**Mathematics.** — *On a generalisation of the formula of* HILLE *and* HARDY *in the theory of Laguerre polynomials.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

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1. The Laguerre polynomials $L_n^{(\alpha)}(x)$ can be defined by means of a generating function

$$\frac{1}{(1-t)^{\alpha+1}}\, e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x) \qquad |t|<1 \quad . \quad . \quad . \quad . \quad (1)$$

or by

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{\alpha+n}) = \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-x)^r}{r!} \quad . \quad . \quad (2)$$

For these polynomials the following theorem holds

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} t^n \frac{n!\, e^{-\frac{1}{2}(x+y)} (xy)^{\frac{1}{2}\alpha}}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = \\ &= \frac{t^{-\frac{1}{2}\alpha}}{1-t} \exp\left(-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right) \cdot I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right) \qquad (|t|<1) \end{aligned}\right\} \quad (3)$$

where $I_\alpha$ is the "Bessel function of imaginary argument". This formula is often called the Hille–Hardy formula, but besides those of HILLE [1]) and HARDY [2]) the names of WIGERT, BATEMAN and MYLLER LEBEDEW have been associated with the discovery of the theorem [3]). HARDY obtained his result by an application of Mellin's inversion formula; the proof of Hille involved the use of infinite integrals containing Bessel functions. A simple proof for (3) has been given by WATSON [4]) by means of generalized hypergeometric functions.

If we write

$$\varphi_n^{(\alpha)}(x) = \left[\frac{e^{-x} x^\alpha n!}{\Gamma(n+\alpha+1)}\right]^{\frac{1}{2}} L_n^{(\alpha)}(x). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the left member of (3) is seen to be

$$\Sigma t^n \varphi_n^{(\alpha)}(x)\, \varphi_n^{(\alpha)}(y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now in recent years series of the form

$$\Sigma c_n t^n \varphi_n^{(\alpha)}(x) \cdot \varphi_n^{(\alpha)}(y) \qquad (|t| \leqq 1) \quad . \quad . \quad . \quad (6)$$

have been investigated by WATSON and by ERDELYI. WATSON [5]) has

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1) HILLE, Proc. Nat. Acad. of Sci., **12**, 261—265, 265—269, 348—352 (1926).

2) HARDY, Journal London Math. Soc. **7**, 138—139 (1932).

3) For the history of the formula see WATSON, Journal London Math. Soc., **8**, 190 (1933), ERDELYI, Compositio Math. **6**, 336—347 (1939), BATEMAN, Zb. Mathem. **21**, 24 (1940).

4) WATSON, l.c. 189—192.

WATSON, Sitzungsber. Ak. Wiss. Wien, **147**, 151—159 (1938).

shown that for $t = 1$, $c_n = \frac{1}{n+1}$ (6) can be expressed by incomplete $\Gamma$-functions, thus generalizing a formula which had been given by R. NEUMANN and by KOSCHMIEDER for $\alpha = 0$. ERDELYI has published several papers on the subject [6]), showing finally that the results obtained by WATSON and by himself can be considered as special cases of a general theorem concerning bilinear series of confluent hypergeometric functions.

The aim of the present note is far more unpretending and it tries to show only that for $c_n = \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+1)}$ where $k$ is an integer and for $|t| < 1$, the series (6) can be written as a sum of $k$ Bessel functions which coefficients are expressions containing Laguerre polynomials of the argument $\frac{(x+y)\,t}{1-t}$.

2. We prove the following generalisation of (3)

$$\left.\begin{aligned}&\sum_{n=0}^{\infty} t^n \frac{n!\, e^{-\frac{1}{2}(x+y)}\, \Gamma(n+\alpha+k+1)\,(xy)^{\frac{1}{2}\alpha}}{\Gamma^2(n+\alpha+1)} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y) = \\ &= \frac{t^{-\frac{1}{2}\alpha}}{(1-t)^{k+1}} \cdot \exp\left(-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right) \cdot k! \sum_{p=0}^{k} \left[\left[\sum_{m=0}^{k-p} \binom{p}{m} \frac{t^m}{(1+t)^m} L_{k-p-m}^{(\alpha+p+m)}\left(\frac{(x+y)\,t}{1-t}\right)\right] \times \right. \\ &\qquad \left. \times \frac{1}{p!}(xyt)^{\frac{p}{2}} \left(\frac{1+t}{1-t}\right)^p I_{\alpha+p}\left(\frac{2\sqrt{xyt}}{1-t}\right)\right] \qquad (|t| < 1,\ k \text{ an integer})\end{aligned}\right\} \quad (7)$$

Since $\binom{p}{m} = 0$ for $m > p$ the second $\Sigma$ on the right has $q + 1$ terms, where $q = \min\,[k-p, p]$. For $k = 0$ we have the Hille–Hardy theorem. The proof is extremely elementary for (7) can be derived from (3) by multiplying with $t^{\alpha+k}$, differentiating $k$ times with respect to $t$ and dividing by $t^{\alpha}$. The only difficulty arises from the arrangement of the righthand member. Once the formula discovered, the proof can best be given by induction. We assume that (7) is valid for $k$, multiply by $t^{\alpha+k+1}$, differentiate with respect to $t$ and divide by $t^{\alpha+k}$. We obtain then the left member of (7) for $k + 1$. For the reduction of the right member we make use of the following relations

$$\frac{d\, I_\beta(z)}{dz} = \frac{\beta}{z} I_\beta(z) + I_{\beta+1}(z) \quad \dots\dots \quad (8)$$

$$z\frac{d\, L_n^{(\alpha)}(z)}{dz} = (n+1)\, L_{n+1}^{(\alpha)}(z) - (\alpha + n + 1 - z)\, L_n^{(\alpha)}(z) \quad .\ . \quad (9)$$

$$L_n^{(\alpha)}(z) + L_{n-1}^{(\alpha+1)}(z) = L_n^{(\alpha+1)}(z) \quad \dots\dots \quad (10)$$

6) ERDELYI, Rend. Acc. Lincei, **24**, 347—350 (1936); S.—B. Akad. Wiss. Wien IIa, **147**, 513—520 (1938); id. **148**, 38—39 (1939); Compositio Math. **6**, 336—347 (1939); id. **7**, 340—352 (1939).

For $I_\beta\left(\frac{2\sqrt{xyt}}{1-t}\right)$ and $L_n^{(\alpha)}\left(\frac{(x+y)\,t}{1-t}\right)$ we obtain accordingly

$$\frac{d}{dt}I_\beta=\frac{\beta(1+t)}{2\,t\,(1-t)}\,I_\beta+\sqrt{xyt}\,\frac{1+t}{t\,(1-t)^2}\,I_{\beta+1}\,. \quad . \quad . \quad . \qquad (8a)$$

$$\frac{d}{dt}L_n^{(\alpha)}=\frac{n+1}{t\,(1-t)}\,L_{n+1}^{(\alpha)}-\frac{\alpha+n+1-u}{t\,(1-t)}\,L_n^{(\alpha)}\,. \quad . \quad . \quad . \qquad (9a)$$

where

$$u=\frac{(x+y)\,t}{1-t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

From (8a) it is obvious that in the right member we obtain an expression

$$\frac{t^{-1/_2\alpha}}{(1-t)^{k+2}}\exp\left(-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right)\cdot k!\sum_{p=0}^{k+1}A_p(x,y,t)\,I_{\alpha+p} \quad . \qquad (12)$$

and the functions $A_p(x,y,t)$ have to be found. If we differentiate

$$\frac{t^{\frac{1}{2}\alpha+k+1}}{(1-t)^{k+1}}\cdot\exp\left(-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right)$$

with respect to $t$ and divide by $t^{\alpha+k}$, we obtain

$$\frac{t^{-1/_2\alpha}}{(-t)^{k+2}}\cdot\exp\left(-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right)\cdot\left(\tfrac{1}{2}\,\alpha+k+1-\tfrac{1}{2}\,\alpha\,t-u\right).$$

Again, differentiating $t^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p$ we have

$$p\cdot t^{\frac{p}{2}-1}\left(\frac{1+t}{1-t}\right)^{p-1}\cdot\frac{1+4t-t^2}{2\,(1-t)^2}.$$

Thus if

$$\sum_{m=0}^{k-p}\binom{p}{m}\frac{t^m}{(1+t)^m}\,L_{k-p-m}^{(\alpha+p+m)}\left(\frac{(x+y)\,t}{1-t}\right)=S_p(t) \quad . \quad . \quad . \qquad (13)$$

it follows that for $0<p<k+1$

$$\left.\begin{aligned}
A_p&=\left(\tfrac{1}{2}\,\alpha+k+1-\tfrac{1}{2}\,\alpha\,t-u\right)\cdot\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p S_p\\
&+\frac{p\,t\,(1-t)}{p!}(xy)^{\frac{p}{2}}\cdot t^{\frac{p}{2}-1}\left(\frac{1+t}{1-t}\right)^{p-1}\cdot\frac{(1+4t-t^2)}{2\,(1-t)^2}\,S_p\\
&+t\,(1-t)\cdot\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p\frac{d}{dt}\,S_p(t)\\
&+t\,(1-t)\,\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)S_p\cdot\frac{(\alpha+p)\,(1+t)}{2\,t\,(1-t)}\\
&+t\,(1-t)\cdot\frac{1}{(p-1)!}(xyt)^{\frac{p-1}{2}}\left(\frac{1+t}{1-t}\right)^{p-1}\cdot S_{p-1}\cdot(xyt)^{1/_2}\,\frac{1+t}{t\,(1-t)^2}\\
&=\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p B_p\,(xyt)
\end{aligned}\right\}\qquad (14)$$

where

$$\left.\begin{aligned} B_p = & \left[\tfrac{1}{2}a + k + 1 - \tfrac{1}{2}a t - u + \frac{p}{2}\cdot\frac{1+4t-t^2}{1+t} + \frac{a+p}{2}(1+t)\right] S_p \\ & + t(1-t)\frac{d}{dt}S_p + p S_{p-1} \\ & = \left(a + k + 1 + p - u + \frac{2pt}{1+t}\right) S_p + t(1-t)\frac{d}{dt}S_p + pS_{p-1} \end{aligned}\right\} \quad (15)$$

Now

$$\left.\begin{aligned} \frac{d}{dt}S_p = & \sum_{m=0}^{k-p}\binom{p}{m}\left[\frac{m\,t^{m-1}}{(1+t)^{m+1}} L_{k-p-m}^{(\alpha+p+m)} + \right. \\ & \left. + \frac{t^{m-1}}{(1-t)(1+t)^m}\{k-p-m+1)\, L_{k-p-m+1}^{(\alpha+p+m)} - (a+k+1-u)\, L_{k-p-m}^{(\alpha+p+m)}\}\right] \end{aligned}\right\}. \quad (16)$$

Thus we obtain

$$\left.\begin{aligned} B_p = & \sum_{m=0}^{k-p}\binom{p}{m}\frac{t^m}{(1+t)^m}\left[\left(a+k+1+p-u+\frac{2pt}{1+t}\right) L_{k-p-m}^{(\alpha+p+m)} + \frac{m(1-t)}{1+t} L_{k-p-m}^{(\alpha+p+m)} + \right. \\ & \left. + (k-p-m+1)\, L_{k-p-m+1}^{(\alpha+p+m)} - (a+1-u)\, L_{k-p-m}^{(\alpha+p+m)} + (p-m)\, L_{k-p+1-m}^{(\alpha+p-1+m)}\right] \\ = & \sum_{m=0}^{k-p+1}\binom{p}{m}\frac{t^m}{(1+t)^m}\left[(a+k+1+p-u)\, L_{k-p-m}^{(\alpha+p+m)} + \frac{2mp}{p-m+1} L_{k-p-m+1}^{(\alpha+p+m-1)} + \right. \\ & + m\, L_{k-p-m}^{(\alpha+p+m)} - \frac{2m(m-1)}{p-m+1} L_{k-p-m+1}^{(\alpha+p+m-1)} + (k-p-m+1)\, L_{k-p-m+1}^{(\alpha+p+m)} - \\ & \left. - (a+k+1-u)\, L_{k-p-m}^{(\alpha+p+m)} + (p-m)\, L_{k-p+1-m}^{(\alpha+p-1+m)}\right] \\ = & \sum_{m=0}^{k-p+1}\binom{p}{m}\frac{t^m}{(1+t)^m}\left[(p+m)\, L_{k-p-m}^{(\alpha+p+m)} + (p+m)\, L_{k-p+1-m}^{(\alpha+p-1+m)} + (k-p-m+1)\, L_{k-p-m+1}^{(\alpha+p+m)}\right] \\ = & \sum_{m=0}^{k+1-p}\binom{p}{m}\frac{t^m}{(1+t)^m}\left[(p+m)\, L_{k+1-p-m}^{(\alpha+p+m)} + (k-p-m+1)\, L_{k+1-p-m}^{(\alpha+p+m)}\right] \\ = & (k+1)\sum_{m=0}^{k+1-p}\binom{p}{m}\frac{t^m}{(1+t)^m} L_{k+1-p-m}^{(\alpha+p+m)} \end{aligned}\right\} \quad (17)$$

It follows that for $0 < p < k+1$

$$A_p = (k+1)\,(x y t)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p \sum_{m=0}^{k+1-p}\binom{p}{m}\frac{t^m}{(1+t)^m} L_{k+1-p-m}^{(\alpha+p+m)}\ . \quad (18)$$

and the same result is valid for $p = 0$ and for $p = k + 1$. Thus from (12) we take the conclusion that (3) is due for $k + 1$.

3. From the special cases derivable from (7) we quote only the case $y \to 0$. We obtain then

$$\sum_{n=0}^{\infty} \binom{n+a+k}{k} t^n L_n^{(\alpha)}(x) = \frac{1}{(1-t)^{k+1+\alpha}} \exp\left(-\frac{xt}{1-t}\right) \cdot L_k^{(\alpha)}\left(\frac{xt}{1-t}\right) \quad (19)$$

a formula we have given elsewhere [7]) and which can be used for the evaluation of some definite integrals involving products of Laguerre polynomials.

[7]) BOTTEMA, Een betrekking voor de polynomen van LAGUERRE en VAN HERMITE, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 65—71 (1946).