therefore divisible by $F_1^{\alpha_1 - 1}$ and has no further common factor with F_1. A similar argument is valid for the polynomials $F_2, \dots, F_\lambda$, so that $G = F_1^{\alpha_1 - 1} \dots F_\lambda^{\alpha_\lambda - 1}$ is the greatest common divisor of F and $\dfrac{dF}{dX}$. Hence it follows, that $F_1 \dots F_\lambda$ is equal to the characteristic divisor.

Corollaries. The characteristic divisor $F^*(X)$ of a polynomial $F(X)$ is simple, being the product of simple, relatively prime polynomials.

$F(X)$ divides $(F^*(X))^\mu$, where μ denotes the degree of $F(X)$, since each $\alpha_\nu \leqq \mu$.

If $D(X)$ is a divisor of $F(X)$, then $D^*(X)$ is a divisor of $F^*(X)$. In fact, we may write $D(X) = F_1^{\beta_1} \dots F_\lambda^{\beta_\lambda}$ where $0 \leqq \beta_\nu \leqq \alpha_\nu$, so that the characteristic divisor $\prod_{\beta_\varrho > 0} F_\varrho$ of $D(X)$ is a divisor of the characteristic divisor $\prod_{\varrho=1}^{\lambda} F_\varrho$ of $F(X)$.

Lemma 6. *If U is the greatest common divisor of F and G, then U^* is the greatest common divisor of F^* and G^*.*

Proof. U is a divisor of both F and G. By corollary of lemma 5 the polynomial U^* is a divisor of both F^* and G^*, hence also of their greatest common divisor.

The greatest common divisor D of F^* and G^* is a divisor of U. Hence the characteristic divisor D^* of D is a divisor of U^*. But D is a divisor of the simple polynomial $F^*(X)$, so that it is also simple, consequently identical with its own characteristic divisor. Hence D is a divisor of U^*. Consequently U^* and D are identical.

Lemma 7. *If q denotes a positive element of Ω, then every interval can be divided in a finite number of subintervals, each with length $\leqq q$.*

In fact, Ω being Archimedeanly ordered, a natural number ϱ exists, such that $\varrho > \frac{b-a}{q}$, where a and b denote the endpoints of the interval. The points $a + \frac{\sigma(b-a)}{q}$ where σ runs through $0, 1, \ldots \varrho$, divide the given interval in subintervals, each with length $\leqq q$.

Lemma 8. *To any two relatively prime polynomials $F(X)$ and $G(X)$ corresponds a positive element $p = p(F, G) \leqq e$ of Ω, satisfying the inequality*

$$|F(X)| + |G(X)| > p$$

for all elements X of Ω.

Proof. We can restrict ourselves to an interval Γ outside which $|F(X)| > e$. Since $F(X)$ and $G(X)$ are relatively prime, the unit element of Ω can be written in the form

$$e = U(X)\,F(X) + V(X)\,G(X),$$

where $U(X)$ and $V(X)$ denote appropiate polynomials. The absolute value of $U(X)$ and $V(X)$ in Γ is less than $\frac{e}{p}$, where p is a suitable positive element of Ω. Then we have for all X of Ω in Γ

$$e < \frac{e}{p}\left(|F(X)| + |G(X)|\right),$$

which implies the required result.

Lemma 9. *To any two relatively prime polynomials $F(X)$ and $G(X)$ corresponds a positive element $q = q(F, G)$ of Ω, such that $G(X)$ is definite in every interval with length $\leqq q$, where $F(X)$ changes sign, and $F(X)$ is definite in every interval with length $\leqq q$, where $G(X)$ changes sign* ("definite" means: always positive or always negative).

Proof. We may restrict ourselves to a bounded interval $\triangle$. Consider the elements $p(F^*, G)$ and $p(F, G^*)$ of lemma 8. Be p_1 the smaller of) these two elements. Ω contains a positive element $m \geqq e$, such that each derivative of $F^*(X) = H(X)$, of $F(X)$, of $G^*(X)$ and of $G(X$ has in $\triangle$ an absolute value $\leqq m$. To prove that the element $q = \frac{p_1}{4m}$ possesses the required property we consider a subinterval $\triangle_1$ of $\triangle$ with length $\leqq q$, where $F(X)$ changes sign. For any two elements x and

$x+h$ of $\triangle_1$ we have $|h| \leqq q \leqq e$, hence, if μ denotes the degree of F,

$$\left.\begin{aligned} |H(x+h)-H(x)| &= |h| \left| \frac{H'(x)}{1!} + \ldots + \frac{h^{\mu-1} H^{(\mu)}(x)}{\mu!} \right| \\ &\leqq |h|\, m \left(\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{\mu!} \right) < 2\,|h|\, m \leqq 2qm = \frac{p_1}{2} \end{aligned}\right\} . \quad (7)$$

Similarly

$$|G(x+h) - G(x)| < \frac{p_1}{2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since $F(X)$ changes sign in $\triangle_1$, this interval contains two elements u and v with $F^*(v) \leqq 0 \leqq F^*(u)$.

Hence from (7) we infer

$$0 \leqq F^*(u) \leqq F^*(u) - F^*(v) < \frac{p_1}{2}.$$

From the preceeding lemma it follows then $|G(u)| > \frac{p_1}{2}$. From (8) we find for each element x of $\triangle_1$

$$G(x) > G(u) - \frac{p_1}{2} > 0, \text{ if } G(u) > \frac{p_1}{2},$$

and

$$G(x) < G(u) + \frac{p_1}{2} < 0, \text{ if } G(u) < -\frac{p_1}{2}.$$

Hence $G(X)$ is definite in $\triangle_1$.

The second part of the theorem follows by exchanging F and G.

Definition. Two polynomials $F(X)$ and $G(X)$ are said to be equivalent in an interval Φ, if two polynomials P and Q, both definite in Φ, exist, such that $F^* P = G^* Q$.

Lemma 10. *If a polynomial $F(X)$ changes sign in an interval Φ, then every multiple $G(X)$ of $F(X)$ changes also sign in Φ. If further is given, that $G(X)$ changes sign at most once in Φ, then $F(X)$ and $G(X)$ are equivalent in Φ* (and therefore both change sign exactly once in Φ).

Proof. G^* is a multiple of F^*. Be $G^*(X) = F^*(X)\, U(X)$. Since $F^*(X)$ changes sign in Φ, by lemma 9 and 3 we can divide Φ in subintervals, each with length $\leqq q$, where q denotes the element $q(F^*, U)$ of lemma 9. In at least one of these subintervals, say Φ_1, the polynomial $F^*(X)$ changes sign. In virtue of the choice of q the polynomial $U(X)$ is definite in Φ_1, hence $G^*(X) = F^*(X)\, U(X)$ changes sign in Φ_1.

Suppose that $G(X)$ changes sign only once in Φ. If $U(X)$ were not definite in Φ, in one of the constructed subintervals, say Φ_2, the polynomial $U(X)$ would change sign and there $F^*(X)$ would be definite.

Hence $G^*(X) = F^*(X)\, U(X)$ would change sign in Φ_2. Since $U(X)$ is definite in Φ_1, but not in Φ_2, the intervals Φ_1 and Φ_2 would be different and $G^*(X)$ would change sign in Φ more than once. This not being the case, we find $U(X)$ to be definite in Φ and therefore $F(X)$ and $G(X)$ to be equivalent in Φ.

Remark. From $(\Gamma, C) = (\Delta, D)$ it follows, that (C, D) changes sign at most once in (Γ, Δ), for (C, D) changes sign there and has a multiple $D(X)$, which changes sign at most once in (Γ, Δ).

Lemma 11. *If $F(X)$ and $G(X)$ are equivalent in Φ, they are also equivalent to their greatest common divisor in Φ.*

Proof. Put $F^* = U^* P$ and $G^* = U^* Q$, where U^* is the greatest common divisor of F^* and G^*, so that P and Q are relatively prime. By lemma 6 the polynomial U^* is the characteristic divisor of the greatest common divisor U of F and G. If $P(X)$ is not definite in Φ, then a subinterval of Φ exists, in which $P(X)$ changes sign and which has a length $< q(U^*, P)$ and $< q(P, Q)$, where q denotes the element of Ω, introduced in lemma 9. By this lemma U^* and Q are definite in this subinterval; hence also G^*, while F^* changes sign in that subinterval. This is impossible, since $F(X)$ and $G(X)$ are equivalent in Φ.

Proof of the transitivity. From $(\Gamma, C) = (\Delta, D)$ and $(\Delta, D) = (\Lambda, L)$ we must deduce $(\Gamma, C) = (\Lambda, L)$. The greatest common divisor (C, D) of C and D changes sign in (Γ, Δ) and possesses a multiple $D(X)$, which changes sign at most once in that interval. From lemma 10 these polynomials are equivalent in (Γ, Δ) and both change sign there. Similarly (D, L) and D are equivalent in (Δ, Λ) and both change sign there. By lemma 11 the polynomial $D(X)$ is in $(\Gamma, \Delta, \Lambda)$ equivalent to the greatest common divisor (C, D, L) of (C, D) and (D, L). In $(\Gamma, \Delta, \Lambda)$ the polynomial $D(X)$ changes sign, for otherwise $D(X)$ would change sign both in (Γ, Δ) and (Δ, Λ) outside $(\Gamma, \Delta, \Lambda)$, consequently more than once in Δ, which is impossible. Hence the polynomial (C, D, L) which is equivalent with $D(X)$, changes sign in $(\Gamma, \Delta, \Lambda)$, and therefore also its multiple (C, L) by lemma 10. Then (C, L) changes sign also in (Γ, Λ).

§ 3. *Definition of sum and product.*

Consider two polynomials

$$F(X) = f_0 + f_1 X + \ldots + f_\mu X^\mu \text{ and } G(X) = g_0 + g_1 X + \ldots + g_\nu X^\nu,$$

where $f_\mu = g_\nu = e$. The products $\Pi(X - Y_\varrho - Z_\sigma)$ and $\Pi(X - Y_\varrho Z_\sigma)$, where ϱ runs through $1, \ldots, \mu$ and where σ runs through $1, \ldots, \nu$, and where X, Y_ϱ and Z_σ denote indeterminates, may be written as integral rational functions of X, of the elementary symmetric functions of $Y_1, \ldots, Y_\mu$ and of the elementary symmetric functions of $Z_1, \ldots, Z_\nu$. If we replace the elementary symmetric functions ΣY_1, $\Sigma Y_1 Y_2, \ldots, Y_1 Y_2 \ldots Y_\mu$, ΣZ_1, $\Sigma Z_1 Z_2, \ldots, Z_1 Z_2 \ldots Z_\nu$ respectively by $-f_{\mu-1}, f_{\mu-2}, \ldots, (-1)^\mu f_0$, $-g_{\nu-1}$, $g_{\nu-2}, \ldots, (-1)^\nu g_0$, these products become polynomials in X, which I denote by $F(X) \dotplus G(X)$ and $F(X) \times G(X)$.

If we put $G(X) = G_1(X) G_2(X)$, the left side of the identity

$$\prod_{\varrho=1}^{\mu} G(X-Y_\varrho) = \prod_{\varrho=1}^{\mu} G_1(X-Y_\varrho) \cdot \prod_{\varrho=1}^{\mu} G_2(X-Y_\varrho)$$

becomes $F(X) \dotplus G(X)$, hence

$$F(X) \dotplus G(X) = (F(X) \dotplus G_1(X))\ (F(X) \dotplus G_2(X)).$$

Herefrom it follows: If $G_1(X)$ is a divisor of $G(X)$, then $F(X) \dotplus G_1(X)$ is a divisor of $F(X) \dotplus G(X)$. In the same way we get: If also $F_1(X)$ is a divisor of $F(X)$, then $F_1(X) \dotplus G_1(X)$ is a divisor of $F(X) \dotplus G_1(X)$, hence of $F(X) \dotplus G(X)$. In particular:

Lemma 12. *If u is a root of $F(X)$ and v is a root of $G(X)$, then $(X-u) \dotplus (X-v) = X-u-v$ is a divisor of $F(X) \dotplus G(X)$, hence $u+v$ is a root of $F(X) \dotplus G(X)$.*

Lemma 13. *A polynomial $F(X)$, the derivative of which is always $\geqq 0$ in an interval Φ, satisfies the inequality $F(u) \leqq F(v)$ for all elements u and v of Φ with $u \leqq v$.*

Proof. The second and higher derivatives of $F(X)$ in Φ are absolutely less than a suitably chosen element m of Ω. Divide the interval with endpoints u and v into σ equal parts, each of length $l = \frac{v-u}{\sigma} \leqq e$. For the endpoints a and b $(a < b)$ of such a part we have

$$F(b) - F(a) = \frac{b-a}{1!} F'(a) + \frac{(b-a)^2}{2!} F''(a) + \ldots + \frac{(b-a)^\mu F^{(\mu)}(a)}{\mu!},$$

where μ denotes the degree of F. From $F'(a) \geqq 0$ it follows

$$F(b) - F(a) \geqq -m\, l^2 \left(\frac{1}{2!} + \ldots + \frac{1}{\mu!}\right) \geqq -m\, l^2,$$

and adding we obtain

$$F(v) - F(u) \geqq -m\, l^2\, \sigma = -\frac{m\,(v-u)^2}{\sigma}.$$

Since the number σ may be taken arbitrary large, we find $F(v) - F(u) \geqq 0$, for otherwise the number σ could be taken as large as to contradict the inequality.

Remark. From this lemma it follows immediately: If a polynomial has a definite derivative in an interval, the polynomial changes sign there at most once.

Lemma 14. *If the polynomial $C(X)$ changes sign in Γ and the polynomial $D(X)$ in Δ, then the polynomial $C(X)+D(X)$ changes sign in $\Gamma+\Delta$.*

Proof. Suppose first that C and D are simple. Put $C(X)+D(X)=F(X)$. If a and b denote the endpoints of $\Gamma+\Delta$, we may assume without loss of generality $F^*(a)\neq 0$ and $F^*(b)\neq 0$, for otherwise the lemma is evident. Ω contains a positive element m, such that the second and higher derivatives of $F^*(X)$ are all absolutely $\leqq m$ in $\Gamma+\Delta$. Choose in Ω a positive element l, satisfying the inequalities

$$l\leqq e\,;\; l\leqq \frac{p}{4m}\,;\; l\leqq \frac{2}{p}F^*(a)\,;\; l\leqq \frac{2}{p}F^*(b), \quad \ldots \quad (9)$$

(where p denotes the element $p\left(F^*, \frac{dF^*}{dX}\right)$, introduced in lemma 8), such that in the interval with endpoints a and $a+l$ the inequality $|F^*(x)|>\frac{1}{2}|F^*(a)|$ is valid and in the interval with endpoints $b-l$ and b similarly $|F^*(x)|>\frac{1}{2}|F^*(b)|$. From lemma 5, corollary we know $(F^*)^\mu = FG$, where μ denotes the degree of F and G is a suitable polynomial. Ω contains a positive element g, such that in $\Gamma+\Delta$ the polynomial G possesses an absolute value $\leqq g$.

If s and t are arbitrary elements of Ω, then

$$H(X,s,t)=\{(C(X)-s)+(D(X)-t)\}-\{C(X)+D(X)\}$$

is a polynomial in X, s and t. In each term of H either a factor s or a factor t occurs, for $H(X,0,0)$ is identically equal to 0. Hence a positive element k of Ω exists, such that from $|s|<k$ and $|t|<k$ it follows

$$|H(w,s,t)|<\frac{1}{g}\left(\frac{pl}{4}\right)^\mu, \quad \ldots \quad (10)$$

for all elements w of $\Gamma+\Delta$. Finally we choose the positive element h of Ω such that in every subinterval of Γ with length $\leqq h$, the oscillation of $C(X)$ is less than k, and that also in each subinterval of Δ with length $\leqq h$, the oscillation of $D(X)$ is less than k. Divide Γ and Δ into subintervals, each of length $\leqq h$. In at least one of these subintervals of Γ, say Γ_1, the polynomial $C(X)$ changes sign and in at least one of the subintervals of Δ, say Δ_1, the polynomial $D(X)$ changes sign.

Then the interval Γ_1 contains two elements u and u_1 with $C(u_1)\leqq 0\leqq C(u)$. Since the oscillation of $C(X)$ in Γ_1 is less than k, it follows

$$0\leqq C(u)\leqq C(u)-C(u_1)<k.$$

Similarly Δ_1 contains a point v with $0\leqq D(v)<k$. Hence inequality (10) is valid for $w=u+v$, $s=C(u)$ and $t=D(v)$. Then u is a root of $C(X)-s$ and v is a root of $D(X)-t$, hence $w=u+v$ is a root of $(C(X)-s)+(D(X)-t)$ by lemma 12. Therefore

$$|F(w)|=|C(w)+D(w)|=|H(w,s,t)|<\frac{1}{g}\left(\frac{pl}{4}\right)^\mu.$$

Hence

$$|F^*(w)|^\mu = |F(w)\,G(w)| < \left(\frac{p\,l}{4}\right)^\mu.$$

Consequently

$$|F^*(w)| < \frac{p\,l}{4} < \tfrac{1}{2}\,p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

From the definition of $p = p\left(F^*, \dfrac{dF^*}{dX}\right)$ it follows

$$\left|\frac{dF^*(w)}{dw}\right| > \tfrac{1}{2}\,p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

From (11) and (9) we infer

$$|F^*(w)| < \tfrac{1}{2}\,|F^*(a)| \quad \text{and} \quad |F^*(w)| < \tfrac{1}{2}\,|F^*(b)|.$$

Therefore it is impossible that w lies either in the interval with endpoints a and $a+l$ or in the interval with endpoints $b-l$ and b. As w lies in the interval with endpoints a and b, it lies in the interval with endpoints $a+l$ and $b-l$. The interval $\Gamma + \Delta$ contains consequently the elements $w-l$ and $w+l$.

Since the second and higher derivatives of F^* are absolutely $\leqq m$ in $\Gamma + \Delta$ and since $l \leqq e$, the Taylor development gives

$$\left|F^*(w \mp l) - F^*(w) \pm l\,\frac{dF^*(w)}{dw}\right| \leqq m\,l^2\left(\frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{\mu!}\right) < m\,l^2 \leqq \tfrac{1}{4}\,p\,l,$$

hence by (11) we get

$$\left|F^*(w \pm l) \mp l\,\frac{dF^*(w)}{dw}\right| < \tfrac{1}{2}\,p\,l.$$

From (12) we infer that $F^*(w-l)$ and $F^*(w+l)$ have different sign; consequently $F(X)$ changes sign in $\Gamma + \Delta$.

Suppose now that C and D are not both simple. Then C^* and D^*, and therefore $C^* + D^*$ change sign resp. in Γ, Δ and $\Gamma + \Delta$; consequently the multiple $C + D$ of $C^* + D^*$ changes sign also in $\Gamma + \Delta$.

Now we pass to the definition of the sum of $\gamma = (\Gamma, C)$ and $\delta = (\Delta, D)$. Put $F = C + D$. A subinterval Γ' of Γ and a subinterval Δ' of Δ can be found, both with length $\leqq \frac{1}{2}\,q$, where q denotes the element $q\left(F^*, \dfrac{dF^*}{dX}\right)$, introduced in lemma 9, such that C changes sign in Γ' and D in Δ'. In the interval $\Gamma' + \Delta'$ with length $\leqq q$ the polynomial F changes sign by lemma 14. By lemma 9 the derivative $\dfrac{dF^*}{dX}$ is definite throughout that interval; therefore F^* changes sign there at most once. So we have proved the existence of subintervals Γ' of Γ and Δ' of Δ, such that C changes sign once in Γ', D in Δ' and $C+D$ in $\Gamma'+\Delta'$. Now we may put

$$\gamma+\delta=(\Gamma'+\Delta',\ C+D),$$

if we show that the couple $(\Gamma'+\Delta',\ C+D)$ is uniquely determined. Suppose

$$(\Gamma, C)=(\Gamma_1, C_1) \text{ and } (\Delta, D)=(\Delta_1, D_1).$$

We have to prove

$$(\Gamma'+\Delta',\ C+D)=(\Gamma_1'+\Delta_1',\ C_1+D_1),$$

where Γ', Δ', Γ_1' and Δ_1' are subintervals respectively of Γ, Δ, Γ_1 and Δ_1, such that C changes sign only once in Γ', D in Δ', C_1 in Γ_1', D_1 in Δ_1', $F=C+D$ in $\Phi=\Gamma'+\Delta'$ and $F_1=C_1+D_1$ in $\Phi_1=\Gamma_1'+\Delta_1'$. We must prove that the greatest common divisor L of $F=C+D$ and $F_1=C_1+D_1$ changes sign in the common part Λ of Φ and Φ_1. Since $S=(C, C_1)+(D, D_1)$ is a divisor both of $F=C+D$ and $F_1=C_1+D_1$, the polynomial S is also a divisor of their greatest common divisor L. By assumption the couples $(\Gamma, C)=(\Gamma' C)$ and $(\Gamma_1, C_1)=(\Gamma_1', C_1)$ are equal, so that (C, C_1) changes sign in (Γ', Γ_1') and similarly (D, D_1) in (Δ', Δ_1'). By lemma 14 S changes sign in $\Sigma=(\Gamma', \Gamma_1')+(\Delta', \Delta_1')$. By lemma 10 the multiple L of S changes sign in Σ, hence certainly in Λ, which contains Σ; in fact each point w of Σ may be written in the form $u+v$, where u lies both in Γ' and Γ_1', and v lies both in Δ' and Δ_1', hence w lies both in $\Gamma'+\Delta'$ and $\Gamma_1'+\Delta_1'$, consequently also in their common part Λ. This establishes the proof.

In a similar way we define the product of two couples.