Mathematics. — *On the theory of linear integral equations.* VIIIa.
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§ 4. *The system with iterated kernel.*

Defining the linear transformations B_p in $[L_2]^n$ by $B_1 = A$, $B_p = AHB_{p-1}$ $(p = 2, 3, \ldots)$, it is not difficult to see that the iterated transformations K^p and $\widetilde{K}^p$ are given by $K^p = B_p H$ and $\widetilde{K}^p = H^{1/2} B_p H^{1/2}$, while moreover $HB_pH = HB_p^* H$ for $p = 1, 2, \ldots$. These transformations stand therefore in the same relation to each other as the original transformations K and $\widetilde{K}$. Since $\widetilde{K}^p$ has the sequence λ_k^p $(k = 1, 2, \ldots)$ of characteristic values $\neq 0$ with the orthonormal sequence $\{\Psi_k\} = H^{1/2}\{\psi_k\}$ of characteristic elements, it follows that K^p has also the sequence λ_k^p of characteristic values $\neq 0$ with the H-orthonormal sequence $\{\psi_k\}$ of characteristic elements. Defining the matrix-kernels $\| K_{ij}^{(p)}(x, y) \|$ $(p \geq 1)$ by

$$K_{ij}^{(1)}(x, y) = K_{ij}(x, y),$$

$$K_{ij}^{(p)}(x, y) = \Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, K_{qj}^{(p-1)}(z, y)\, dz,$$

the Theorems 2—10 hold therefore for the system of integral equations with matrix-kernel $\| K_{ij}^{(p)}(x, y) \|$, replacing everywhere λ_k by λ_k^p. In particular

$$\Sigma_{j=1}^n \int_\Delta K_{ij}^{(p)}(x, y)\, f^j(y)\, dy \backsim \Sigma_k \lambda_k^p a_k \psi_k^i(x) + q_{(p)}^i(x) \qquad (i = 1, \ldots, n),$$

where $H\{q_{(p)}\} = \{0\}$, and

$$K_{ij}^{(p)}(x, y) - q_{ij}^{(p)}(x, y) \backsim \Sigma_k \lambda_k^p \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n),$$

where $\Sigma_{r=1}^n h_{ir}(x)\, q_{rj}^{(p)}(x, y) = 0$.

We shall show now that, for $p \geq 2$, all functions $q_{(p)}^i(x)$ and $q_{ij}^{(p)}(x, y)$ vanish (For the proof that this is not necessarily true for $p = 1$, even in the simple case that $n = 1$, we refer to III, § 3).

Theorem 11. *For $p \geq 2$, we have*

$$\Sigma_{j=1}^n \int_\Delta K_{ij}^{(p)}(x, y)\, f^j(y)\, dy \backsim \Sigma_k \lambda_k^p a_k \psi_k^i(x) \qquad (i = 1, \ldots, n), \quad (13)$$

$$K_{ij}^{(p)}(x, y) \backsim \Sigma_k \lambda_k^p \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n). \quad . \; . \; . \; (14)$$

Proof. The first part of the theorem follows from I, Theorem 15. To prove the second part, we observe that, by SCHWARZ's inequality,

$$K_{qj}(z, y) - p_{qj}(z, y) \backsim \Sigma_k \lambda_k \psi_k^q(z)\, \overline{\chi_k^j(y)}$$

(cf. (8)) implies

$$\Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, K_{qj}(z, y)\, dz - \Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, p_{qj}(z, y)\, dz \backsim$$

$$\Sigma_k \lambda_k \overline{\chi_k^j(y)} \left(\Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, \psi_k^q(z)\, dz\right),$$

or, since by (10)

$$\Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, p_{qj}(z, y)\, dz = \Sigma_{q, r=1}^n \int_\Delta A_{ir}(x, z)\, h_{rq}(z)\, p_{qj}(z, y)\, dz = 0,$$

$$K_{ij}^{(2)}(x, y) \backsim \Sigma_k \lambda_k^2 \psi_k^i(x)\, \overline{\chi_k^j(y)}.$$

The proof for $p > 2$ follows easily by induction.

Theorem 12. *For $p \geq 2$ we have*

$$\Sigma_{i=1}^n \int_\Delta K_{ii}^{(p)}(x, x)\, dx = \Sigma_k \lambda_k^p.$$

Proof. Denoting the matrix-kernel, corresponding with the transformation B_p, by $\| B_{ij}^{(p)}(x, y) \|$, we have

$$\Sigma_{i=1}^n K_{ii}^{(p)}(x, x) = \Sigma_{i,j=1}^n B_{ij}^{(p)}(x, x)\, h_{ji}(x) =$$

$$\Sigma_{i,j,r=1}^n h_{ri}^{(1/2)}(x)\, B_{ij}^{(p)}(x, x)\, h_{jr}^{(1/2)}(x) = \Sigma_{r=1}^n \widetilde{K}_{rr}^{(p)}(x, x),$$

hence (cf. VII, Theorem 14)

$$\Sigma_{i=1}^n \int_\Delta K_{ii}^{(p)}(x, x)\, dx = \Sigma_{i=1}^n \int_\Delta \widetilde{K}_{ii}^{(p)}(x, x)\, dx = \Sigma_k \lambda_k^p.$$

§ 5. *The case that all $A_{ij}(x, y)$ are continuous in mean and all $h_{ij}(x)$ are continuous.*

We shall suppose now that $A_{ij}(x, y)$ $(i, j = 1, \ldots, n)$ is continuous in mean in $\Delta \times \Delta$, that is, $\int_\Delta |A_{ij}(x, y)|^2\, dy$ is finite for every $x \in \Delta$, $\int_\Delta |A_{ij}(x, y)|^2\, dx$ is finite for every $y \in \Delta$, and

$$\lim_{x_2 \to x_1} \int_\Delta |A_{ij}(x_2, y) - A_{ij}(x_1, y)|^2\, dy =$$

$$\lim_{y_2 \to y_1} \int_\Delta |A_{ij}(x, y_2) - A_{ij}(x, y_1)|^2\, dx = 0,$$

We observe that these conditions imply that $\int_\Delta |A_{ij}(x,y)|^2\,dy$ is a continuous function of x in Δ, and $\int_\Delta |A_{ij}(x,y)|^2\,dx$ a continuous function of y in Δ. Furthermore we shall suppose that $h_{ij}(x)$ $(i,j=1,\ldots,n)$ is continuous in Δ, so that, by Lemma 1, 3°., $h_{ij}^{(1/2)}(x)$ $(i,j=1,\ldots,n)$ is also continuous in Δ. It is not difficult to see that, under these conditions, $K_{ij}(x,y)$ $(i,j=1,\ldots,n)$ and $\widetilde{K}_{ij}(x,y)$ $(i,j=1,\ldots,n)$ are also continuous in mean in $\Delta\times\Delta$. Finally we observe that for every $\{f\}\in[L_2]^n$ the functions $g^i(x)=\Sigma_{j=1}^n \int_\Delta K_{ij}(x,y)\,f^j(y)\,dy$ $(i=1,\ldots,n)$ are continuous in Δ, since

$$\left|\int_\Delta K_{ij}(x_2,y)\,f^j(y)\,dy-\int_\Delta K_{ij}(x_1,y)\,f^j(y)\,dy\right|\leqslant$$
$$\left(\int_\Delta |K_{ij}(x_2,y)-K_{ij}(x_1,y)|^2\,dy\right)^{1/2}\cdot\|f^j\|.$$

As a consequence, all characteristic functionsets $\psi_k^i(x)$ $(i=1,\ldots,n)$ consist of continuous functions.

Theorem 13 (*Expansion Theorem*). *If*

$$a_k=(\{f\},\{\chi_k\})=\Sigma_{i=1}^n\int_\Delta f^i(x)\,\overline{\chi_k^i(x)}\,dx$$

for an arbitrary $\{f\}\in[L_2]^n$, *then*

$$\Sigma_{j=1}^n\int_\Delta K_{ij}(x,y)\,f^j(y)\,dy=\Sigma_k\lambda_k\,a_k\,\psi_k^i(x)+\widetilde{p}^i(x)\qquad(i=1,\ldots,n),\ (15)$$

uniformly in Δ, *where* $\{\widetilde{p}\}=\{\widetilde{p}^1(x),\ldots,\widetilde{p}^n(x)\}$ *consists of continuous functions, and satisfies*

$$\Sigma_{j=1}^n h_{ij}(x)\,\widetilde{p}^j(x)=0$$

for every $x\in\Delta$.

Proof. First of all we prove that $\Sigma_k|a_k|^2$ converges. This follows from

$$a_k=(\{f\},\{\chi_k\})=(\{f\},H^{1/2}\{\Psi_k\})=(H^{1/2}\{f\},\{\Psi_k\}),$$

hence, since the system $\{\Psi_k\}$ is orthonormal, by BESSEL's inequality,

$$\Sigma_k|a_k|^2\leqslant\|H^{1/2}\{f\}\|^2=(H\{f\},\{f\})\leqslant\|H\|\cdot\|\{f\}\|^2.$$

Furthermore, we see that the matrix-kernel $\|D_{ij}(x,y)\|$, corresponding with the transformation $D=AH^{1/2}$, is also continuous in mean, so that $\int_\Delta |D_{ij}(x,y)|^2\,dy$ is a continuous function of x in Δ for $i,j=1,\ldots,n$. Since, consequently, for $i=1,\ldots,n$, and for every $x\in\Delta$,

$$\{d_i\}=\{d_i^1(y),\ldots,d_i^n(y)\},\ \text{where}\ d_i^j(y)=D_{ij}(x,y),$$

belongs to $[L_2]^n$, the inequality (11) holds now for every $x\in\Delta$, which shows

that there exists a constant M, such that $\Sigma_k\lambda_k^2\,|\psi_k^i(x)|^2\leqslant M$ $(i=1,\ldots,n)$. Hence, if $\varepsilon>0$ is given, for k_1 and k_2 sufficiently large,

$$|\Sigma_{k_1}^{k_2}\lambda_k\,a_k\,\psi_k^i(x)|\leqslant(\Sigma_k\lambda_k^2|\psi_k^i(x)|^2)^{1/2}(\Sigma_{k_1}^{k_2}|a_k|^2)^{1/2}<\varepsilon$$

for every $x\in\Delta$. The series $\Sigma_k\lambda_k\,a_k\,\psi_k^i(x)$ of continuous functions converges, therefore, uniformly in Δ, to a continuous sumfunction. Since, by what we have already remarked, $\Sigma_{j=1}^n\int_\Delta K_{ij}(x,y)\,f^j(y)\,dy$ is also continuous, the same holds for the difference $\widetilde{p}^i(x)$. But $\widetilde{p}^i(x)$ is, as follows from (5), almost everywhere in Δ equal to the function $p^i(x)$ occurring in that formula, hence $\Sigma_{j=1}^n h_{ij}(x)\,\widetilde{p}^j(x)=0$ almost everywhere in Δ. The functions $h_{ij}(x)$ $(i,j=1,\ldots,n)$ and $\widetilde{p}^j(x)$ $(j=1,\ldots,n)$ being however continuous, we have $\Sigma_{j=1}^n h_{ij}(x)\,\widetilde{p}^j(x)=0$ for every $x\in\Delta$.

Theorem 14 (*Expansion Theorem for the iterated kernels*). *For* $p\geqslant 2$ *we have*

$$K_{ij}^{(p)}(x,y)=\Sigma_k\lambda_k^p\,\psi_k^i(x)\,\overline{\chi_k^j(y)}\qquad(i,j=1,\ldots,n),$$

uniformly in $\Delta\times\Delta$.

Proof. The Hermitian matrix-kernel $\|\widetilde{K}_{ij}(x,y)\|$ is continuous in mean in $\Delta\times\Delta$; hence, by the Remark in VII, § 4,

$$\widetilde{K}_{qr}^{(2)}(x,y)=\Sigma_k\lambda_k^2\,\Psi_k^q(x)\,\overline{\Psi_k^r(y)}\qquad(q,r=1,\ldots,n),$$

uniformly in $\Delta\times\Delta$. Then also, for $i,j=1,\ldots,n$,

$$\Sigma_{q,r=1}^n h_{iq}^{(1/2)}(x)\,\widetilde{K}_{qr}^{(2)}(x,y)\,h_{rj}^{(1/2)}(y)=\Sigma_{q,r=1}^n h_{iq}^{(1/2)}(x)\,\widetilde{K}_{qr}^{(2)}(x,y)\,\overline{h_{jr}^{(1/2)}(y)}$$
$$=\Sigma_k\lambda_k^2\,\chi_k^i\ x)\,\overline{\chi_k^j(y)},$$

uniformly in $\Delta\times\Delta$. This shows in particular that, for $i=1,\ldots,n$, the series $\Sigma_k\lambda_k^2\,|\chi_k^i(x)|^2$ converges uniformly in Δ.

In the proof of Theorem 13 we have already seen that

$$\Sigma_k\lambda_k^2\,|\psi_k^i(x)|^2\leqslant M\ \ (i=1,\ldots,n),$$

where M does not depend on x; hence, if $\varepsilon>0$ is given,

$$\Sigma_{k=N}|\lambda_k^2\,\psi_k^i(x)\,\overline{\chi_k^j(y)}|\leqslant M^{1/2}(\Sigma_{k=N}\lambda_k^2\,|\chi_k^j(y)|^2)^{1/2}<\varepsilon$$

for sufficiently large N. The series $\Sigma_k\lambda_k^2\,\psi_k^i(x)\,\overline{\chi_k^j(y)}$ of continuous functions converges, therefore, uniformly in $\Delta\times\Delta$, to a continuous sumfunction, which is, by (14), equal to $K_{ij}^{(2)}(x,y)$ almost everywhere in $\Delta\times\Delta$. Since however, as may easily be seen, $K_{ij}^{(p)}(x,y)$ is, for $p\geqslant 2$, continuous in $\Delta\times\Delta$, equality holds everywhere in $\Delta\times\Delta$.

The proof for $p>2$ follows immediately by induction.

§ 6. *Comparison with the results of* J. E. WILKINS.

In this paragraph we shall compare our results with those of WILKINS. As we have already remarked in § 1, our hypotheses a. and c. on $h_{ij}(x)$ and $A_{ij}(x,y)$ are weaker than the corresponding hypotheses in WILKINS' paper. It is only in our § 5, where all $h_{ij}(x)$ are continuous and all $A_{ij}(x,y)$ continuous in mean, that our and WILKINS' hypotheses are comparable.

We shall first consider some of WILKINS' results which are proved by us under the weaker hypotheses a. and c. WILKINS' Theorem 2.2 is the first part of our Theorem 2. WILKINS' Theorem 2.3, stating that if $\chi_1,\ldots,\chi_k$ are linearly independent solutions of $K\chi = AH\chi = \lambda\chi$, then $H\chi,\ldots,H\chi_k$ are linearly independent solutions of $K^*\psi = \lambda\psi$, is contained in the first part of the proof of I, Theorem 12, where the same is proved for general symmetrisable transformations T (not necessarily of the form $T = AH$), if only $Hf = 0$ implies $Tf = 0$. WILKINS' Theorem 2.5 states that if $a_k = (\{f\}, \{\chi_k\}) = (H\{f\}, \{\psi_k\}) = 0$ for $k = 1, 2, \ldots$, then $(H\{f\}, \{g\}) = 0$ for every $\{g\} = K\{g_1\}$, where $\{g_1\} = \{g_1^1(x),\ldots,g_1^n(x)\}$ consists of continuous functions. This continuity is a superfluous condition, and the theorem in question is the latter part of our Corollary to Theorem 5.

We do not find WILKINS' Theorem 3.1 about the zero's of the FREDHOLM determinant $D(\lambda)$ in the present paper.

In his paper WILKINS considers a class of functionsets called by him the class L. This class consists of all functionsets $\{f\} = K\{g\}$, where $\{g\} = \{g^1(x),\ldots,g^n(x)\}$ consists of continuous functions. The first part of his Theorem 5.1 is our formula (6), but his hypothesis that $\{f\} \in L$ is superfluous. The second part of his Theorem 5.1, stating that if $\{f\} \in L$, then $(H\{f\}, \{f\}) = \Sigma_k |a_k|^2$, where $a_k = (\{f\}, \{\chi_k\})$, is an easy consequence of (6). Here it is necessary that $\{f\} = K\{g\}$, but not that $\{g\}$ consists of continuous functions. WILKINS' Theorem 5.2 and Theorem 5.3 together form essentially our Theorem 6. Here again it is not necessary that $\{f\} \in L$.

We shall next compare WILKINS' expansion theorems with ours in § 5. We recall that the hypotheses about $h_{ij}(x)$ and $A_{ij}(x,y)$ are now equivalent. WILKINS' Theorem 4.1 is our Theorem 13 and his Theorem 4.2 is an immediate consequence. In his Theorem 4.1 it is necessary that $\{f\} = K\{g\}$, and in his Theorem 4.2 that $\{f\} = K^2\{g\}$, but in both cases it is not necessary that $\{g\}$ consists of continuous functions.

Finally we shall say a few words about WILKINS' Theorem 7.1. In the terminology of HILBERT space, this theorem runs as follows:

Let H be a bounded, self-adjoint transformation of positive type, and let the self-adjoint transformation M be such that $M^2 = H$. Let, furthermore, the linear, completely continuous transformation A satisfy $MAM = MA^*M$. Denoting the characteristic values $\neq 0$ of $K = AH$ by λ_k and a corresponding H-orthonormal system of characteristic

elements by ψ_k $(k = 1, 2, \ldots)$, we have for every element f of the HILBERT *space*

$$g = AMf = \Sigma_k (Hg, \psi_k)\,\psi_k + p,$$

where $Mp = 0$.

This theorem may easily be proved by the methods of our paper I, using the following facts:

A. $(Hg, \psi_k) = (HAMf, \psi_k) = (MAMf, M\psi_k) = (f, MAH\psi_k) = \lambda_k (f, M\psi_k)$.

B. The self-adjoint transformation MAM has the same sequence λ_k of characteristic values $\neq 0$ as $K = AH$ with the corresponding orthonormal sequence $M\psi_k$ of characteristic elements (Proof as in I, Theorem 18). Hence, by a well-known theorem about self-adjoint, completely continuous transformations, $(h, M\psi_k) = 0$ $(k = 1, 2, \ldots)$ implies $MAMh = 0$.

Starting now with

$$f = \Sigma_k (f, M\psi_k)\, M\psi_k + h,$$

where f is arbitrary and $(h, M\psi_k) = 0$ $(k = 1, 2, \ldots)$, we find by *A*.

$$g = AMf = \Sigma_k \lambda_k (f, M\psi_k)\,\psi_k + AMh = \Sigma_k (Hg, \psi_k)\,\psi_k + AMh,$$

so that only $MAMh = 0$ remains to be proved. This however is a consequence of $(h, M\psi_k) = 0$ $(k = 1, 2, \ldots)$ and *B*.