Mathematics. — *Distance geometry. Curvature in abstract metric spaces.* By J. HAANTJES. (Communicated by Prof. J. A. SCHOUTEN.)

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The problem of treating the local properties of geometrical figures metrically, i.e. of developing a differential geometry without the use of coordinates, has recently received the attention of some mathematicians. The local properties of curves and surfaces have been studied in differential geometry almost entirely by means of analytic methods, which make it necessary to impose some conditions (e.g. conditions concerning differentiability) upon the entities involved. In a metrical theory the differential geometry can be freed from many of these restrictions, which are often geometrically unessential and serve merely to make the application of differential calculus possible [1]).

In this paper we are concerned with a metric treatment of the local property of curvature for arcs in abstract metric spaces. A metric definition of curvature has been given by MENGER [2]). A somewhat more general notion of curvature is due to ALT [3]). Both definitions impose, however, a euclidean notion of curvature upon general metric spaces (See § 2 and § 3). In the following a definition is given, which is free from this aesthetic imperfection. This definition of curvature proves to be more general than the notion of Menger curvature, though it can be shown that both definitions are equivalent for arcs in euclidean planes.

§ 1. *The abstract metric space.*

A metric space M is a set of abstract elements, called points, such that to each pair of elements p, q is attached a non-negative real number pq, called the *distance* of p and q, satisfying the conditions:

1. $pq = 0$ if and only if $p = q$.
2. $pq + qr \geq pr$ (triangle inequality).

An infinite sequence of points $\{p_\nu\}$ is said to have the limit p if and only if $\lim_{\nu \to \infty} p_\nu p = 0$. From the triangle inequality follows the continuity of the metric, which means that if $\{q_\nu\} \to q$ and $\{p_\nu\} \to p$ $\lim p_\nu q_\nu = pq$.

A neighbourhood $U(p, d)$ of the point p is defined as the set of points q for which $pq < d$.

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1) See L. M. BLUMENTHAL. Distance Geometries, A study of the development of abstract metrics, The University of Missouri studies, Vol. 13, nr 2 (1938).

2) K. MENGER. Untersuchungen über allgemeine Metrik, Vierte Untersuchung. Zur Metrik der Kurven. Math. Annalen **103**, 466—501 (1930). Referred to as: Menger IV.

3) F. ALT. Ueber eine metrische Definition der Krümmung einer Kurve, Vienna Dissertation (1931).

§ 2. *The Menger curvature.*

Because of the triangle inequality each set of three pairwise distinct points q, r, s in M is congruent with three points q', r', s' of the euclidean plane. The inverse of the radius of the circumscribed circle of the triangle with vertices q', r', s' can be expressed in terms of the distances of the points q', r', s', which equal the distances of the points q, r, s in M. This expression is according to a formula of elementary geometry

$$K(q,r,s) = \frac{\sqrt{(qr+rs+sq)(qr+rs-sq)(qr-rs+sq)(-qr+rs+sq)}}{qr \, . \, rs \, . \, sq} \quad (1)$$

and is called the curvature of q, r, s. Let K be a sub-set of M. Then MENGER defines the curvature at an accumulation point p of K as follows [4])

Definition. *The set K has, at an accumulation point p, a curvature $K(p) \geqq 0$ provided that, corresponding to each $\varepsilon > 0$ there is a $\delta > 0$, such that $|K(p) - K(q,r,s)| < \varepsilon$ for every triple q, r, s in the neighbourhood $U(p,\delta)$.*

This curvature is called the *Menger curvature*; throughout this paper it will be denoted by K_M. The relation of K_M with the classical curvature is expressed in [5])

Theorem 1. *If a curve in a euclidean plane has a Menger curvature K_M at a point p, then the classical curvature exists at p and is equal to K_M. The converse is not true.*

The second part of this theorem follows from the fact that the Menger curvature is a continuous function of p, which is not necessarily the case for the classical curvature.

PAUC [6]) answered the question which continua possess Menger curvature by proving

Theorem 2. *If a metric continuum K has a Menger curvature at one of its points p, then K is, in a neighbourhood of p, a rectifiable curve.*

§ 3. *The Alt curvature.*

The existence of a Menger curvature at p requires that $K(q,r,s)$ has a limit as the points q, r, s independently approach to p. ALT takes p as one of these points. This leads to the following

Definition [7]). *The set K has, at an accumulation point p, a curvature $K(p) \geqq 0$ provided that corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that $|K(p) - K(p,r,s)| < \varepsilon$ for every pair of points r, s in the neighbourhood $U(p,\delta)$.*

This curvature is called the *Alt curvature* and is denoted by K_A. It can be shown

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[4]) MENGER IV, p. 480.

[5]) O. HAUPT and F. ALT. Zum Krümmungsbegriff, Ergebnisse eines mathematischen Kolloquiums (Wien) Heft 3 (1932), 4—5.

[6]) C. PAUC. Courbure dans les espaces métriques, Atti Acad. di Lincei, Serie 6, 24 (1936) 109—115.

[7]) F. ALT. l.c. [3]).

Theorem 3. *For euclidean plane curves the existence of classical curvature implies the existence of Alt curvature and the two are equal. The converse is not true, at least when the classical curvature of the curve* $y = f(x)$ *is defined by the expression* $f''(1 + f'^2)^{-3/2}$ [8])

PAUC showed (see [6]))

Theorem 4. *A metric continuum K that has an Alt curvature at one of its points p is, in a neighbourhood of p, either the sum of finitely many rectifiable arcs, which have pairwise only the end-point p in common, or the sum of a denumerable infinity of such arcs, the diameters of which converge to zero.*

We see that both definitions impose a euclidean notion upon general metric spaces. This can however be avoided as will be shown in the next paragraph, where a third definition of curvature is given. This definition applies only to rectifiable arcs, though an extension to other sub-sets could be given. But for continua theorems 2 and 4 show that for Menger curvature and Alt curvature too we may confine ourselves to rectifiable arcs.

§ 4. *The curvature K.*

Let B be a rectifiable arc (topological image of a segment) in a metric space M. Then to each pair of points q, s of B are attached two numbers: the distance $d = \overline{qs}$ and the length l of the arc qs, part of the arc B.

Definition. *The arc B has at a point p a finite curvature* $K(p) \geqq 0$, *provided that, corresponding to each* $\varepsilon > 0$ *there is a* $\delta > 0$, *such that*

$$\left| 4! \left(\frac{l-d}{l^3}\right)_{q,s} - K^2(p) \right| < \varepsilon. \quad \ldots \ldots \quad (2)$$

for every pair of points q, s of B in the neighbourhood $U(p, \delta)$.

In this paper this curvature will be denoted by K. It is natural to ask how this curvature, applied to arcs of spaces where a classical curvature can be defined (Riemannian spaces) compares with the classical curvature. The answer is given in

Theorem 5. *For rectifiable arcs in Riemannian spaces, which defining equations are differentiable to a sufficiently high order, the classical curvature is equal to K.*

In order to prove this theorem we introduce in the n-dimensional Riemannian space V_n polar coordinates with the point P (one of the points of the given arc) as pole. The coordinates of a point Q are determined by the geodesic distance z to P together with the unit vector $i^\varkappa$ at P, which is tangent to the geodesic PQ. The polar coordinates of Q are defined as [9])

$$\xi^h = \delta^h_\varkappa \, i^\varkappa \, z.$$

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[8]) See K. MENGER. La géométrie des distances et ses relations avec les autres branches mathématiques. L'Enseignement Mathématique, Nos 5—6, 348—372 (1936).

[9]) J. A. SCHOUTEN. Einführung in die neueren Methoden der Differentialgeometrie I, Noordhoff, Groningen 1935, p. 101.

In a neighbourhood of P the arc is given by

$$\xi^h = a_1^h s + \frac{1}{2!} a_2^h s^2 + \frac{1}{3!} a_3^h s^3 + \ldots$$

where s is the arc length. The classical curvature ϱ is defined as the length of the vector

$$\frac{d^2 \xi^h}{ds^2} + \Gamma^h_{ij} \frac{d\xi^i}{ds} \cdot \frac{d\xi^j}{ds}$$

at P. As a result of the vanishing of Γ^h_{ij} at P (see SCHOUTEN p. 103) this vector is a_2^k. The distance z to P is given by

$$z^2 = (g_{hi})_P \xi^h \xi^i = s^2 + (a_1 a_2) s^3 + \{\tfrac{1}{3}(a_1 a_3) + \tfrac{1}{4}(a_2 a_2)\} s^4 + \ldots \quad (3)$$

where (uv) stands for $(g_{hi})_P u^h v^i$. Now

$$\frac{d\xi^h}{ds} = a_1^h + a_2^h s + \frac{1}{2!} a_3^h s^3 + \ldots$$

is a unit vector. Using the formula (SCHOUTEN p. 138)

$$g_{hi} = (g_{hi})_P + \tfrac{1}{3} \xi^k \xi^j K_{kijh}$$

it follows

$$(a_1 a_2) = 0; (a_1 a_3) + (a_2 a_2) = 0.$$

So equation (3) gets the form

$$z^2 = s^2 - \tfrac{1}{12} \varrho^2 s^4 + \ldots$$

from which we have — in harmony with (2) —

$$\varrho^2 = 4! \lim_{s \to 0} \frac{s-z}{s^3}$$

We will not investigate here which curves in Riemannian spaces possess a curvature K. Theorem 5 has only been proved to show that the classical curvature in Riemannian spaces is connected in the same way with the same limit as K is; therefore, in Riemannian spaces our definition of curvature cannot lead to paradoxical results.

Theorem 6. *The curvature K is a continuous function of p wherever it exists.*

Let us denote the expression $4!(l-d)\, l^{-3}$ for the pair of points q, r by $K^2(q, r)$. According to the definition of curvature there corresponds to each ε a δ such that

$$|K(p) - K(r, s)| < \frac{\varepsilon}{2} \quad \text{for} \quad r, s \subset U(p, 2\delta). \quad . \quad . \quad . \quad . \quad (4)$$

Let q be a point of the arc B contained in $U(p, \delta)$. There exists a neighbourhood $U(q, \delta_1)$ such that for each pair of arc points r, s contained in this neighbourhood we have

$$|K(q) - K(r, s)| < \frac{\varepsilon}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The intersection of $U(q, \delta_1)$ and $U(p, 2\delta)$ contains an infinity of arc points, q being an accumulation point of B. For two of these points both the inequalities (4) and (5) hold, from which it follows

$$|K(p)-K(q)|<\varepsilon \quad \text{for} \quad q \subset U(p, \delta).$$

This means however that $K(p)$ is a continuous function.

It is natural to investigate how the curvature K compares with the curvatures K_A and K_M. One of the results is given by the following theorem:

Theorem 7. *If both K_A (or K_M) and K exist, these curvatures are equal.*

Consider three points p, q, r of the arc, for which $pq = qr = d$. The points q and r are supposed to lie on the same side of the fixed point p. If the distance pr is denoted by a the following expression for the curvature K_A (or K_M) at p is obtained from (1)

$$K_A^2 = \lim_{d \to 0} \frac{(2d+a)\, a^2\, (2d-a)}{d^4\, a^2} = \lim_{d \to 0} \frac{2d+a}{d} \cdot \frac{2d-a}{d^3} = \lim_{d \to 0} \frac{4}{d^2}\left(1-\frac{a^2}{4d^2}\right)$$

It follows that $a \to 2d$ as d tends to zero and therefore

$$K_A^2 = 4 \lim_{d \to 0} \frac{2d-a}{d^3} \quad \ldots \ldots \ldots \quad (6)$$

Let the lengths of the arcs pq and qr be denoted by l_1 and l_2 respectively. According to the assumption that the curvature K exists at p, we have

$$\lim_{d \to 0} \frac{l_1-d}{l_1^3} = \lim_{d \to 0} \frac{l_2-d}{l_2^3} = \lim_{d \to 0} \frac{l_1+l_2-a}{(l_1+l_2)^3} = \frac{K^2}{4l} \cdot \ \ldots \quad (7)$$

Now

$$\frac{l_1+l_2-a}{(l_1+l_2)^3} = \frac{l_1-d}{l_1^3} \cdot \frac{l_1^3}{(l_1+l_2)^3} + \frac{l_2-d}{l_2^3} \cdot \frac{l_2^3}{(l_1+l_2)^3} + \frac{2d-a}{d^3} \cdot \frac{d^3}{(l_1+l_2)^3} \cdot \quad (8)$$

If it is taken into consideration that $l_1/d \to 1$ and $l_2/d \to 1$ as $d \to 0$ it is seen from (7) and (8)

$$\tfrac{1}{4} K^2 = \lim_{d \to 0} \frac{2d-a}{d^3},$$

which gives for K exactly the same expression as (6) gives for K_A. So K and K_A (or K_M) if both existing are equal.

Further investigation to the relations existing between the curvatures K_M, K_A and K leads to the following theorem:

Theorem 8. *If the curvature K_A of the arc at p exists, it equals $\lim_{q \leftarrow p} K(p, q)$.*

It is easily seen that the preceding theorem 7 is an immediate consequence of theorem 8. In order to prove this latter theorem we consider the triangle in the euclidean plane corresponding to the arc points p, q, r, i.e.

the triangle with the sides pq, qr and pr. Let $e(p, q, r)$ stand for the sum of the smallest two angles of this triangle. PAUC [10]) has shown that if K_A exists at p the quantity $e(p, q, r) \to 0$ as both q and r approach the point p. Thus there is a neighbourhood $U(p, \delta_1)$ such that for each pair q, r contained in this neighbourhood $e(p, q, r) < \frac{\pi}{2}$ which means that the triangle is obtuse-angled. If at the same time two of the sides are equal (b), the length of the third side will exceed $b\sqrt{2}$ (*property P*). By means of this property it can be shown that the distance d of the arc points q ($q \subset U(p, \delta_1)$ to p will increase steadily if q traverses the arc B starting from the point p. For d is a continuous function of q and thus of the parameter on the arc. Suppose this function is not monotonic. Then it has relative maxima or minima and it is possible to find two different points which have the same distance to p but a smaller mutual distance, contrary to property P. Hence this cannot occur. It is equally impossible that d remains constant for a while, so d will increase.

Another consequence of the existence of K_A at p follows immediately from the definition. There is a $\delta_2 > 0$ such that for each pair of arc points q, r in $U(p, \delta_2)$

$$K_A^2(1-\eta) < K^2(p, q, r) < K_A^2(1+\eta). \quad \dots\dots \quad (9)$$

Consider a point q of B in the neighbourhood $U(p, \delta)$, where $\delta < \delta_1$ and $\delta < \delta_2$. From the proof of theorem 4 it follows that the arc pq (part of B) is rectifiable. Let l be its length and d the distance of the end points. A finite subset $(p_1, \dots, p_N)$ of the arc pq (B') is called an ε-subset provided that each pair of two consecutive points has the distance ε and $pp_1 \leq \varepsilon$ and $p_N q \leq \varepsilon$. It has been proved by MENGER [11]) that corresponding to each positive ζ there is a positive number ε_0 such that for every ε-subset ($\varepsilon \leq \varepsilon_0$)

$$l \geqslant pp_1 + \dots + p_{N-1}p_N > l - \zeta. \quad \dots\dots \quad (10)$$

The number N depends of course on ε.

Let r and s be two points of B' such that in traversing B' from p to q the point r is encountered first. Then as we have seen $ps > pr$. Suppose $pr \geq rs$. Putting for shortness $ps = a$, $pr = b$, $rs = c$, $K(p, r, s) = R^{-1}$ we obtain from some formulae of elementary geometry

$$a = b\sqrt{1 - \frac{c^2}{4R^2}} + c\sqrt{1 - \frac{b^2}{4R^2}}. \quad \dots\dots \quad (11)$$

By using the first of the following inequalities

$$\sqrt{1-a} \leqslant 1 - \tfrac{1}{2}a. \quad \dots\dots \quad (12)$$

$$\sqrt{1-a} \geqslant 1 - \tfrac{1}{2}a - a^2 \text{ for } a < \tfrac{1}{2} \quad \dots\dots \quad (13)$$

$$\sqrt{1-a} \geqslant 1 - a \quad \dots\dots \quad (14)$$

[10]) See footnote [6]).

[11]) MENGER IV, p. 469.

it is readily seen from (11)

$$a < b + c - \frac{1}{8R^2} cb^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

whereas the second inequality (13) leads to

$$a > b + c - \frac{1}{8R^2}(bc^2 + cb^2) - \frac{1}{8R^4} cb^4. \; . \; . \; . \; . \quad (16)$$

This it is true is only right if b and c are small enough but we may choose δ smaller than $K_A(1+\eta)$ and then it is correct. Yet another inequality will be used. It is obtained from (11) and (14) and runs as follows

$$a > (b+c)\left(1 - \frac{dc}{4R^2}\right). \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

for b and c are smaller than d, d being the distance pq.

The inequalities (15), (16) and (17) are now applied succesively to the sets pp_ip_{i+1} $(i = 1, \ldots, N-1)$ where the subset p_i is supposed to be an ε-subset (with $pp_1 = \varepsilon$) satisfying (10). Let us start with the third inequality. It will be proved that

$$pp_j > j\,\varepsilon\,\{1 - \tfrac{1}{8}(j+1)K_A^2(1+\eta)\,d\,\varepsilon\}. \quad . \; . \; . \; . \quad (18)$$

This is true for $j = 1$. Suppose it is true for $j = m$ and let us prove it for $j = m+1$. From (17) it follows

$$\begin{aligned} pp_{m+1} &> (pp_m + \varepsilon)\{1 - \tfrac{1}{4}\,d\,\varepsilon K_A^2(1+\eta)\} \\ &> (m+1)\,\varepsilon\{1 - \tfrac{1}{8}\,m\,\varepsilon\,d\,K_A^2(1+\eta)\}\{1 - \tfrac{1}{4}\,\varepsilon\,d\,K_A^2(1+\eta)\} \\ &> (m+1)\,\varepsilon\{1 - \tfrac{1}{8}(m+2)\,\varepsilon\,d\,K_A^2(1+\eta)\} \end{aligned}$$

which is exactly the inequality (18) for $j = m+1$.

In a similar way it can be proved from (15)

$$pp_m \leqq m\,\varepsilon - \tfrac{1}{8}\,\varepsilon^3 K_A^2(1-\eta)\sum_{\nu=1}^{m-1}\nu^2 + \tfrac{1}{64}K_A^4(1+\eta)^2\,d\,\varepsilon^4 f(m) \quad . \quad (19)$$

with $f(m) = \frac{1}{2}m^4 - \frac{1}{3}m^3 - \frac{1}{2}m^2 + \frac{1}{3}m$.

For we know it is true for $m = 1$. Suppose it holds for m. Then it follows from (15) under consideration of the inequality (18)

$$\begin{aligned} pp_{m+1} &< (m+1)\,\varepsilon - \tfrac{1}{8}\,\varepsilon^3 K_A^2(1-\eta)\sum_{\nu=1}^{m-1}\nu^2 + \tfrac{1}{64}K_A^4(1+\eta)^2\,d\,\varepsilon^4 f(m) \\ &\quad - \tfrac{1}{8}K_A^2(1-\eta)\,\varepsilon^3 m^2\{1 - \tfrac{1}{8}(m+1)K_A^2(1+\eta)\,d\,\varepsilon\}^2 \\ &< (m+1)\,\varepsilon - \tfrac{1}{8}\,\varepsilon^3 K_A^2(1-\eta)\sum_{\nu=1}^{m}\nu^2 + \tfrac{1}{64}K_A^4(1+\eta)^2\,d\,\varepsilon^4 f(m+1) \end{aligned}$$

since

$$f(m+1) = f(m) + 2\,m^2(m+1).$$

Application of the inequality (16) to the sets pp_ip_{i+1} leads in much the same way to

$$pp_m \geqq m\,\varepsilon - \tfrac{1}{8}K_A^2(1+\eta)\,\varepsilon^3\sum_{\nu=1}^{m-1}(\nu^2+\nu) - \tfrac{1}{8}K_A^4(1+\eta)^2\,\varepsilon^5\sum_{\nu=1}^{m-1}\nu^4 \quad (20)$$

Again it is true for $m = 1$. If it is true for m we have as a result from (16)

$$pp_{m+1} \geqslant (m+1)\,\varepsilon - \tfrac{1}{8} K_A^2 (1+\eta)\,\varepsilon^3 \sum_{\nu=1}^{m-1} (\nu^2+\nu) - \tfrac{1}{8} K_A^4 (1+\eta)^2 \varepsilon^5 \sum_{\nu=1}^{m-1} \nu^4 +$$
$$- \tfrac{1}{8} K_A^2 (1+\eta)(\varepsilon^3 m + \varepsilon^3 m^2) - \tfrac{1}{8} K_A^4 (1+\eta)^2 \varepsilon^5 m^4$$

since

$$pp_m \leqslant m\,\varepsilon$$

according to the triangle inequality. But this shows that (20) holds still if m is replaced by $m+1$.

We turn back to (19). Putting $m = N$ it gives

$$pp_N \leqslant N\varepsilon - \tfrac{1}{24} K_A^2 (1-\eta)(N\varepsilon)^3 \left\{ 1 - \frac{3}{2N} + \frac{1}{2N^2} \right\} +$$
$$+ 2^{-7} K_A^4 (1+\eta)^2 d\,(N\varepsilon)^4 \left\{ 1 - \frac{2}{3N} - \frac{1}{N^2} + \frac{1}{3N^3} \right\}$$

So we have in connection with (10)

$$d \leqslant pp_N + \varepsilon \leqslant \varepsilon + l - \tfrac{1}{24} K_A^2 (1-\eta)(l-\zeta)^3 \left\{ 1 - \frac{3}{2N} + \frac{1}{2N^2} \right\} +$$
$$+ 2^{-7} K_A^4 (1+\eta)^2 d l^4 \left\{ 1 - \frac{2}{3N} - \frac{1}{N^2} + \frac{1}{3N^3} \right\}$$

Now after fixing the point q, the quantities ζ and ε can be chosen arbitrarily small. Thus we have

$$d \leqslant l - \tfrac{1}{24} K_A^2 (1-\eta)\, l^3 + 2^{-7} K_A^4 (1+\eta)^2 d l^4$$

from which it follows

$$K^2(p,q) = 4!\,\frac{l-d}{l^3} \geqslant K_A^2 (1-\eta) - 2^{-7}\, 4!\, K_A^4 (1+\eta)^2 d l. \quad . \quad (21)$$

Another inequality is obtained from (20) by putting $m = N$. Since

$$\sum_{\nu=1}^{N-1} \nu^4 < N^5 \text{ and } pp_N \leqslant d$$

we have again in connection with (10)

$$d \geqslant l - \zeta - \tfrac{1}{24} K_A^2 (1+\eta)\, l^3 \left(1 - \frac{1}{N^2} \right) - \tfrac{1}{8} K_A^4 (1+\eta)^2 l^5$$

which leads to

$$d \geqslant l - \tfrac{1}{24} K_A^2 (1+\eta)\, l^3 - \tfrac{1}{8} K_A^4 (1+\eta)^2 l^5$$

or

$$K^2(p,q) \leqslant K_A^2 (1+\eta) + 3 K_A^4 (1+\eta)^2 l^2. \quad . \quad . \quad . \quad . \quad (22)$$

From (21) and (22) together it follows that $|K^2(p,q) - K_A^2|$ is smaller than an arbitrarily given ε' provided that pq is sufficiently small ($d < \delta'$), in other words

$$K_A = \lim_{q \to p} K(p,q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which proves theorem 8.

So the existence of the Alt curvature implies the existence of the limit (23). The existence of a curvature K at a point p requires however that the function $K(q, r)$ has a limit as the two points q, r independently approach the point p. This is more than the existence of the limit (23). A curve may possess Alt curvature at a point p without having a curvature K. For according to theorem (6) the curvature $K(p)$ is a continuous function, which is not the case for the Alt curvature. E.g. the plane curve $(y = x^4 \sin \frac{1}{x},\ x \neq 0;\ y = 0, x = 0)$ in the euclidean plane has an Alt curvature, but not a curvature K at the origin.

If however the Menger curvature exists at p the inequalities (9) are not only true for $K^2(p, q, r)$ but even for $K^2(q, r, s)$, providing that the triple q, r, s is contained in $U(p, \delta_2)$. As a consequence of this the inequalities (21) and (22) are true not only for the pair p, q, but for every pair of points q, r contained in $U(p, \delta)$. So we see that the limit of $K(q, r)$ exists if q and r approach p, which means that the curvature K exists and

$$K_M = \lim_{\substack{q \to p \\ r \to p}} K\,(q, r) = K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence we have the theorem

Theorem 9. *The existence of the Menger curvature of an arc implies the existence of the curvature K and the two are equal.*

§ 5. *Further investigations as to how the curvature K compares with K_M.*

In § 4 it has been shown that the existence of K_M implies the existence of the curvature K so far as arcs in metric spaces are concerned.

The converse of this theorem is not necessarily true. A curve may possess a curvature K at a point without having a Menger curvature or even an Alt curvature. We give an example.

Consider a space M formed by the points of the interval $0 \leqq x \leqq 1$, where the distance xy of two points x, y is defined by

$$x\,y = t - \frac{1}{3!}\,t^3 + \frac{1}{4!}\,t^4 \sin\frac{1}{t} \equiv f(t) \qquad t = |x - y|. \quad . \quad . \quad (25)$$

The first thing to show is that M is indeed a metric space, thus that the triangle inequality is satisfied. Let x, y, z be three points of M and $y - x = u,\ z - y = t,\ z - x = u + t\ (u > 0,\ t > 0)$.

Since

$$\frac{d f(t)}{d t} = 1 - \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 \sin\frac{1}{t} - \tfrac{1}{24}\,t^2 \cos\frac{1}{t} > 1 - \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{24} > 0$$

the distance will increase with t. So $xz > xy$ and $xz > yz$ and it is sufficient to show that $xy + yz \geqq xz$. According to (25)

$$xy + yz - xz = \tfrac{1}{2}\,ut(u + t) - \frac{1}{4!}\{\varphi(u + t) - \varphi(u) - \varphi(t)\};\ \varphi(u) = u^4 \sin\frac{1}{u} \quad (26)$$

Suppose $u \leq t$ (this may be supposed because the expression is symmetrical in u and t). Then

$$|\varphi(u+t)-\varphi(t)-\varphi(u)| = |u\varphi'(t+\theta u)-\varphi(u)|$$
$$< 4u(u+t)^3 + u(u+t)^2 + u^4 < (8+2+1)\,ut(u+t)$$

from which it follows that the expression (26) is positive.

The metric of M is topologically equivalent to the euclidean metric, hence M is an *arc*.

Next it will be shown that the arc is rectifiable. Let E' $(p_0 = x, p_1, \ldots, p_N = y)$ be a finite subset of the arc with end points x and y and $|p_{i+1} - p_i| = \varepsilon = t/N,\ t = |x-y|$. Then

$$\lambda(E') \equiv \sum_{i=1}^{N} p_{i-1}\,p_i = t\left(1 - \frac{1}{3!}\varepsilon^2 + \frac{1}{4!}\varepsilon^3 \sin\frac{1}{\varepsilon}\right)$$

which expression converges to t if $\varepsilon \to 0$. Now the length $l(x, y)$ of the arc is defined as the least upper bound of the numbers $\lambda(E)$ if E describes all finite subsets of the arc. So we see $l(x, y) \geq t$ for $\lambda(E') \to t$, but from (25) it is clear that for any finite subset E the number $\lambda(E) < t$. Hence

$$l(x, y) = t.$$

If d denotes the distance of x and y, we have therefore

$$K^2(x, y) \equiv 4!\,\frac{l-d}{l^3} = 4 - t\sin\frac{1}{t}.$$

Hence

$$|K(x, y) - 2| < \eta \quad \text{for} \quad |x-y| < 2\eta.$$

So the curvature K exists at every point of the arc and is equal to 2. The arc has constant curvature.

In order to show that the Alt curvature does not exist we consider again the points x, y and z $(y - x = u,\ z - y = t,\ z - x = u + t)$ and study the function $K(x, y, z)$ as x and z approach the fixed point y. According to (1)

$$K^2(x, y, z) = \frac{(d + d_1 + d_2)(d - d_1 + d_2)(d + d_1 - d_2)(d_1 + d_2 - d)}{d^2 d_1^2 d_2^2}$$

where

$$d_1 = f(u), \quad d_2 = f(t), \quad d = f(u+t).$$

Put

$$p = \frac{d+d_1+d_2}{u+t} = 2 - \frac{1}{3!}\left\{(u+t)^2 + \frac{u^3}{u+t} + \frac{t^3}{u+t}\right\} + \frac{1}{4!}\left\{\frac{\varphi(u+t)+\varphi(u)+\varphi(t)}{u+t}\right\}$$

$$q = \frac{d-d_1+d_2}{t} = 2 - \frac{1}{3!}\{2t^2 + 3u(u+t)\} + \frac{1}{4!}\left\{t^3\sin\frac{1}{t} + \varphi'(u+\theta_1 t)\right\}$$

$$r = \frac{d+d_1-d_2}{u} = 2 - \frac{1}{3!}\{2u^2 + 3t(u+t)\} + \frac{1}{4!}\left\{u^3\sin\frac{1}{u} + \varphi'(t+\theta_2 u)\right\}$$

$$s = \frac{d_1+d_2-d}{ut(u+t)} = \tfrac{1}{2} - \frac{1}{4!}\left\{\frac{\varphi(u+t)-\varphi(u)-\varphi(t)}{ut(u+t)}\right\} = \tfrac{1}{2} - \frac{1}{4!}\psi(u, t).$$

It is easily seen that

$$\lim p = \lim q = \lim r = 2$$

as x and z independently approach to y. Thus the Alt curvature will exist if and only if $\lim s$ exists.

For $u = t$

$$\psi(u, u) = \frac{\varphi(2u) - 2\varphi(u)}{2u^3} = 8u \sin\frac{1}{2u} - u \sin\frac{1}{u}$$

which expression goes to zero if $u \to 0$. For $u^{-1} = 2n\pi$, $t^{-1} = 2n^3\pi$ however

$$\psi(u, t) = -\frac{(u+t)^3}{ut} \sin\frac{2\pi n}{n^2+1} = -\frac{(n^2+1)^3}{2\pi n^5} \sin\frac{2\pi n}{n^2+1}$$

and the limit of the expression for $n \to \infty$ is -1. Hence the limit of $\psi(u, t)$ as u and t independently approach zero does not exist from which it follows that the arc has no Alt curvature. The Menger curvature does not exist either because the existence of the Menger curvature implies the existence of the Alt curvature.

This example shows that the notion of curvature K is more general than the notion of Menger curvature. It may of course be possible that for certain metric spaces the definitions are equivalent. Without giving a set of necessary and sufficient conditions for such spaces it is shown in the following that for arcs in a euclidean plane both definitions are equivalent. Because of theorem (9) it will be sufficient to prove that in this space the existence of the curvature K implies the existence of the Menger curvature. As a first result the following theorem will be proved.

Theorem 10. *An arc in a euclidean space with a curvature K at a point p has a tangent in a neighbourhood of this point.*

Let q and r be two arc points lying on the same side of p such that the arcs pq, qr and pr have the lengths l, l_1 and $l + l_1$ respectively. Suppose further that q and r lie in a neighbourhood $U(p, \delta)$ with the property that for each pair of arc points s, t in $U(p, \delta)$

$$|K^2(r, s) - K^2| < 4!\,\varepsilon \quad \ldots \ldots \quad (27)$$

where ε is an arbitrarily chosen number. Putting $pq = c$, $qr = a$, $pr = b$, we have from (27)

$$\left.\begin{aligned} c &= l - \sigma l^3 + \eta_1 l^3, \qquad \sigma = \frac{K^2}{4!} \\ a &= l_1 - \sigma l_1^3 + \eta_2 l_1^3 \\ b &= l + l_1 - \sigma (l + l_1)^3 + \eta_3 (l + l_1)^3 \end{aligned}\right\} \quad \ldots \ldots \quad (28)$$

with $|\eta_i| < \varepsilon$. Let α denote the angle qpr. Then

$$4 \sin^2 \tfrac{1}{2}\alpha = \frac{(b + a - c)(c + a - b)}{bc} \quad \ldots \ldots \quad (29)$$

Suppose $l_1 \leq l$. In this case

$$b + a - c < 2l + 10\,\varepsilon\, l^3,$$
$$c + a - b < (6\sigma + 10\,\varepsilon) l^3.$$

Hence

$$4\sin^2\tfrac{1}{2}\alpha < \frac{(6\sigma + 10\varepsilon)(2 + 10\varepsilon l^2)l^2}{(1-\sigma l^2-\varepsilon l^2)(1-8\sigma l^2-8\varepsilon l^2)}. \quad . \quad . \quad . \quad (30)$$

As the angle $\frac{1}{2}\alpha$ is acute

$$\tfrac{1}{2}\alpha < \frac{\pi}{2}\sin\tfrac{1}{2}\alpha$$

From this inequality and (30) it is seen that if l is sufficiently small $(l < \lambda)$

$$\alpha < A\,l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where A is a fixed number. Consider the points q_i $(q_0 = q)$ on the arc pq for which the lengths of the arcs pq_i are equal to $2^{-i}l$. If the angle $q_i\,p\,q_{i-1}$ is denoted by α_i we have from (31)

$$\alpha_i < A\,l\,2^{-i}$$

as a result of which the angle q_ipq $(= \varphi_i)$ satisfies the inequality

$$\varphi_i \leqslant \alpha_1 + \alpha_2 + \ldots + \alpha_i < A\,l\left(\tfrac{1}{2} + \tfrac{1}{4} + \ldots + \frac{1}{2^i}\right) < A\,l.$$

Let s be an arbitrarily chosen point on the arc pq. Then s lies on one of the arcs q_iq_{i+1} and so we have

$$\angle s\,p\,q \leqslant \varphi_i + \angle s\,p\,q_i < A\,l\left(\tfrac{1}{2} + \ldots + \frac{1}{2^i} + \frac{1}{2^i}\right) = A\,l.$$

Thus corresponding to each $\xi > 0$ there is a $\varrho > 0$ such that for every pair of points q, s of the arc on the same side of p for which each of the distances pq and ps is less than ϱ, the angle spq is smaller than ξ. In a similar way it can be shown that the angle $spq \to 2\pi$ if s and q approach p from different sides. Hence the tangent at p exists, but not only at p for in the above account we may replace p by any other point in the neighbourhood $U(p, \delta)$, which completes the proof of theorem 10.

In the following we confine ourselves to arcs in a euclidean *plane* and it will be proved

$$\varphi_i = \alpha_1 + \ldots \alpha_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It is sufficient to prove that $\varphi_2 = \alpha_1 + \alpha_2$. As will be shown later in this paper $\alpha_2 < \alpha_1$. So there are only two possibilities $\varphi_2 = \alpha_1 + \alpha_2$ and $\varphi_2 = \alpha_1 - \alpha_2$. Suppose $\varphi_2 = \alpha_1 - \alpha_2$. If a point r traverses the arc from q to q_1, the angle $r\,p\,q$ is a continuous function of the parameter and this function runs from 0 to α_1. So it passes the value $\varphi_2 = \alpha_1 - \alpha_2$. (The angle cannot pass π because it is acute if l is small enough). Let r be the point on the arc $q\,q_1$ for which $\angle r\,p\,q = \varphi_2$. Then the points p_1q_2 and r lie on a straight line and we have

$$(p\,q_2 + q_2\,r - p\,r)(p\,q_2 - q_2\,r + p\,r)(-p\,q_2 + q_2\,r + p\,r) = 0. \quad . \quad (32)$$

Let u be the arc length of pq_2 and t the length of the arc q_2r. Thus $u = \frac{1}{4}l$, $\frac{1}{2}l < t < l$ and

$$p\,q_2 + q_2\,r - p\,r > 3\,\sigma\,u\,t\,(u + t) - \varepsilon\,\{u^3 + t^3 + (u + t)^3\}$$
$$> 18\,\sigma\,u^3 - 190\,\varepsilon\,u^3 > 0$$

if ε has been chosen small enough. It is equally impossible that one of the other factors in (32) vanish. So $\varphi_2 = \alpha_1 - \alpha_2$ cannot occur and there remains only the other possibility $\varphi_2 = \alpha_1 + \alpha_2$.

By substituting the value of a, b and c for $l_1 = l$ given in (28) into the formula (29) it is seen that corresponding to a ε' there is a δ' such that

$$\left|\frac{a^2}{l^2} - 6\,\sigma\right| < \varepsilon' \text{ for } l < \delta'$$

Hence

$$2^{-i}\, l \sqrt{6\,\sigma - \varepsilon'} < a_i < 2^{-i}\, l \sqrt{6\,\sigma + \varepsilon'}$$

from which it follows firstly that (again if ε' is small enough) $a_{i+1} < a_i$ (already used for proving (31)) and secondly that the angle φ between pq and the tangent at p satisfies the inequalities

$$l\sqrt{6\,\sigma - \varepsilon'} < \varphi < l\sqrt{6\,\sigma + \varepsilon'} \quad \ldots\ldots \quad (33)$$

The same is true however for the angle between rs and the tangent at r providing that r and s are contained in $U(p, \delta)$ and the length of the arc rs is smaller than δ'. Let q, r, s be three points in a neighbourhood $U(p, d)$ where d is chosen smaller than δ and so that the length of the arc between two arc points in $U(p, d)$ is smaller than δ'. The lengths of the arcs qr, rs and qs denoting by l_1, l and $l + l_1$ respectively it follows from (33) that the angle $r\,q\,s$ (β) satisfies the inequalities

$$\beta < (l + l_1)\sqrt{6\,\sigma + \varepsilon'} - l_1\sqrt{6\,\sigma - \varepsilon'} < l\sqrt{6\,\sigma + \varepsilon'} + l_1\frac{\sqrt{2}\,\varepsilon'}{\sqrt{6\,\sigma}}$$

$$\beta > l\sqrt{6\,\sigma - \varepsilon'} - l_1\frac{\sqrt{2}\,\varepsilon'}{\sqrt{6\,\sigma}}.$$

Now suppose $l_1 \leq l$ (For $l_1 \geq l$ one proceeds in the same way with the angle $r\,s\,q$).

The inverse of the radius of the circumscribed circle of the triangle with vertices q, r, s is given by

$$K(q, r, s) = \frac{2\sin\beta}{r\,s} = \frac{2\sin\beta}{l}\cdot\frac{l}{r\,s} \quad \ldots\ldots \quad (34)$$

Since $\beta > \sin\beta > \beta - \frac{\beta^3}{3!}$, $K = 2\sqrt{6\sigma}$ (see (28)) it is seen from (34) that

$$\lim K(q, r, s) = K$$

as q, r and s independently approach p. So the Menger curvature exists and equals K. The result is

Theorem 11. *For arcs in a euclidean plane the notion of curvature K is equivalent with the notion of Menger curvature.*

It should be observed that this proof is only valid for a euclidean plane, not for euclidean spaces of higher dimension, though it is natural to conjecture that theorem 11 will be true for every euclidean space, whatever its dimension may be.