Mathematics. — *On sets of integers.* (Third communication.) By J. G. VAN DER CORPUT.

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§ 4. *Theorems on more than two sets.*

The "sumset" $A_1 + \dots + A_n$ of n sets $A_1, \dots, A_n$, each consisting of integers $\geqq 0$, is defined as the set of all integers of the form $a_1 + \dots + a_n$, where, for each value of h, the integer a_h is a term of A_h.

By $A(m)$ I denote again the number of positive elements $\leqq m$ of A.

In order to deduce a general result, I consider l systems $s_1, \dots, s_l$ ($l \geqq 1$), where s_h, for each value of h, is formed by n_h sets A_{hk} ($k = 1, \dots, n_h$), each consisting of integers $\geqq 0$, each containing zero and satisfying the inequality

$$A_{h,1}(m) + \dots + A_{h,n_h}(m) \geqq \gamma_h m \qquad (m = 1, \dots, g), \quad . \quad . \quad (46)$$

where $\gamma_1 + \dots + \gamma_l \leqq 1$. Further I consider a sum of the form $\Sigma_1 T(m)$, extended over a finite number of sets T with the property, that any of these sets T is the sumset of a number of systems A_{hk}. I suppose that in any T each system A_{hk} is counted at most once, so that for instance T may have the form A_{11} or $A_{12} + A_{13}$ or $A_{12} + A_{41} + A_{52}$, but not $A_{11} + A_{11}$. In connection with the fact that the given inequalities (46) are symmetrical in the sets A_{1h} belonging to s_1, I will assume that $\Sigma_1 T(m)$ is symmetrical in these sets, also symmetrical in the sets belonging to $s_2, \dots$, finally symmetrical in the sets, belonging to s_l. Then it is possible to deduce for $m = 1, \dots, g$ a convenient lower bound for the sum $\Sigma_1 T(m)$, namely

Theorem 8. *Under these conditions we have for* $m = 1, \dots, g$

$$\Sigma_1 T(m) \geqq m \, \Sigma_1 \tau(T),$$

if

$$\tau(T) = \frac{\lambda_1 \gamma_1}{n_1} + \dots + \frac{\lambda_l \gamma_l}{n_l},$$

where λ_h *denotes, for each value of* h, *the number of terms* A_{hk}, *occurring in* T.

As corollaries of the special case $l = 1$ we obtain the theorems 9 and 10, found by F. J. DYSON:

Theorem 9. *If each system* $A_1, \dots, A_n$ *contains zero and*

$$A_1(m) + \dots + A_n(m) \geqq \gamma m \qquad (m = 1, \dots, g),$$

where $\gamma \leqq 1$, *then*

$$(A_1 + \dots + A_n)(m) \geqq \gamma m \qquad (m = 1, \dots, g).$$

In fact, the left hand side is symmetrical in $A_1, \dots, A_n$ and is the sum of n terms $T(m)$, where $\tau(T) = \frac{\gamma}{n}$

Theorem 10. *If the conditions of the preceding proposition are satisfied, we have for every natural number $r \leqq n$ and for $m = 1, \dots, g$*

$$\Sigma_2 (A_{t_1} + \dots + A_{t_r})(m) \geqq \binom{n-1}{r-1} \gamma m,$$

where Σ_2 is extended over the $\binom{n}{r}$ systems of natural numbers $t_1, \dots, t_r$ with $t_1 < t_2 < \dots < t_r \leqq n$.

In fact, the left hand side is symmetrical in $A_1, \dots, A_n$ and is the sum Σ_1 of $\binom{n}{r}$ terms $T(m)$, where $\tau(T) = \frac{r\gamma}{n}$, so that

$$\Sigma_1 \tau(T) = \binom{n}{r} \cdot \frac{r\gamma}{n} = \binom{n-1}{r-1} \gamma.$$

Another example of theorem 8: If each of the systems A, B, C and D contains zero,

$$A(m) + B(m) \geqq \gamma m \text{ and } C(m) + D(m) \geqq \delta m \qquad (m = 1, \dots, g),$$

where $\gamma + \delta \leqq 1$, then we obtain for $m = 1, \dots, g$

$$(A + C)(m) + (A + D)(m) + (B + C)(m) + (B + D)(m) \geqq 2(\gamma + \delta)m.$$

In fact, the left hand side is symmetrical in A and B, also in C and D and is the sum of four terms $T(m)$, where $\tau(T) = \frac{1}{2}(\gamma + \delta)$.

We have treated the last problem already in § 2.

I propose not only to prove theorem 8, but simultaneously the following proposition, involving positive weights $f(m)$. These weights satisfy the inequalities

$$\left. \begin{array}{r} f(m+1) \geqq f(m) \quad (m = 1, \dots, g-1) \text{ and } f^2(m+1) \geqq f(m) f(m+2) \\ (m = 1, \dots, g-2), \end{array} \right\} \quad (29)$$

whereas $A(m)$ denotes the sum $\sum_{a \leqq m} f(a)$, extended over all positive elements $a \leqq m$ of A.

Theorem 11. *If the positive numbers $f(m)$ $(m = 1, \dots, g)$ satisfy the inequalities (29), if the sets $A_1, \dots, A_n$ consist of integers ≥ 0, if each of these sets contains zero and if*

$$A_1(m) + \dots + A_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \dots, g),$$

where $\gamma \leqq 1$, then for these values of m

$$(A_1 + \dots + A_n)(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \dots, g).$$

To prove these theorems I make use of two lemmas. In these lemmas $A(m)$ denotes the sum of the weigths of the positive elements $\leqq m$ of A, where for $m > g$ the weight $f(m)$ is defined such that

$$f(m+1) \geqq f(m) \text{ and } f^2(m+1) \geqq f(m) f(m+2) \qquad (m = 1, \ldots, g).$$

Let us consider $n (n \geqq 2)$ systems $A_1, \ldots, A_n$ each of which consists of integers $\geqq 0$ and $\leqq g$ and contains the number zero, whereas A_n contains moreover at least one positive integer. Be e the smallest integer $\geqq 0$, such that a natural number $q \leqq n - 1$ can be found with the property that e is an element of A_q and that A_n contains at least one positive element b, such that $e + b$ is not contained in A_q. Such an element e exists, since the greatest element a of A_1 and any positive element b of A have the property that $a + b$ does not belong to A_1. After having fixed e and q, I cancel in A_n a positive element b, such that $e + b$ does not belong to A_q, and I add to A_q this integer $e + b$, if it is $\leqq g$. If A_q and A_n are transformed in this manner into A'_q and A'_n and if further $A'_h = A_h$ for $h \neq q$ and $\neq n$, I say that the system $A_1, \ldots, A_n$ is transformed by an elementary transformation into $A'_1, \ldots, A'_n$. Thus A'_n is the set of elements $\neq b$ of A_n; if $e + b > g$, then $A'_q = A_q$, and if $e + b \leqq g$, then A'_q is the set formed by $e + b$ and the elements of A_q.

Lemma 8. *If each of the systems $A_1, \ldots, A_n$ $(n \geqq 2)$ contains zero, if A_n contains moreover at least one positive integer $\leqq g$, and if*

$$A_1(m) + \ldots + A_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g), \quad . \quad . \quad (47)$$

where $\gamma \leqq 1$, then each elementary transformation transforms $A_1, \ldots, A_n$ into a system $A'_1, \ldots, A'_n$, with

$$A'_1(m) + \ldots + A'_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g).$$

We obtain even for any integer $r \geqq 0$

$$\sum_{a'_1 \leqq m} f(a'_1 + r) + \ldots + \sum_{a'_n \leqq m} f(a'_n + r) \geqq \gamma \sum_{h=1}^{m} f(h + r) \quad (m = 1, \ldots, g), \; (48)$$

if $\sum\limits_{a'_h \leqq m}$ is extended, for each value of h, over the positive elements $a'_h \leqq m$ of A'_h.

Let us suppose that this assertion is not true. Be k the smallest positive integer $\leqq g$, for which the lemma is not valid, more precisely: the positive integer $k \leqq g$ possesses the following properties:

1. It is possible to find a system $A_1, \ldots, A_n$, such that each set A_h contains zero, that A_n contains at least one positive element $\leqq g$, that the inequalities (47) are true for $m = 1, \ldots, k$ and that a suitably chosen

elementary transformation transforms the system $A_1, \ldots, A_n$ into a system $A'_1, \ldots, A'_n$, for which at least one of the inequalities

$$\sum_{a'_1 \leqq m} f(a'_1 + r) + \ldots + \sum_{a'_n \leqq m} f(a'_n + r) \geqq \gamma \sum_{h=1}^{m} f(h + r) \qquad (m = 1, \ldots, k), \quad (49)$$

where r denotes a convenient integer $\geqq 0$, is not true.

2. Be l an arbitrary positive integer $< k$. If each of the systems $C_1, \ldots, C_n$ contains zero, if moreover C_n contains at least one positive integer $\leqq g$ and if finally

$$C_1(m) + \ldots + C_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, l),$$

then each elementary transformation transforms $C_1, \ldots, C_n$ into $C'_1, \ldots, C'_n$, such that

$$\sum_{c'_1 \leqq m} f(c'_1 + t) + \ldots + \sum_{c'_n \leqq m} f(c'_n + t) \geqq \gamma \sum_{h=1}^{m} f(h + t) \qquad (m = 1, \ldots, l) \quad (50)$$

for each integer $t \geqq 0$.

The special case $l = k - 1$, $t = r$, $C_h = A_h$ gives, that the inequalities (49) are true for $m = 1, \ldots, k - 1$. Consequently (49) is not valid for $m = k$, that is

$$\sum_{a'_1 \leqq m} f(a'_1 + r) + \ldots + \sum_{a'_n \leqq m} f(a'_n + r) < \gamma \sum_{h=1}^{m} f(h + r). \quad . \quad . \quad (51)$$

From (47) and lemma 5 we conclude for $m = 1, \ldots, k$

$$\sum_{a_1 \leqq m} f(a_1 + r) + \ldots + \sum_{a_n \leqq m} f(a_n + r) \geqq \gamma \sum_{h=1}^{m} f(h + r) \quad . \quad . \quad (52)$$

Just as in the proof of lemma 6 (§ 3) we show, that the number b, cancelled in A_n, is $\leqq k$ and that the number $e + b$, possibly added to A_q, is $> k$.

From the definition of e it follows that each element $a < e$ of each set A_h ($h = 1, \ldots, n - 1$) has the property, that $a + b$ belongs to A_h; in particular b belongs to each of the sets $A_1, \ldots, A_{n-1}$. The system A_h ($h < n$) contains $1 + A_h(k - b)$ elements $a \leqq k - b < e$ and each of these elements furnishes an element $a + b$ of A_h, which is $\geqq b$ and $\leqq k$, so that we find for $h = 1, \ldots, n - 1$

$$\sum_{a_h \leqq k} f(a_h + r) - \sum_{a_h \leqq b-1} f(a_h + r) \geqq f(b + r) + \sum_{a_h \leqq k-b} f(a_h + r + b). \quad (53)$$

The inequality

$$\sum_{a_1 \leqq b-1} f(a_1 + r) + \ldots + \sum_{a_n \leqq b-1} f(a_n + r) \geqq \gamma \sum_{h=1}^{b-1} f(h + r). \quad . \quad (54)$$

is obvious for $b = 1$ and follows for $b > 1$ from (52). For $h = 1, \ldots, n - 1$ the sets A_h and A'_h contain the same integers $\leqq k$; the two sets A_n and A'_n contain the same integers $\leqq b-1$, so that we obtain for $h=1, \dots, n-1$ from (53)

$$\sum_{a'_h \leqq k} f(a'_h+r) \geqq f(b+r) + \sum_{a'_h \leqq b-1} f(a'_h+r) + \sum_{a_h \leqq k-b} f(a_h+r+b) \quad (55)$$

and further from (54)

$$\sum_{a'_1 \leqq b-1} f(a'_1+r) + \dots + \sum_{a'_n \leqq b-1} f(a'_n+r) \geqq \gamma \sum_{h=1}^{b-1} f(h+r). \quad . \quad (56)$$

The proof is established if we show

$$\sum_{a_1 \leqq k-b} f(a_1+r+b) + \dots + \sum_{a_{n-1} \leqq k-b} f(a_{n-1}+r+b) \geqq \gamma \sum_{h=1}^{k-b} f(h+r+b), \quad (57)$$

for then we may replace in this inequality $a_1, \dots, a_{n-1}$ by $a'_1, \dots, a'_{n-1}$, hence by (55) and (56)

$$\sum_{a'_1 \leqq k} f(a'_1+r) + \dots + \sum_{a'_n \leqq k} f(a'_n+r)$$

$$\geqq \sum_{a'_1 \leqq k} f(a'_1+r) + \dots + \sum_{a'_{n-1} \leqq k} f(a'_{n-1}+r) + \sum_{a'_n \leqq b-1} f(a'_n+r)$$

$$\geqq (n-1) f(b+r) + \gamma \sum_{h=1}^{b-1} f(h+r) + \gamma \sum_{h=1}^{k-b} f(h+r+b)$$

$$\geqq \gamma \sum_{h=1}^{k} f(h+r)$$

(in virtue of $n-1 \geqq 1 \geqq \gamma$), contrary to (51).

The proof of (57) runs precisely as that of (6) in lemma 1 (§ 1). Thus we find the assertion of our lemma.

Let us again consider n ($n \geqq 2$) systems $A_1, \dots, A_n$, each of which consists of integers $\geqq 0$ and $\leqq g$ and contains the number zero, whereas A_n contains moreover at least one positive integer. Be e again the smallest integer $\geqq 0$ such that a natural number $q \leqq n-1$ can be found with the property that e is an element of A_q and that A_n contains at least one positive element b, such that $e+b$ is not contained in A_q. I cancel in A_n all these elements b and I add $e+b$ to A_q, as far as they are $\leqq g$. In this manner A_n and A_q are transformed into the sets A^*_n and A^*_q; I put $A^*_h = A_h$ for $h \neq q$ and $\neq n$. I say that the system $A_1, \dots, A_n$ is transformed into the system $A^*_1, \dots, A^*_n$ by an e-transformation. Since this transformation can be decomposed into a finite number of elementary transformations, the preceding lemma gives:

Lemma 9. *Suppose that each of the n ($n \geqq 2$) systems $A_1, \dots, A_n$ consists of integers $\geqq 0$ and $\leqq g$, and contains the number zero, whereas moreover A_n contains at least one positive integer. If the inequalities* (47) *are true, where $\gamma \leqq 1$, we have also*

$$A^*_1(m) + \dots + A^*_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m=1, \dots, g). \quad . \quad (58)$$

The proof of theorem 11 can now be given in a few lines. The assertion of this theorem is obvious if $n = 1$, so that I may suppose $n > 1$ and I may assume that the theorem is true for every value of n less than the actual value. Consequently the theorem holds, if A_n consists only of the number zero. I may therefore suppose, that A_n contains at least one positive element $\leqq g$ and that the theorem is true with the actual value of n, whenever the value of $A_n(g)$ is less than its actual value. I transform the system $A_1, \dots, A_n$ by an e-transformation into a system $A_1^*, \dots, A_n^*$. Then $A_n^*(g) < A_n(g)$ and the preceding lemma gives the inequalities (58), so that it follows from the inductive hypothesis

$$(A_1^* + \dots + A_n^*)(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \dots, g).$$

We have $A_h^* = A_h$ for each value $h \leqq n - 1$, with the exception of one value $h = q$ and by lemma 3 (§ 1) $A_q + A_n$ contains each element $\leqq g$ of $A_q^* + A_n^*$, hence

$$(A_1 + \dots + A_n)(m) \geqq (A_1^* + \dots + A_n^*)(m).$$

This proves theorem 11.

Now the proof of theorem 8. I put $n_1 + \dots + n_l = p$. If $p = 1$, then $l = 1$ and $n_1 = 1$, so that each set T, occurring in the sum Σ_1, denotes the set A_{11} with $\tau(T) = \gamma_1$, hence for $m = 1, \dots, g$

$$T(m) = A_{11}(m) \geqq \gamma_1 m = m\tau(T).$$

Consequently I may suppose $p > 1$ and assume that the theorem is true for any value of p less than the actual value. If one of the p sets A_{hk} consists only of the number zero, each set T, occurring in the sum Σ_1, is the sumset of $p - 1$ systems A_{hk}, so that the assertion of theorem 8 follows from the inductive hypothesis. Consequently I may suppose, that each set A_{hk} contains at least one positive integer and I may assume that the theorem holds with the actual value of p, whenever the value of $A_{l,n_l}(g)$ is less than its actual value. I distinguish two cases:

1. If $n_1 = \dots = n_l = 1$, then by (46)

$$A_{h,1}(m) \geqq \gamma_h m$$

and T is the sumset $\Sigma_3 A_{h,1}$ of a number of sets $A_{h,1}$. In virtue of $\gamma_1 + \dots + \gamma_l \leqq 1$ and $n_h = 1$ it follows from the theorem of Mann

$$T(m) \geqq m \Sigma_3 \gamma_h = m(\lambda_1\gamma_1 + \dots + \lambda_l\gamma_l) = m\tau(T),$$

consequently

$$\Sigma_1 \, T(m) \geqq m \, \Sigma_1 \, \tau(T).$$

2. In the remaining case at least one of the integers $n_1, \dots, n_l$ is greater than 1. Without loss of generality I may assume $n_l > 1$. I transform the system $A_{l,1}, \dots, A_{l,n_l}$ by an e-transformation into a system $A_{l,1}^*, \dots, A_{l,n_l}^*$. Then $A_{l,n_l}^*(g) < A_{l,n_l}(g)$. By the preceding lemma, applied with $f(m) = 1$,

the inequalities (46) remain true, if $A_{l,h}$ $(1 \leqq h \leqq n_l)$ is replaced by $A^*_{l,h}$. Consequently we deduce from the inductive hypothesis for $m = 1, \ldots, g$

$$\Sigma_1 T^*(m) \geqq m \, \Sigma_1 \tau(T^*) = m \, \Sigma_1 \tau(T);$$

here T^* denotes the set into which T is transformed, if $A_{l,h}$ $(1 \leqq h \leqq n_l)$ is replaced by $A^*_{l,h}$. It is therefore sufficient to deduce the inequalities

$$\Sigma_1 T(m) \geqq \Sigma_1 T^*(m) \qquad (m = 1, \ldots, g).$$

It follows from the definition of the e-transformation, that $A^*_{l,h} = A_{l,h}$ for each $h \leqq n_l - 1$, only one value $h = q$ excepted. I write

$$\Sigma_1 T = \Sigma_4 T + \Sigma_5 T + \Sigma_6 T;$$

The summation Σ_4 is extended over the sets T, involving neither $A_{l,q}$ nor A_{l,n_l}; the summation Σ_5 over the sets T, involving one and only one of the two sets $A_{l,q}$ and A_{l,n_l}; finally Σ_6 is extended over the sets T involving both $A_{l,q}$ *and* A_{l,n_l}.

For the sets T, occurring in Σ_4, we have $T = T^*$, hence $\Sigma_4 T(m) = \Sigma_4 T^*(m)$. Each set T, occurring in Σ_6, is a sumset containing both sets $A_{l,q}$ and A_{l,n_l}; by convention, made at the beginning of this section, each of these two sets is counted only once, hence

$$T = A_{l,q} + A_{l,n_l} + U \text{ and } T^* = A^*_{l,q} + A^*_{l,n_l} + U.$$

By lemma 3 (§ 1) $A_{l,q} + A_{l,n_l}$ contains each element $\leqq g$ of $A^*_{l,q} + A^*_{l,n_l}$, so that

$$T(m) \geqq T^*(m), \text{ hence } \Sigma_6 T(m) \geqq \Sigma_6 T^*(m).$$

The sum $\Sigma_5 T(m)$, which is symmetrical in $A_{l,q}$ and A_{l,n_l} can be written as a sum of terms of the form $(U + A_{l,q})(m) + (U + A_{l,n_l})(m)$, and lemma 4 (§ 2) gives for $m = 1, \ldots, g$

$$(U + A_{l,q})(m) + (U + A_{l,n_l})(m) \geqq (U + A^*_{l,q})(m) + (U + A^*_{l,n_l})(m),$$

hence

$$\Sigma_5 T(m) \geqq \Sigma_5 T^*(m).$$

This completes the proof.