Mathematics. — *On the G-function.* VI. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 17. Further investigation of the expansions of § 16.

I may recall that the function $G_{p,q}^{m,n}(z)$ satisfies the homogeneous linear differential equation (34) and that, if $q > p$ and the conditions (36), (37) and (38) are satisfied, a system of fundamental solutions of this equation valid in the neighbourhood of $z = \infty$ is formed by the p functions (39) together with the $q-p$ functions (40) [48].

In this § I will express the function $G_{p,q}^{m,n}(z)$ $(q > p)$ as a linear combination of these fundamental solutions; the expressions in question appear to be special cases of the expansion formulae (145), (148), (149), (150) and (152). My results can be stated as follows:

Theorem 11. Assumptions: *m, n, p and q are integers with*

$$1 \leqq n \leqq p < q, \quad 2 \leqq m \leqq q \quad \text{and} \quad m+n \geqq q+1; \quad . \quad . \quad (153)$$

the number z satisfies the inequality

$$-(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi < \arg z < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi; \quad . \quad (154)$$

the numbers $a_1, \dots, a_n$, and $b_1, \dots, b_m$ fulfil the conditions (1) *and* (20); *λ is an arbitrary integer which satisfies the inequalities* [49])

$$0 \leqq \lambda \leqq m+n-q-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

$$(m+n+\tfrac{1}{2}p-\tfrac{3}{2}q-2)\pi - \arg z < 2\lambda\pi < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi - \arg z. \quad (156)$$

Assertion: *The function $G_{p,q}^{m,n}(z)$ can by means of* (145) *be expressed in terms of fundamental solutions valid near $z = \infty$.*

Theorem 12 A. Assumptions: *m, n, p and q are integers which fulfil the conditions* (153); *the number z satisfies the inequality*

$$(m+n+\tfrac{1}{2}p-\tfrac{3}{2}q-2)\pi < \arg z < (m+n-p+\varepsilon)\pi; \quad . \quad (157)$$

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[48]) Comp. also the Remark at the end of § 4.

[49]) By (154) we have

$$0 < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi - \arg z,$$

$$(m+n+\tfrac{1}{2}p-\tfrac{3}{2}q-2)\pi - \arg z < (2m+2n-2q-2)\pi;$$

from these relations it follows that there exists at least one integer λ satisfying (155) and (156).

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); r *is an arbitrary integer which satisfies the inequalities*[50])

$$r \geqq 0, \quad \ldots \ldots \ldots \quad (158)$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z<2r\pi<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z. \quad (159)$$

Assertion: *The function* $G_{p,q}^{m,n}(z)$ *can by means of* (148) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

Theorem 12B. Assumptions: *m, n, p and q are integers which fulfil the conditions* (153); *the number z satisfies the inequality*

$$-(m+n-p+\varepsilon)\pi<\arg z<-(m+n+\tfrac{1}{2}p-\tfrac{3}{2}q-2)\pi;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); r *is an arbitrary integer which satisfies the inequalities*

$$r \geqq 0,$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi-\arg z<2r\pi<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi-\arg z.$$

Assertion: *The function* $G_{p,q}^{m,n}(z)$ *can by means of* (149) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

Theorem 13A. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q, \; 1 \leqq m \leqq q \quad \ldots \ldots \quad (160)$$

and

$$\tfrac{3}{4}p+\tfrac{1}{4}q-\tfrac{1}{2}\varepsilon<m+n\leqq q+1; \quad \ldots \ldots \quad (161)$$

the number z satisfies the inequality

$$-(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi<\arg z<(m+n-p+\varepsilon)\pi; \quad . \quad (162)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); r *is an arbitrary integer which satisfies the inequalities*[51])

$$r \geqq q-m-n+1, \quad \ldots \ldots \ldots \quad (163)$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z<2r\pi<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z. \quad (164)$$

Assertion: *The function* $G_{p,q}^{m,n}(z)$ *can by means of* (148) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

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[50]) From (157) it follows

$$(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z>0;$$

hence there exists at least one integer r satisfying (158) and (159).

[51]) Because of (162) we have

$$(2q-2m-2n+2)\pi<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z;$$

from this it appears that there exists at least one integer r which satisfies (163) and (164).

Theorem 13 B. Assumptions: *m, n, p and q are integers which fulfil the conditions* (160) *and* (161); *the number z satisfies the inequality*

$$-(m+n-p+\varepsilon)\pi < \arg z < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*

$$r \geqq q-m-n+1,$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi-\arg z < 2r\pi < (\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi-\arg z.$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (149) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

Theorem 14. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q, \; 1 \leqq m \leqq q$$

and

$$p+1 \leqq m+n \leqq \tfrac{3}{4}q+\tfrac{1}{4}p-\tfrac{1}{2}\varepsilon+1; \quad \ldots \quad (165)$$

the number z satisfies the inequality

$$-(m+n-p+\varepsilon)\pi < \arg z < (m+n-p+\varepsilon)\pi; \quad \ldots \quad (166)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*

$$0 \leqq r \leqq q-m-n+1, \quad \ldots \quad (167)$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z < 2r\pi < (\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z. \quad (168)$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (150) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

Theorem 15. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q \text{ and } 0 \leqq m \leqq q;$$

λ *is an arbitrary integer; the number z satisfies the inequality*

$$(m+n-p+\varepsilon+2\lambda-2)\pi \leqq \arg z < (m+n-p+\varepsilon+2\lambda)\pi; \quad (169)$$

the numbers $a_1, \ldots, a_p$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (38); μ *is an arbitrary integer which satisfies the condition*

$$(m+n-\tfrac{3}{2}p+\tfrac{1}{2}q+2\lambda-2)\pi-\arg z < 2\mu\pi < (m+n-\tfrac{5}{2}p+\tfrac{3}{2}q+2\lambda)\pi-\arg z. \; (170)$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (152) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

If λ runs through the sequence of all positive and negative integers (zero included), we find by means of theorem 15 for all values of $\arg z$ an expression of $G^{m,n}_{p,q}(z)$ $(q>p)$ in terms of fundamental solutions valid near $z=\infty$.

If $m+n \geqq p+1$ and at the same time $|\arg z| < (m+n-p+\varepsilon)\pi$, we may obtain in a simpler way such an expression by means of the theorems 11, 12, 13 and 14.

For instance:

If $m+n \geqq q+1$ and $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, we may use theorem 11.

If $m+n \geqq q+1$ and $(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi \leqq \arg z < (m+n-p+\varepsilon)\pi$, we may use the theorems 12 A, B.

If $\frac{1}{2}p+\frac{1}{2}q < m+n \leqq q+1$ and $|\arg z| < (m+n-p+\varepsilon)\pi$, we may use the theorems 13 A, B.

If $p+1 \leqq m+n \leqq \frac{1}{2}p+\frac{1}{2}q$ and $|\arg z| < (m+n-p+\varepsilon)\pi$, we may use theorem 14.

Proof of theorem 11. From (156) it follows

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi < \arg z+(q-m-n+2\lambda+1)\pi < (\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi.$$

Hence condition (36) (with $-\lambda$ instead of λ) holds for the functions $G_{p,q}^{q,1}$ on the right of (145) and so these functions are fundamental solutions.

Proof of theorem 12 A. By (159) we have

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi < \arg z+(q-m-n-2r+1)\pi < (\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi.$$

The functions $G_{p,q}^{q,1}$ on the right-hand side of (148) are therefore fundamental solutions.

From (159) and (157) it follows

$$2r\pi < (\tfrac{3}{2}q-\tfrac{3}{2}p+\varepsilon+2)\pi,$$

consequently

$$r < q-p+1.$$

Hence the number of the functions $G_{p,q}^{q,0}$ occurring on the right of (148) is at most equal to $q-p$. These functions satisfy the condition (37) (with $\psi=s$); for we have by (157)

$$\arg z+(q-m-n)\pi < (q-p+\varepsilon)\pi$$

and by (159)

$$-(q-p+\varepsilon)\pi < -(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg z+(q-m-n-2r+2)\pi.$$

The functions $G_{p,q}^{q,0}$ in (148) are therefore also fundamental solutions.

Proof of theorem 12 B. Similar to that of 12 A.

Proof of theorem 13 A. The inequality (162) has a meaning; for it follows from (161) that

$$-(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi < (m+n-p+\varepsilon)\pi.$$

For the rest the proof is similar to that of theorem 12 A.

Proof of theorem 13 B. Similar to that of 13 A.

Proof of theorem 14. From (166) it follows

$$(\tfrac{3}{2}q+\tfrac{1}{2}p-2m-2n-\varepsilon+2)\pi<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z \quad (171)$$

and

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z<(\tfrac{1}{2}q-\tfrac{1}{2}p+\varepsilon)\pi. \quad (172)$$

Because of (165) we have

$$0\leqq\tfrac{3}{2}q+\tfrac{1}{2}p-2m-2n-\varepsilon+2 \quad (173)$$

and

$$\tfrac{1}{2}q-\tfrac{1}{2}p+\varepsilon\leqq 2+2q-2m-2n. \quad (174)$$

By combining (171) and (173) we obtain

$$0<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z; \quad (175)$$

similarly by combining (172) and (174)

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z<(2+2q-2m-2n)\pi. \quad (176)$$

From (175) and (176) it appears that there exists at least one integer r satisfying (167) and (168).

Now on the right-hand side of (150) there occur $q-m-n+1$ functions of the type $G_{p,q}^{q,0}(\zeta)$, the values of $\arg\zeta$ being

$$\arg z+(m+n-q)\pi, \arg z+(m+n-q+2)\pi, \dots, \arg z+(q-m-n)\pi. \quad (177)$$

But by (165) we have $q-m-n+1\leqq q-p$. Hence the number of these functions $G_{p,q}^{q,0}(\zeta)$ is at most equal to $q-p$. It is easily seen that these functions are fundamental solutions. For it follows from (166) that the values (177) lie between $-(q-p+\varepsilon)\pi$ and $(q-p+\varepsilon)\pi$. The condition (37) is therefore satisfied and so the functions $G_{p,q}^{q,0}(\zeta)$ on the right-hand side of (150) are fundamental solutions. The functions $G_{p,q}^{q,1}$ are also fundamental solutions. This may be established in the same manner as in the proof of theorem 12 A.

Proof of theorem 15. From (170) and (169) it follows

$$(\tfrac{1}{2}q-\tfrac{1}{2}p-\varepsilon-2)\pi<2\mu\pi<(\tfrac{3}{2}q-\tfrac{3}{2}p-\varepsilon+2)\pi,$$

hence

$$-1<\mu<q-p+1;$$

the condition (151) of theorem 10 is therefore satisfied.

The functions $G_{p,q}^{q,1}$ on the right-hand side of (152) are fundamental solutions, since we have by (170)

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi<\arg z+(2p-q-m-n-2\lambda+2\mu+1)\pi<(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi.$$

We will still show that the $q-p$ functions $G_{p,q}^{q,0}(\zeta)$ on the right-hand side of (152) are fundamental solutions. Now the values of $\arg\zeta$ are

$$\arg z + (2p-q-m-n-2\lambda+2)\pi,\ \arg z + (2p-q-m-n-2\lambda+4)\pi,\ \ldots,\ \arg z + (q-m-n-2\lambda)\pi,$$

and these values lie because of (169) between $-(q-p+\varepsilon)\pi$ and $(q-p+\varepsilon)\pi$. The functions $G_{p,q}^{q,0}(\zeta)$ satisfy therefore the condition (37) and so they are fundamental solutions.

With this the theorem has been proved.

§ 18. The asymptotic expansion of the function $G_{p,q}^{m,n}(z)$ $(q>p)$.

We are now able, for all values of m, n, p, q and $\arg z$, to investigate the behaviour of $G_{p,q}^{m,n}(z)$ $(q>p)$ as $|z|\to\infty$. For the theorems of § 17 in connection with the expansion formulae (145), (148), (149), (150) and (152) enable us to express the function $G_{p,q}^{m,n}(z)$ linearly in terms of functions $G_{p,q}^{q,1}$ and $G_{p,q}^{q,0}$ of which the asymptotic expansions can immediately be deduced from the theorems A and C of § 2.

In order to get an asymptotic expansion of $G_{p,q}^{m,n}(z)$ for $|z|\to\infty$ I investigate all the functions $G_{p,q}^{q,0}$ and $G_{p,q}^{q,1}$ on the right-hand side of one of the mentioned expansion formulae and I determine the dominant[52]) or the dominants among them[53]). Unless the coefficients of all the dominant functions vanish, we need only take account of the asymptotic expansions of these dominants and may neglect the others. Now these coefficients are functions of the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ and these functions are in general not zero. Such a function is only zero if the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ satisfy a certain equation. Since these parameters are mutually independent, there exists in general no relation between them.

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[52]) I say that $\phi(z)$ is dominant compared with $\psi(z)$ if the leading term of the asymptotic expansion of $\psi(z)$ is of an order less than the error term of the asymptotic expansion of $\phi(z)$.

For instance: If $\phi_1(z),\ \ldots,\ \phi_6(z)$ possess the asymptotic expansions

$$\Phi_1(z) \backsim e^z\left(a_{1,0}+\frac{a_{1,1}}{z}+\ldots\right),\quad \Phi_2(z) \backsim z^5\left(a_{2,0}+\frac{a_{2,1}}{z}+\ldots\right),$$

$$\Phi_3(z) \backsim z^{-\frac{1}{2}}\left(a_{3,0}+\frac{a_{3,1}}{z}+\ldots\right),\quad \Phi_4(z) \backsim e^{iz}z^{-2}\left(a_{4,0}+\frac{a_{4,1}}{z}+\ldots\right),$$

$$\Phi_5(z) \backsim e^{-z}\left(a_{5,0}+\frac{a_{5,1}}{z}+\ldots\right),\quad \Phi_6(z) \backsim e^{-2z}\left(a_{6,0}+\frac{a_{6,1}}{z}+\ldots\right)$$

and z is positive, then $\phi_1(z)$ is dominant compared with $\phi_2(z),\ldots,\phi_6(z)$; $\phi_2(z)$, $\phi_3(z)$ and $\phi_4(z)$ are dominant compared with $\phi_5(z)$ and $\phi_6(z)$; $\phi_5(z)$ is dominant compared with $\phi_6(z)$; but $\phi_2(z)$ is not dominant compared with $\phi_3(z)$ and $\phi_4(z)$. Among the functions $\phi_2(z),\ldots,\phi_6(z)$ there are three dominants, viz. $\phi_2(z)$, $\phi_3(z)$ and $\phi_4(z)$.

[53]) In many cases there is only one dominant function, viz. a function $G_{p,q}^{q,0}$.

If there is only one dominant function, I will suppose in this § that the coefficient of this function is not zero; if there are two or more than two dominant functions, I assume that at least one of them possesses a coefficient which is not zero. So in formula (195) it is tacitly supposed that the coefficient $D_{p,q}^{m,n}(\lambda)$ does not vanish; in formula (196) that at most one of the coefficients $D_{p,q}^{m,n}(\lambda)$ and $D_{p,q}^{m,n}(\lambda-1)$ vanishes.

If the coefficients of all the dominant functions are zero, it is necessary to make a closer investigation, with which I will not occupy myself.

Except in some simple cases the asymptotic behaviour of the function $G_{p,p+1}^{m,n}(z)$ is quite different from that of the function $G_{p,q}^{m,n}(z)$ with $q \geqq p+2$ (comp. the theorems 20 and 21).

Substantially the results run as follows:

1. If $n \geqq 1$ and $m+n > \frac{1}{2}p + \frac{1}{2}q$, then $G_{p,q}^{m,n}(z)$ has for large values of $|z|$ with $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion of algebraic order (This is the case of theorem B of § 2).

2. If $m > \frac{1}{2}p + \frac{1}{2}q$, then $G_{p,q}^{m,0}(z)$ has for large values of $|z|$ with $|\arg z| < (m-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion which is exponentially zero.

3. If $q \geqq p+2$ and $m+n > \frac{1}{2}p+\frac{1}{2}q$, then $G_{p,q}^{m,n}(z)$ has for large values of $|z|$ with $|\arg z| > (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion which is exponentially infinite.

4. If $q \geqq p+2$ and $m+n \leqq \frac{1}{2}p+\frac{1}{2}q$, then $G_{p,q}^{m,n}(z)$ has for large values of $|z|$ an asymptotic expansion which is exponentially infinite [54]).

5. If $m+n \geqq p+1$ and λ is either an arbitrary integer $\geqq 0$ or an arbitrary integer $\leqq p-m-n$, then $G_{p,p+1}^{m,n}(z)$ has for large values of $|z|$ with

$$(m+n-p+2\lambda-\tfrac{1}{2})\pi < \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi \quad . \quad (178)$$

an asymptotic expansion which is exponentially infinite.

6. If λ is an arbitrary integer, then $G_{p,p+1}^{m,n}(z)$ has for large values of $|z|$ with

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi < \arg z < (m+n-p+2\lambda-\tfrac{1}{2})\pi$$

an asymptotic expansion of algebraic order.

7. If $m+n \leqq p$ and λ is an arbitrary integer, then $G_{p,p+1}^{m,n}(z)$ has for large values of $|z|$ in the sector (178) an asymptotic expansion which is exponentially infinite.

I will now state my results. The simplest case is afforded by

Theorem 16. Assumptions: *m, n, p and q are integers with*

$$1 \leqq n \leqq p < q, \ 1 \leqq m \leqq q \text{ and } m+n > \tfrac{1}{2}p + \tfrac{1}{2}q;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *satisfy the conditions* (1) *and* (20).

[54]) There are certain special values of arg z for which 3. and 4. are not true; comp. assertion 4 of theorem 18 and assertion 3 of theorem 20.

Assertion: *The function* $G_{p,q}^{m,n}(z)$ *possesses for large values of* $|z|$ *with*

$$-(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi < \arg z < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\ . \quad (179)$$

the asymptotic expansion [55])

$$G_{p,q}^{m,n}(z) \backsim \sum_{t=1}^{n} e^{(m+n-q-1)\pi i a_t} \triangle^{m,n}_{q}(t) E_{p,q}(z e^{(q-m-n+1)\pi i} \| a_t). \quad (180)$$

Remark. This theorem is equivalent to theorem B of § 2.

Proof: We may distinguish two cases:

First case: $m+n \geqq q+1$ [56]). We apply theorem 11. The asymptotic expansions of the functions $G_{p,q}^{q,1}$ on the right of (145) can be deduced from theorem A and (15) (with $\gamma=\lambda$); the result is

$$G_{p,q}^{q,1}(z e^{(q-m-n+2\lambda+1)\pi i} \| a_t) \backsim e^{2\lambda\pi i a_t} E_{p,q}(z e^{(q-m-n+1)\pi i} \| a_t).$$

From this relation and (145) follows (180).

Second case: $\frac{1}{2}p+\frac{1}{2}q < m+n \leqq q+1$. We use theorem 13 A. The asymptotic behaviour of the functions $G_{p,q}^{q,0}(z e^{(q-m-n-2s)\pi i})$ on the right of (148) can be determined by means of theorem C. Now it follows from (179)

$$\arg(z e^{(q-m-n)\pi i}) < (\tfrac{1}{2}q-\tfrac{1}{2}p)\pi$$

and from (164)

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg(z e^{(q-m-n-2r+2)\pi i}).$$

Hence we have for $s=0, 1, \ldots, r-1$

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg(z e^{(q-m-n-2s)\pi i}) < (\tfrac{1}{2}q-\tfrac{1}{2}p)\pi.$$

From this relation and (26) and (25) it appears that the functions $G_{p,q}^{q,0}(z e^{(q-m-n-2s)\pi i})$ on the right of (148) tend exponentially to zero as $|z| \to \infty$.

The functions $G_{p,q}^{q,1}$ in (148) yield by means of theorem A and (15) the same asymptotic expansions of algebraic order as in the first case. With this the theorem is established.

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[55]) Comp. footnote [12]).

[56]) If $m+n \geqq q+1$, then $m \geqq 2$, since $n < q$.