Mathematics. — *A note on 0-dimensional spaces.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of January 25, 1947.)

1. *Introduction.* An 0-dimensional space is, as is well known, a space in which any point has arbitrarily small neighbourhoods with vacuous boundary, or — which is the same thing — in which for any point p and any neighbourhood $U \supset p$ may be found a both open and closed set S with $p \subset S \subset U$.

Among spaces in general these 0-dimensional spaces naturally have a special place: for 0-dimensional spaces some problems may easily be solved, while other ones — by total lack of connection — may present particular difficulties. From this special place among spaces of higher dimension the 0-dimensional spaces derive the right of a separate treatment. This is, however, not intended by this note. We only want to justify the particular position of the 0-dimensional spaces by one special question, which we may put as follows: *which axioms must we impose on a general space* (space of neighbourhoods or topological space) *such that this space be 0-dimensional and metrisable* (= separable metric) *and how far is it easier to prove the metrization for these spaces than for general separable spaces?*

In 2. we shall give the answer to this question and offer a proof of the metrization of the 0-dimensional separable spaces, which is much simpler than that of URYSOHN's well-known theorem of metrization for separable spaces in general.

In 3. we shall consider a totally different problem, connected with 0-dimensional spaces; here we shall study the *space of quasicomponents* $Q(M)$, *corresponding with an arbitrary separable space* M. This space $Q(M)$ is always of dimension 0. Some properties and examples will be mentioned without comment. The more complicated proofs and an extensive discussion will be given on another occasion. Incidentally a so far unsolved problem of KURATOWSKI's, put in 1938 (comp. [2]), will be solved by the study of the space of quasicomponents.

2. *Definition and metrization of 0-dimensional separable spaces.* In a (non-vacuous) point-set N we consider a certain system $\{S\}$ of subsets $S \subset N$ — called *lump-sets* —: this system $\{S\}$ satisfies the conditions

1°. N itself and the vacuous set are contained in $\{S\}$;

2°. if $\{S\}$ contains S, then $\{S\}$ also contains $N - S$;

3°. if S_1 and S_2 belong to $\{S\}$, then the intersection $S_1 \cdot S_2$ and the sum $S_1 + S_2$ also belong to $\{S\}$.

If in N is defined a system of lump-sets $\{S\}$ and if every S is understood to be a neighbourhood of every point n belonging to S, then $N = N_{\{S\}}$ is called an 0-dimensional topological space. Moreover two 0-dimensional topological spaces $N_{\{S_1\}}$ and $N_{\{S_2\}}$, corresponding with a point-set N, are considered to be identical: $N_{\{S_1\}} \equiv N_{\{S_2\}} \equiv N$, if for any point $n \subset N$ and any S_1 with $n \subset S_1$ it is possible to find an S_2 with $n \subset S_2 \subset S_1$, and conversely (interchanging the indices 1 and 2).

Let $N_{\{S\}}$ be an 0-dimensional topological space and let $\{U\}$ be a system of subsets of N such that for any point $n \subset N$ and any S containing n may be found a U with $n \subset U \subset S$; and conversely, that for any $U \supset n$ may be found an S with $n \subset S \subset U$. In this case $\{U\}$ is called a *base* of $N_{\{S\}}$.

$\{S\}$ itself, but possibly also subsystems of $\{S\}$ are e.g. bases (*lump-bases*) of $N_{\{S\}}$.

Two spaces N and N' are called *topologically equivalent* if there exists a one to one mapping f of N on N': $f(N) = N'$, such that a base $\{S\}$ of N is mapped on a base $\{f(S)\}$ of N'.

An 0-dimensional topological space $N = N_{\{S\}}$ is called by definition an *0-dimensional separable space*, if the following two conditions are satisfied:

(α) (HAUSDORFF *axiom*) For two different points n_1 and n_2 of N there exist disjunct lump-sets S_1 and S_2:

$$n_1 \subset S_1 \quad , \quad n_2 \subset S_2 \quad , \quad S_1 \cdot S_2 = 0$$

(β) (*axiom of countability*) Among the different bases of N there exists a *countable base* $\{S_i\}$ $(i = 1, 2, \ldots)$ [1]).

Theorem. *An 0-dimensional separable space N is metrisable; in particular N is topologically equivalent with a subset of the discontinuum of* CANTOR [2]).

Proof. Be $\{S_j\}$ $(j = 1, 2, \ldots)$ a countable base of N; moreover $\{S_j\}$ will be a sub-system of a system of lump-sets $\{S\}$, which defines N. Each point $n \subset N$ we map on one point $n' = f(n)$ of the real axis, defined by:

$$f(n) = n' = \sum_{i=1}^{\infty} \frac{\delta_i}{3^i} \qquad \begin{pmatrix} \delta_i = 0 \text{ if } n \not\subset S_i \\ \delta_i = 2 \text{ if } n \subset S_i \end{pmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

The set N' of the points n' is apparently a subset of the discontinuum D. We shall prove that N' is topologically equivalent with N. $f(N) = N'$ is one to one, for, according to (α), for different points n_1 and n_2 exist disjunct lump-sets S_k and S_l such that n_1' and n_2' determine different points.

---

1) It is easy to prove that in this case $\{S\}$ contains a countable sub-system which is a countable base (and therefore even a lumpbase) of $N_{\{S\}}$.

2) The discontinuum is, as known, the set of those real numbers of the interval [0,1] which may be expanded in the triadic number system using only the digits 0 and 2.

We shall prove that f is a topological mapping. For fixed arbitrary m all points of the shape

$$V_m = \sum_{i=1}^{m} \frac{\delta_i}{3^i} + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^{m+j}} \qquad \begin{pmatrix} \delta_i = 0, \text{ if } n \not\subset S_i;\ \delta_i = 2, \text{ if } n \subset S_i \\ \varepsilon_j = 0, 2. \end{pmatrix} \quad (2)$$

form a neighbourhood $V_m = V_m(n')$ of n' in N'. The system $\{V_m\}$ $(m = 1, 2, \ldots)$ is apparently a system of neighbourhoods of n'. Let S_{a_i} $(i = 1, 2, \ldots, k)$, resp. S_{b_i} $(i = 1, 2, \ldots, l)$ be all those sets of S_i $(i = 1, 2, \ldots, m)$ which contain, resp. do not contain n. Because of the definition of f and because of (2) now apparently

$$f^{-1}(V_m) = \left[\prod_{i=1}^{k} S_{a_i}\right] \cdot \left[N - \sum_{i=1}^{l} S_{b_i}\right] \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we put

$$f^{-1}(V_m) = U_m,$$

then, according to 2. 1°, 2° and 3°, $\{U_m\}$ is a sub-system of $\{S\}$. n has a system of neighbourhoods $\{S_{r_i}\}$ $(i = 1, 2, \ldots)$. Now obviously $n \subset U_{r_i} \subset S_{r_i}$ $(i = 1, 2, \ldots)$. From this follows that the countable system of *all* sets U_m corresponding with *all* points $n \subset N$ is a base of N. Therefore N and N' are topologically equivalent, q.e.d.

Remark 1. By the afore-said not only the metrization of the 0-dimensional separable spaces has been shown, but also the topological equivalence with a subset of the discontinuum, which fact was shown first — in another way — by SIERPIŃSKI.

Remark 2. It is obvious that our definition of 0-dimensional separable spaces, which differs from the usual one, indeed corresponds with the usually given definition (viz. as a separable space (i.e., a normal space with countable base, comp. [1]), where every point has arbitrarily small neighbourhoods with vacuous boundary). For this well-known definition one may give the same proof of metrization that we gave.

3. *Definition and properties of the space of quasicomponents $Q(M)$ of a separable space M.*

Let M be an arbitrary separable space and let $\{U\}$ be a (for instance countable) base of M, consisting of open sets (the term "base" in this case is used in the usual sense of the word to characterize M and must not be taken for the special base mentioned in 2.).

We consider a system of lump-sets $\{S\}$ of M, *whose topology is weaker than that of M*, i.e., for each point $m \subset M$ and each $S \supset m$ may be found a U of $\{U\}$ with $m \subset U \subset S$ (but not necessarily conversely). — There always exist such lump-systems $\{S\}$; for instance the system consisting of M and the vacuous set is such a lump-system. — Any S of such a $\{S\}$ is q.q. a set open in M; therefore, since $M - S$ also belongs to $\{S\}$, $M - S$

is also open and therefore S is likewise closed, in other words: *the lump-systems $\{S\}$, whose topology is weaker than that of M, consist of (in M) both open and closed sets.*

In the following we shall consider only such lump-systems and may call an (in M) both open and closed set a *lump-set* by definition. — From 2. we already know when to call two lump-systems $\{S\}$ and $\{S'\}$ topologically equivalent. — The topology of $\{S\}$ is *weaker* than that of $\{S'\}$ (or that of $\{S'\}$ is stronger than that of $\{S\}$), if for each point $m \subset M$ and each $S \supset m$ may be found an S' with $m \subset S' \subset S$, but not necessarily conversely. — A lump-system $\{S^*\}$ which is stronger or just as strong (topologically equivalent) as any other lump-system (but weaker than the base $\{U\}$ of M) consists of all lump-sets of M, or of a sub-system of $\{S^*\}$ which is topologically equivalent with $\{S^*\}$. If $\{U\}$ is not stronger, but topologically equivalent with $\{S^*\}$, M is 0-dimensional (and conversely). In the other extreme case that $\{S^*\}$ consists only of M and the vacuous set, M is connected. $\{S^*\}$ leads to a new space $Q(M)$: two points of M are in the same class only if both of them either belong or do not belong to S^* for any $S^* \subset \{S^*\}$. Thus the points of M are divided into disjunct classes. These classes Q are called *quasicomponents* (HAUSDORFF [4] [3])) and are taken as points q of the space of quasicomponents $Q(M)$. The set $Q(M)$ of the points q becomes a space by considering the system $\{S^*\}$ as a system of neighbourhoods of $Q(M)$ (while at the same time one has to consider the sets Q as points q). It is easy to prove that $Q(M)$ *is always an 0-dimensional regular space* (regular in the sense of [1]). Further *the 0-dimensional separable spaces M are exactly those separable spaces which are identical with their spaces of quasicomponents $Q(M)$.*

For $Q(M)$ to be separable it is obviously only necessary that $Q(M)$ has a countable base, in other words $\{S^*\}$ contains a countable sub-system which is at the same time a base of $Q(M)$. We shall now, by means of a general example, show that such a countable base of $Q(M)$ does not always exist.

M by definition is called *totally disconnected*, if the quasicomponents Q considered as subsets of M, all consist of one point. This does not necessarily, however, infer that M is of dimension 0, since the base $\{U\}$ of M may be stronger than the base $\{S^*\}$ of $Q(M)$. There exist (MAZURKIEWICZ [5], SIERPIŃSKI [3]) totally disconnected n-dimensional ($n \geqq 0$) separable spaces M. Now the following theorem holds true.

Theorem. *Any totally disconnected n-dimensional ($n > 0$) separable space M has a space of quasicomponents $Q(M)$ without a countable base.*

P r o o f. Suppose $Q(M)$ did have a countable base; then the system $\{S^*\}$, defining $Q(M)$, would satisfy 2. (α) and (β) ((α) because $Q(M)$ is regular) and $Q(M)$ would, according to the theorem and remark 2 of 2.,

---

[3]) The quasicomponent $Q \subset M$, corresponding with a point a of M, may be shortly defined as the intersection of all those both open and closed subsets of M, which contain a.

be an 0-dimensional separable metrisable space (in the usual sense of the word), contrary to the given dimension n.

R e m a r k. By this theorem we have solved a problem put by KURATOWSKI in 1938 and communicated by SIERPIŃSKI [2]. This problem ran as follows: is it possible for any separable metric space M to find a countable number of both open and closed subsets of M such that any both open and closed subset of M is the finite or infinite sum of a suitably chosen number of the first-mentioned subsets? It is easy to see that this question is identical with the following one: has any separable M a $Q(M)$ with countable base? It follows from our last mentioned theorem that this question must be answered in the negative.

Now the following further question arises: when *is* $Q(M)$ separable, in other words, when does $Q(M)$ have a countable base? To me this question seems to present great difficulties.

I am, however, able to answer the following question: *when is $Q(M)$ a compact separable space*? For this the following condition is necessary and sufficient: the intersection of any decreasing sequence of lump-sets of M

$$S_1 \supset S_2 \supset S_3 \ldots$$

has a non-vacuous intersection in M. This expresses nothing else than the compactness of $Q(M)$. Thus we have the following

Theorem. *A separable space M has a compact 0-dimensional separable space $Q(M)$ of quasicomponents, only if $Q(M)$ is compact.*

The rather complicated proof of this theorem will be given on another occasion. Special cases of this theorem (for which were given sufficient conditions that were, however, not necessary) were already known: for compact separable M the theorem has been proved by L. E. J. BROUWER [6] (for compact M the space of components and the space of quasicomponents coincide), whereas in FREUDENTHAL [7], and in [8] the compactness of M is replaced by the double, weaker, condition: M semicompact and $Q(M)$ compact. It is now clear from our theorem that it is unnecessary to ask that M be semicompact.

REFERENCES.

1. P. ALEXANDROFF und H. HOPF, Topologie I, Berlin (1935).
2. W. SIERPIŃSKI, Fund. Math. **30**, 129—131 (1938).
3. ———, Fund. Math. **2**, 81 (1921).
4. F. HAUSDORFF, Grundzüge der Mengenlehre, First Edition, Leipzig (1914).
5. ST. MAZURKIEWICZ, Fund. Math. **10**, 311 (1927).
6. L. E. J. BROUWER, Versl. Kon. Akad. Amsterdam, March 26, (1910).
7. H. FREUDENTHAL, Neuaufbau der Endentheorie, (1941).
8. J. DE GROOT, Top. Studiën, Thesis, Groningen (1942).