Mathematics. — *On the G-function.* VIII. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 30, 1946.)

Theorem 21. Assumptions: *m, n and p are integers with* [65]

$$p \geqq 1, \; 0 \leqq n \leqq p \text{ and } 1 \leqq m \leqq p+1;$$

λ is an arbitrary integer;
the number z satisfies the inequality

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi \leqq \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi;$$

the numbers $a_1, \ldots, a_p$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (38) *in the formulae* (203), (207) *and* (208) (*assertions* 1, 2, 3, 6, 7, 8 *and* 9); *in formula* (206) (*assertions* 4 *and* 5) *I assume that they satisfy the condition* (1).

Assertions: 1. *The function* $G^{m,n}_{p,p+1}(z)$ *possesses for large values of* $|z|$ *with*

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi < \arg z < (m+n-p+2\lambda-\tfrac{1}{2})\pi \qquad (202)$$

the asymptotic expansion

$$G^{m,n}_{p,p+1}(z) \backsim \sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i a_\sigma}\, T^{m,n}_{p,p+1}(\sigma;\lambda)\, E_{p,p+1}(z e^{(p-m-n+2)\pi i} \,||\, a_\sigma). \qquad (203)$$

Formula (203) *does not hold when* $n=\lambda=0$ *and* $m=p+1$ [66].

2. *The expansion* (203) *is also true if the following conditions are satisfied*:

$$m+n \geqq p+2, \; p-m-n < \lambda < 0,$$

$$(m+n-p+2\lambda-\tfrac{1}{2})\pi \leqq \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi. \qquad (204)$$

3. *The expansion* (203) *is further valid if the following conditions a e satisfied*:

$$m+n \geqq p+2, \; p-m-n+1 < \lambda < 1, \; \arg z = (m+n-p+2\lambda-\tfrac{3}{2})\pi.$$

4. *If* $m+n \geqq p+2$ *and* λ *is either an arbitrary integer* $\geqq 0$ *or an arbitrary integer* $\leqq p-m-n$, *then for large values of* $|z|$ *with*

$$(m+n-p+2\lambda-\tfrac{1}{2})\pi < \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi \qquad (205)$$

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65) We need not give attention to the cases with $p=0$ and $m=0$, since $G^{1,0}_{0,1}(z\,|\,\lambda) = z^\lambda e^{-z}$ and $G^{0,n}_{p,p+1}(z) = 0$.

66) The asymptotic expansion of $G^{p+1,0}_{p,p+1}(z)$ for $|\arg z| < \frac{1}{2}\pi$ is $G^{p+1,0}_{p,p+1}(z) \backsim H_{p,p+1}(z)$ (see theorem C).

the following asymptotic expansion holds

$$G^{m,n}_{p,p+1}(z) \backsim D^{m,n}_{p,p+1}(\lambda)\, H_{p,p+1}(z\, e^{(p-m-n-2\lambda+1)\pi i}). \qquad (206)$$

5. *The asymptotic expansion* (206) *is also valid if the following conditions are satisfied*:

$m+n \leqq p+1$, λ *is an arbitrary integer*, $\arg z$ *satisfies* (205).

6. *If* $m+n \geqq p+2$ *and* λ *is either an arbitrary integer* $\geqq 0$ *or an arbitrary integer* $\leqq p-m-n$, *then for large values of* $|z|$ *with* $\arg z = (m+n-p+2\lambda-\frac{1}{2})\pi$ *the following asymptotic expansion holds*

$$\left.\begin{aligned} &G^{m,n}_{p,p+1}(z) \backsim D^{m,n}_{p,p+1}(\lambda)\, H_{p,p+1}(z\, e^{(p-m-n-2\lambda+1)\pi i}) \\ &+ \sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i\, a_\sigma}\, T^{m,n}_{p,p+1}(\sigma;\lambda)\, E_{p,p+1}(z\, e^{(p-m-n+2)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \qquad (207)$$

7. *The asymptotic expansion* (207) *is also valid if the following conditions are satisfied*:

$m+n \leqq p+1$, λ *is an arbitrary integer*, $\arg z = (m+n-p+2\lambda-\frac{1}{2})\pi$

8. *If* $m+n \geqq p+2$ *and* λ *is either an arbitrary integer* $\geqq 1$ *or an arbitrary integer* $\leqq p-m-n+1$, *then for large values of* $|z|$ *with* $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$ *the following asymptotic expansion holds*

$$\left.\begin{aligned} &G^{m,n}_{p,p+1}(z) \backsim D^{m,n}_{p,p+1}(\lambda-1)\, H_{p,p+1}(z\, e^{(p-m-n-2\lambda+3)\pi i}) \\ &+ \sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i\, a_\sigma}\, T^{m,n}_{p,p+1}(\sigma;\lambda)\, E_{p,p+1}(z\, e^{(p-m-n+2)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \qquad (208)$$

9. *The asymptotic expansion* (208) *is also valid if the following conditions are satisfied*:

$m+n \leqq p+1$, λ *is an arbitrary integer*, $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$.

Remark. Precisely as by theorem 20 there are by theorem 21 certain cases wherein an expansion coincides with an expansion given in one of the previous theorems.

For instance: Formula (180) with $q=p+1$ is a particular case of (203). It follows namely from (86), (87) and (54):

If $1 \leqq \sigma \leqq n$ and $p-m-n+1 \leqq \lambda \leqq 0$, then

$$T^{m,n}_{p,p+1}(\sigma;\lambda) = e^{(m+n-p+2\lambda-1)\pi i\, a_\sigma}\, \triangle^{m,n}{}_{p+1}(\sigma).$$

If $n+1 \leqq \sigma \leqq p$ and $p-m-n+1 \leqq \lambda \leqq 0$, then $T^{m,n}_{p,p+1}(\sigma;\lambda) = 0$.

Hence, formula (203) with $n \geqq 1$, $m+n \geqq p+1$ and $p-m-n+1 \leqq \lambda \leqq 0$ is equivalent to (180) with $q=p+1$.

Proof of theorem 21. This proof rests, like that of theorem 20, on an application of the theorems 15 and 10. The number ε, occurring in condition (169) of theorem 15 is now equal to $\frac{1}{2}$, since $q=p+1$.

The inequality (170) reduces for $q=p+1$ to

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi - \arg z < 2\mu\pi < (m+n-p+2\lambda+\tfrac{3}{2})\pi - \arg z. \qquad (209)$$

Hence it is easily seen:

If $\arg z$ satisfies (204), then $\mu = 0$.

If $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$, then $\mu = 1$.

If $\arg z$ fulfils the condition (202), then the inequality (209) is satisfied by both $\mu = 0$ and $\mu = 1$.

Now on the right of (152) there occurs for $q = p+1$ only one function $G_{p,p+1}^{p+1,0}$, viz. the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$, the coefficient of this function being

$$R_{p,p+1}^{m,n}(1;\lambda) \text{ or } \overline{R}_{p,p+1}^{m,n}(1;p-m-n-\lambda+1)$$

according as $\mu = 0$ or $\mu = 1$; these coefficients may because of (91) and (92) also be written in the form $D_{p,p+1}^{m,n}(\lambda)$, respect.

$$\exp\left\{2\pi i\left(\sum_{h=1}^{p+1} b_h - \sum_{h=1}^{p} a_h\right)\right\} D_{p,p+1}^{m,n}(\lambda-1).$$

On account of the just determined values of μ we see that the coefficient in question is equal to

$$D_{p,p+1}^{m,n}(\lambda) \text{ if } \arg z \text{ satisfies (204)} \quad \ldots\ldots \quad (210)$$

and equal to

$$\exp\left\{2\pi i\left(\sum_{h=1}^{p+1} b_h - \sum_{h=1}^{p} a_h\right)\right\} D_{p,p+1}^{m,n}(\lambda-1) \text{ if } \arg z = (m+n-p+2\lambda-\tfrac{3}{2})\pi; \quad (211)$$

the coefficient depends on the choice of μ if $\arg z$ fulfils the condition (202).

Now if $\arg z$ satisfies (205), then

$$\tfrac{1}{2}\pi < \arg(z\,e^{(p-m-n-2\lambda+1)\pi i}) < \tfrac{3}{2}\pi;$$

hence it follows from (26) and (25) that the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ is exponentially infinite for $|z| \to \infty$.

If $\arg z$ satisfies (202), then

$$-\tfrac{1}{2}\pi < \arg(z\,e^{(p-m-n-2\lambda+1)\pi i}) < \tfrac{1}{2}\pi;$$

in this case the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ tends exponentially to zero for $|z| \to \infty$. If

$$\arg z = (m+n-p+2\lambda-\tfrac{1}{2})\pi \text{ or } \arg z = (m+n-p+2\lambda-\tfrac{3}{2})\pi,$$

the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ behaves like $e^{\mp i|z|}|z|^{\vartheta}$ for $|z| \to \infty$.

Hence, if $\arg z$ satisfies (205), we may, in writing down the asymptotic expansion of the right-hand side of (152) with $q = p+1$, neglect the asymptotic expansions of algebraic order which are caused by the functions $G_{p,p+1}^{p+1,1}(z\,e^{(p-m-n-2\lambda+2\mu)\pi i}\,||\,a_\sigma)$ $(\sigma = 1,\ldots,p)$; we need only consider the exponential expansion of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$, provided that the coefficient of this function does not vanish.

On the other hand, if $\arg z$ satisfies (202), the expansion of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$, that is exponentially zero for $|z| \to \infty$, may be neglected; unless the coefficients of all the functions

$G^{p+1,1}_{p,p+1}(z\,e^{(p-m-n-2\lambda+2\mu)\pi i}\,||\,a_\sigma)$ vanish, we need only take account of the algebraic expansions of these functions.

If $\arg z=(m+n-p+2\lambda-\frac{1}{2})\pi$ or $\arg z=(m+n-p+2\lambda-\frac{3}{2})\pi$, we must take account of both the exponential and the algebraic expansions.

Now the coefficients $D^{m,n}_{p,p+1}(\lambda)$ and $D^{m,n}_{p,p+1}(\lambda-1)$ vanish identically if[67] $p-m-n<\lambda<0$, respect. $p-m-n+1<\lambda<1$. In these cases the coefficient of the function $G^{p+1,0}_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ is zero, so that this function does not at all occur on the right-hand side of (152) with $q=p+1$; asymptotic expansions which are exponentially infinite are then certainly impossible.

Hence we see in virtue of (210) and (211) that the asymptotic expansion of the right-hand side of (152) with $q=p+1$ contains no expansions which are exponentially infinite but only algebraic expansions in the two following cases:

1. $m+n\geqq p+2$, $p-m-n<\lambda<0$, $\arg z$ satisfies (204).
2. $m+n\geqq p+2$, $p-m-n+1<\lambda<1$, $\arg z=(m+n-p+2\lambda-\frac{3}{2})\pi$.

Of course, the algebraic expansions are also predominant in the above mentioned case that $\arg z$ satisfies (202).

Since the coefficients $\Theta^{m,n}_p(l;\lambda)$ are zero for $\lambda=-1,-2,-3,\ldots$, it follows from (87) that the coefficients $T^{m,n}_{p,q}(l;\lambda)$ of the second type $(n+1\leqq l\leqq p)$ vanish identically if $p-m-n+1\leqq\lambda\leqq 0$. On the other hand we deduce from (86) that the coefficients $T^{m,n}_{p,q}(l;\lambda)$ of the first type $(1\leqq l\leqq n)$ are in general not zero if $p-m-n+1\leqq\lambda\leqq 0$, since $\Delta^{m,n}_{\ q}(l)$ is only zero if the parameters $b_{m+1},\ldots,b_q$ and a_l satisfy a certain equation. If $n=0$, there occur no coefficients $T^{m,n}_{p,q}(l;\lambda)$ of the first type. Hence, if $p-m+1\leqq\lambda\leqq 0$, the coefficients $T^{m,0}_{p,q}(l;\lambda)$ vanish for $1\leqq l\leqq p$. In the case before us the coefficients of the functions $G^{p+1,1}_{p,p+1}(z\,e^{(p-m-n-2\lambda+2\mu)\pi i}\,||\,a_\sigma)$ on the right-hand side of (152) with $q=p+1$ are equal to $e^{(1-2\mu)\pi i a_\sigma}\,T^{m,n}_{p,p+1}(\sigma;\lambda)$; we further have $m\leqq p+1$. So we see that the coefficients of all the functions $G^{p+1,1}_{p,p+1}$ on the right-hand side of (152) with $q=p+1$ are zero if we take $n=0$, $m=p+1$ and $\lambda=0$. The function $G^{p+1,0}_{p,p+1}(z)$ possesses therefore for $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$ no expansion of algebraic order, but instead of that an expansion which is exponentially zero for $|z|\to\infty$. The expansion in question is $G^{p+1,0}_{p,p+1}(z)\backsim H_{p,p+1}(z)$ (see theorem C).

Now the expansions of algebraic order, which are caused by the sum $\sum\limits_{\sigma=1}^{p}$ in (152), have according to (18) (with $q=p+1$) and (15) (with $\gamma=\mu-\lambda-1$) the form

$$\sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i a_\sigma}\,T^{m,n}_{p,p+1}(\sigma;\lambda)\,E_{p,p+1}(z\,e^{(p-m-n+2)\pi i}\,||\,a_\sigma).$$

[67]) Comp. the second Remark in § 8.

The assertions 1, 2 and 3 have therefore been proved.

The asymptotic expansion of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ is in virtue of (26)

$$G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i}) \backsim H_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i}).$$

Hence the exponential expansion of the right-hand side of (152) with $q=p+1$ is

$$D_{p,p+1}^{m,n}(\lambda)\, H_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i})$$

or

$$\exp\left\{2\pi i\left(\sum_{h=1}^{p+1} b_h - \sum_{h=1}^{p} a_h\right)\right\} D_{p,p+1}^{m,n}(\lambda-1)\, H_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i}),$$

according as we find ourselves in the case (210) or (211). The second expansion may because of (27) with $q=p+1$ be written in the simpler form

$$D_{p,p+1}^{m,n}(\lambda-1)\, H_{p,p+1}(z\,e^{(p-m-n-2\lambda+3)\pi i}).$$

We can now easily verify the assertions 4, 5, 6, 7, 8 and 9.

§ 19. The analytic continuation of $G_{p,p}^{m,n}(z)$ (general case).

The function $G_{p,p}^{m,n}(z)$ satisfies the differential equation (34) with $q=p$. As I have proved in § 4 the p functions (43) form, provided that the conditions (41), (42) and (38) are satisfied [68], a system of fundamental solutions in the vicinity of $z=\infty$.

In this § I will give the expression of $G_{p,p}^{m,n}(z)$ in terms of these fundamental solutions. From this expression we may derive by means of theorem F the analytic continuation of $G_{p,p}^{m,n}(z)$ outside the circle $|z|=1$. The result runs as follows:

Theorem 22. Assumptions: *m, n and p are integers with*

$$p \geqq 1,\ 0 \leqq n \leqq p \text{ and } 0 \leqq m \leqq q;$$

the numbers $a_1,\dots,a_p$ and $b_1,\dots,b_m$ satisfy the conditions (1) *and* (38); *λ is an arbitrary integer.*

Assertions: 1. *The function $G_{p,p}^{m,n}(z)$ can in the sector*

$$(m+n-p+2\lambda-2)\pi < \arg z < (m+n-p+2\lambda)\pi \quad . \quad (212)$$

by means of

$$G_{p,p}^{m,n}(z) = \sum_{\sigma=1}^{p} T_{p,p}^{m,n}(\sigma;\lambda)\, G_{p,p}^{p,1}(z\,e^{(p-m-n-2\lambda+1)\pi i} \,||\, a_\sigma) \quad . \quad . \quad (213)$$

be expressed in terms of fundamental solutions valid near $z=\infty$.

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68) Comp. also the Remark at the end of § 4.

2. *The function $G^{m,n}_{p,p}(z)$ possesses in the sector* (212) *an analytic continuation outside the circle $|z|=1$ which can be expressed in the form*

$$G^{m,n}_{p,p}(z)=\sum_{\sigma=1}^{p} e^{-2\lambda\pi i a_\sigma}\, T^{m,n}_{p,p}(\sigma;\lambda)\, E_{p,p}(z\,e^{(p-m-n+1)\pi i}\,||\,a_\sigma). \quad . \quad (214)$$

Proof. If we put $\mu=0$ and $q=p$ in (152), we find (213). The functions $G^{p,1}_{p,p}$ on the right of (213) satisfy because of (212) the conditions (41) and (42). Hence they are fundamental solutions.

The analytic continuation of $G^{p,1}_{p,p}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)$ outside the circle $|z|=1$ is in virtue of (33)

$$G^{p,1}_{p,p}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)=E_{p,p}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)$$

and this relation is on account of (15) equivalent to

$$G^{p,1}_{p,p}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)=e^{-2\lambda\pi i a_\sigma}\,E_{p,p}(z\,e^{(p-m-n+1)\pi i}\,||\,a_\sigma).$$

Formula (214) follows therefore from (213), so that the theorem has been proved.

I now consider the particular case with $m+n\geqq p+1$ of formula (214) and I suppose that λ satisfies the inequality $p-m-n+1\leqq\lambda\leqq 0$. Then it is clear on account of (86), (87) and (54)

$$T^{m,n}_{p,p}(\sigma;\lambda)=e^{(m+n-p+2\lambda-1)\pi i a_\sigma}\,\Delta^{m,n}_{\;\;p}(\sigma) \text{ for } 1\leqq\sigma\leqq n$$

and

$$T^{m,n}_{p,p}(\sigma;\lambda)=0 \text{ for } n+1\leqq\sigma\leqq p.$$

It follows therefore from (214): If $m+n\geqq p+1$, $|\arg z|<(m+n-p)\pi$ and

$$\arg z\neq(p-m-n+2)\pi,\ (p-m-n+4)\pi,\ldots,(m+n-p-2)\pi,$$

then the analytic continuation of $G^{m,n}_{p,p}(z)$ outside the circle $|z|=1$ has the form

$$G^{m,n}_{p,p}(z)=\sum_{\sigma=1}^{n} e^{(m+n-p-1)\pi i a_\sigma}\,\Delta^{m,n}_{\;\;p}(\sigma)\,E_{p,p}(z\,e^{(p-m-n+1)\pi i}\,||\,a_\sigma).$$

This formula is equivalent to formula (32) of theorem E*. But the present result is slightly less general than that of theorem E*, since we now have excluded the values

$$\arg z=(p-m-n+2)\pi,\ (p-m-n+4)\pi,\ldots,(m+n-p-2)\pi,$$

which are not omitted in theorem E*.

§ 20. The asymptotic expansion of the generalized hypergeometric function ${}_pF_q(z)$ $(q\geqq p)$.

Preliminary Remarks.

Remark 1. It is easily seen in view of (27) with $q+1$ instead of q that the expressions

$$(-2\pi i)^{p-q}\exp\left\{\pi i\left(\sum_{h=1}^{p}a_h-\sum_{h=1}^{q+1}b_h\right)\right\}H_{p,q+1}(z\,e^{(q-p+1)\pi i}) \quad . \quad (215)$$

and

$$(2\pi i)^{p-q} \exp\left\{\pi i\left(\sum_{h=1}^{q+1} b_h - \sum_{h=1}^{p} a_h\right)\right\} H_{p,q+1}(z e^{(p-q-1)\pi i}) \quad . \quad (216)$$

are equal one to another.

We now put

$$\left.\begin{array}{ll} a_j = 1-\alpha_j & (j=1,\ldots,p), \\ b_1 = 0 \text{ and } b_j = 1-\beta_{j-1} & (j=2,\ldots,q+1). \end{array}\right\} \quad . \quad . \quad (217)$$

It follows from (25) and (23) that the expressions (215) and (216) then reduce to

$$\exp\left\{(q-p+1) z^{\frac{1}{q-p+1}}\right\} z^{\gamma} \left\{\frac{(2\pi)^{\frac{p-q}{2}}}{\sqrt{q-p+1}} + \frac{N_1}{z^{\frac{1}{q-p+1}}} + \frac{N_2}{z^{\frac{2}{q-p+1}}} + \ldots\right\}, (218)$$

where

$$\gamma = \frac{1}{q-p+1}\left\{\tfrac{1}{2}(q-p) + \sum_{h=1}^{p} \alpha_h - \sum_{h=1}^{q} \beta_h\right\}; \quad . \quad . \quad . \quad (219)$$

the values of the coefficients N can be deduced from those of the corresponding coefficients M in (25).

The expression (218), where γ is defined by (219), will, for brevity, be denoted by $K_{p,q}(z)$.

It is now easily seen, on account of (45) and (46), that the expressions

$$A^{1,p}_{q+1} H_{p,q+1}(z e^{(q-p+1)\pi i})$$

and

$$\overline{A}^{1,p}_{q+1} H_{p,q+1}(z e^{(p-q-1)\pi i})$$

also reduce to $K_{p,q}(z)$ when we make the substitution (217).

Remark 2. We consider

$$\sum_{t=1}^{p} e^{(p-q-1)\pi i a_t} \Delta^{1,p}_{q+1}(t) E_{p,q+1}(z e^{(q-p+1)\pi i} \| a_t) \quad . \quad . \quad (220)$$

and

$$\sum_{t=1}^{p} e^{(q-p+1)\pi i a_t} \Delta^{1,p}_{q+1}(t) E_{p,q+1}(z e^{(p-q-1)\pi i} \| a_t). \quad . \quad . \quad (221)$$

These sums are equal one to another because of (15) with $q+1$ instead of q and $\gamma = q-p+1$.

After being transformed by means of the substitution (217) the sums (220) and (221) will be denoted by $L_{p,q}(z)$. It follows from (17) and (13) that (220) and (221), after the substitution (217) has been applied, may be written in the form

$$\sum_{t=1}^{p} \frac{z^{-\alpha_t} \prod\limits_{\substack{j=1 \\ j\neq t}}^{p} \Gamma(\alpha_j-\alpha_t)}{\prod\limits_{j=1}^{q} \Gamma(\beta_j-\alpha_t)} \sum_{h=0}^{\infty} \frac{\{(-1)^{q-p} z\}^{-h} \Gamma(\alpha_t+h) \prod\limits_{j=1}^{q} \{(1+\alpha_t-\beta_j)(2+\alpha_t-\beta_j)\ldots(h+\alpha_t-\beta_j)\}}{h! \prod\limits_{\substack{j=1 \\ j\neq t}}^{p} \{(1+\alpha_t-\alpha_j)(2+\alpha_t-\alpha_j)\ldots(h+\alpha_t-\alpha_j)\}}.$$

This expression will therefore be denoted by $L_{p,q}(z)$.

The behaviour of the generalized hypergeometric function ${}_pF_q(z)$ $(q \geqq p)$ for large values of $|z|$ has already been investigated in various ways and by several authors. I mention here the researches of STOKES, BARNES, WATSON, FOX, WRINCH and WRIGHT [69].

Now the function ${}_pF_q(z)$ is a special case of the function $G^{m,n}_{p,q+1}(z)$ and so it must be possible to deduce asymptotic expansions for ${}_pF_q(z)$ from those of $G^{m,n}_{p,q+1}(z)$. Indeed it follows from (7) [70]

$$\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;-z\end{matrix}\right)=G^{1,p}_{p,q+1}\left(z\,\middle|\,\begin{matrix}1-\alpha_1,\ldots,1-\alpha_p\\ 0,1-\beta_1,\ldots,1-\beta_q\end{matrix}\right)\tag{222}$$

and

$$\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;z\end{matrix}\right)=G^{1,p}_{p,q+1}\left(z\,e^{\pi i}\,\middle|\,\begin{matrix}1-\alpha_1,\ldots,1-\alpha_p\\ 0,1-\beta_1,\ldots,1-\beta_q\end{matrix}\right).\tag{223}$$

I will now write down the asymptotic expansions of the function ${}_pF_q(z)$. I suppose that the parameters $\alpha_1,\ldots,\alpha_p$ fulfil the condition

$$\alpha_j \neq 0,-1,-2,\ldots \qquad (j=1,\ldots,p);$$

in the formulae (224), (225), (226) and (227) I will assume that they satisfy besides the condition

$$\alpha_j-\alpha_t \neq 0,\pm 1,\pm 2,\ldots \qquad (j=1,\ldots,p;\, t=1,\ldots,p;\, j\neq t).$$

We now distinguish two cases:

First case: $0 \leqq p \leqq q-1$. We apply theorem 18.

If $-\pi < \arg z < \pi$, it follows from (223) and (185) (with $m=1$, $n=p$, $q+1$ instead of q and $z e^{\pi i}$ instead of z) on account of Remark 1

$$\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;z\end{matrix}\right)\backsim K_{p,q}(z).$$

If $0 \leqq p < q-1$ and $z>0$, it follows from (222) and (187) (with $m=1$, $n=p$ and $q+1$ instead of q) on account of Remark 1

$$\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;-z\end{matrix}\right)\backsim K_{p,q}(z\,e^{-\pi i})+K_{p,q}(z\,e^{\pi i}).$$

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[69]) STOKES, [28]; BARNES, [3], 68, 83, 115 and [4]; WATSON [30], 37; FOX, [9]; WRINCH, [37], [38] and [39]; WRIGHT, [35]; comp. also WRIGHT, [36].

[70]) Comp. footnote [4]).

If $p \geqq 0$ and $z > 0$, it follows from (222) (with $q = p+1$) and (188) (with $m = 1$, $n = p$ and $q = p+2$) on account of the Remarks 1 and 2

$$\frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{p+1} \Gamma(\beta_j)} \, {}_pF_{p+1}\begin{pmatrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_{p+1}; -z \end{pmatrix} \backsim K_{p,p+1}(z e^{-\pi i}) + K_{p,p+1}(z e^{\pi i}) + L_{p,p+1}(z). \quad (224)$$

Second case: $q = p \geqq 1$. We apply the theorems 16 and 19.

If $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$, it follows from (222) (with $q = p$) and (180) (with $m = 1$, $n = p$ and $q = p+1$) on account of Remark 2

$$\frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} \, {}_pF_p\begin{pmatrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_p; -z \end{pmatrix} \backsim L_{p,p}(z). \quad . \quad . \quad . \quad (225)$$

If $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$, it follows from (223) and the first assertion of theorem (19) (formula (185) with $m = 1$, $n = p$, $q = p+1$ and $z e^{\pi i}$ instead of z) on account of Remark 1

$$\frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} \, {}_pF_p\begin{pmatrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_p; z \end{pmatrix} \backsim K_{p,p}(z).$$

If $\arg z = \frac{1}{2}\pi$, it follows from (222) and (191) (with $m = 1$, $n = p$ and $q = p+1$) on account of the Remarks 1 and 2

$$\frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} \, {}_pF_p\begin{pmatrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_p; -z \end{pmatrix} \backsim K_{p,p}(z e^{-\pi i}) + L_{p,p}(z) \quad . \quad (226)$$

If $\arg z = -\frac{1}{2}\pi$, it follows from (222) and (192) (with $m = 1$, $n = p$ and $q = p+1$) on account of the Remarks 1 and 2

$$\frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} \, {}_pF_p\begin{pmatrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_p; -z \end{pmatrix} \backsim K_{p,p}(z e^{\pi i}) + L_{p,p}(z). \quad . \quad (227)$$

Closing Remark. If it is desirable to express the function ${}_pF_q(z)$ in terms of fundamental solutions near $z = \infty$ of the differential equation satisfied by it, we may use the formulae (222) and (223) in connection with the theorems 13 and 14.

§ 21. The asymptotic expansion of $W_{k,m}(z)$.

As another application of the theorems of § 18 I will write down the asymptotic expansions of WHITTAKER's function $W_{k,m}(z)$. The well-known expansion [71])

$$W_{k,m}(z) \backsim e^{-\frac{1}{2}z} z^k \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right) \quad . \quad (228)$$

holds only when $|\arg z| < \frac{3}{2}\pi$. This formula is because of (143) equivalent to

$$G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}-k \\ m, -m \end{matrix}\right) \backsim e^{-z} z^{k-\frac{1}{2}} \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right).$$

Taking account of (26) the last expression may be interpreted as follows:

If $a_1 = \frac{1}{2} - k$, $b_1 = m$ and $b_2 = -m$, then

$$H_{1,2}(z) = e^{-z} z^{k-\frac{1}{2}} \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right).$$

We now suppose $(2\lambda + \frac{1}{2})\pi < \arg z < (2\lambda + \frac{3}{2})\pi$, where λ is an arbitrary integer. Then we have for large values of $|z|$ because of (143), the fifth assertion of theorem 21 and lemma 17

$$W_{k,m}(z) \backsim (-1)^{\lambda} e^{-2\lambda k\pi i} e^{-\frac{1}{2}z} z^k \left\{\frac{\sin 2(\lambda+1)m\pi}{\sin 2m\pi} + e^{-2k\pi i}\frac{\sin 2\lambda m\pi}{\sin 2m\pi}\right\} \times$$
$$\times \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right).$$

This formula may also be proved by means of (144) and (228); it reduces for $\lambda = 0$ and $\lambda = -1$ to (228).

We now suppose $(2\lambda - \frac{1}{2})\pi < \arg z < (2\lambda + \frac{1}{2})\pi$, where λ is an arbitrary positive or negative integer ($\lambda \neq 0$). Then it follows from (143), the first assertion of theorem 21 and lemma 18

$$W_{k,m}(z) \backsim (-1)^{\lambda-1} e^{2\lambda k\pi i} 2\pi i e^{\frac{1}{2}z} z^{-k} \times$$
$$\times \frac{\sin 2\lambda m\pi}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\sin 2m\pi} \, {}_2F_0\left(\tfrac{1}{2}+k+m, \tfrac{1}{2}+k-m; \frac{1}{z}\right).$$

The asymptotic expansion of $W_{k,m}(z)$ when $\arg z = (2\lambda + \frac{1}{2})\pi$ or $\arg z = (2\lambda - \frac{1}{2})\pi$, where λ is any integer, can be deduced from the seventh, respect. the ninth assertion of theorem 21. These expansions run as follows

$$W_{k,m}(z) \backsim (-1)^{\lambda} e^{-2\lambda k\pi i} e^{-\frac{1}{2}z} z^k \left\{\frac{\sin 2(\lambda+1)m\pi}{\sin 2m\pi} + e^{-2k\pi i}\frac{\sin 2\lambda m\pi}{\sin 2m\pi}\right\} \times$$
$$\times \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right) + (-1)^{\lambda-1} e^{2\lambda k\pi i} 2\pi i \, e^{\frac{1}{2}z} z^{-k} \times$$
$$\times \frac{\sin 2\lambda m\pi}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\sin 2m\pi} \, {}_2F_0\left(\tfrac{1}{2}+k+m, \tfrac{1}{2}+k-m; \frac{1}{z}\right)$$

[71]) WHITTAKER and WATSON, [32], § 16.3—16.4.

respect.

$$W_{k,m}(z) \backsim (-1)^{\lambda-1} e^{-2(\lambda-1)k\pi i} e^{-\frac{1}{2}z} z^k \left\{ \frac{\sin 2\lambda m\pi}{\sin 2m\pi} + e^{-2k\pi i} \frac{\sin 2(\lambda-1) m\pi}{\sin 2m\pi} \right\} \times$$

$$\times {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right) + (-1)^{\lambda-1} e^{2\lambda k\pi i}\, 2\pi i\, e^{\frac{1}{2}z} z^{-k} \times$$

$$\times \frac{\sin 2\lambda m\pi}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)\,\sin 2m\pi}\, {}_2F_0\left(\tfrac{1}{2}+k+m, \tfrac{1}{2}+k-m; \frac{1}{z}\right).$$