Mathematics. — *On the minimum determinant and the circumscribed hexagons of a convex domain.* By K. MAHLER. (Communicated by Prof. J. G. VAN DER CORPUT.)

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In his "Diophantische Approximationen", MINKOWSKI gave a simple rule for obtaining the critical lattices of a convex domain by means of the *inscribed* hexagons (see Lemma 2). I study here an analogous method based instead on the *circumscribed* hexagons. In the special case of a convex polygon, a simple rule for finding all critical lattices and the minimum determinant is obtained. I also show the surprising result that the boundary of an irreducible convex domain not a parallelogram has in all points a continuous tangent. Finally the lower bound of $Q(K)$ is evaluated for all convex octagons.

§ 1. Notation.

The same notation as in earlier papers of mine is used [1]. In particular, the determinant of a lattice Λ is called $d(\Lambda)$; $V(K)$ and $\Delta(K)$ are the area and the minimum determinant of a domain K, and $Q(K)$ is the absolute affine invariant

$$Q(K) = \frac{V(K)}{\Delta(K)}.$$

The letter L is used for straight lines not passing through the origin $O = (0, 0)$, and $-L$ is then the line symmetrical to L in O.

All domains K considered in this paper are assumed to be symmetrical in O; the boundary of K is called C. A convex polygon of $2n$ sides and symmetrical in O will be denoted by Π_n, its boundary by Γ_n. The indices of its vertices P_k and its sides L_k are always chosen in such a way that if Γ_n is described in positive direction, then the successive vertices are

$$Q_1, Q_2, \ldots, Q_n, Q_{n+1} = -Q_1, Q_{n+2} = -Q_2, \ldots, Q_{2n} = -Q_n,$$

and the successive sides are

$$L_1 = Q_1 Q_2, \quad L_2 = Q_2 Q_3, \ldots, L_n = Q_n Q_{n+1},$$

$$L_{n+1} = Q_{n+1} Q_{n+2} = -L_1, \; L_{n+2} = Q_{n+2} Q_{n+3} = -L_2, \ldots,$$
$$L_{2n} = Q_{2n} Q_1 = -L_n.$$

§ 2. Basic lemmas.

The following lemmas are essential for our investigations.

[1] See, e.g. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 50, 98—107 and 108—118 (1947). These two papers will be quoted as A and B, respectively.

Lemma 1: *Let K be a convex domain; let $\mp P_1$, $\mp P_2$, $\mp P_3$ be six points on C such that $P_1 + P_3 = P_2$, and let Λ be the lattice generated by P_1 and P_2. Then Λ is K-admissible.*

Proof: Obvious from the convexity.

Lemma 2: *Let Λ be any critical lattice of the convex domain K. Then Λ contains three points P_1, P_2, P_3 on C such that, (i) P_1, P_2 is a basis of Λ, and (ii) $OP_1P_2P_3$ is a parallelogram of area $d(\Lambda) = \Delta(K)$. Conversely, if P_1, P_2, P_3 are three points on C such that $OP_1P_2P_3$ is a parallelogram, then the area of this parallelogram is not less than $\Delta(K)$, and it is equal to $\Delta(K)$ if and only if the lattice of basis P_1, P_2 is critical* [2].

Lemma 3: *The convex domain K is irreducible if and only if every boundary point of K belongs to a critical lattice of K* [3].

Lemma 4: *For every parallelogram Π_2,*

$$\Delta(\Pi_2) = \tfrac{1}{4} V(\Pi_2), \quad Q(\Pi_2) = 4.$$

Moreover, every such parallelogram is an irreducible domain [4].

Lemma 5: *For every convex hexagon Π_3,*

$$\Delta(\Pi_3) = \tfrac{1}{4} V(\Pi_3), \quad Q(\Pi_3) = 4.$$

Moreover, every such hexagon has only one critical lattice, and this lattice has just six points on Γ_3, viz. the midpoints of the six sides of Π_3 [5].

§ 3. Two formulae for $\Delta(K)$.

Let K be a convex domain symmetrical in O. From Lemma 2, we immediately obtain the formula

$$\text{(I):} \qquad \Delta(K) = \tfrac{1}{3} \operatorname*{fin\,inf}_{h \in I_K} V(h)$$

for $\Delta(K)$; here I_K denotes the set of all hexagons h which have their six vertices $\mp P_1$, $\mp P_2$, $\mp P_3$ on the boundary C of K and for which

$$P_1 + P_3 = P_2.$$

For this relation implies evidently that

$$V(h) = 3V(p)$$

[2] This is Lemma 3 of paper A.

[3] See Lemmas 8 and 12 of paper A.

[4] The first part of the assertion is equivalent to MINKOWSKI's theorem on linear forms; for the second part see Lemma 1 of paper A.

[5] The assertion follows from the fact that the whole plane can be covered in just one way without overlapping by means of hexagons congruent to Π_3; see paper B, § 7.

An entirely different result holds for non-convex hexagonal star domains Π_3 symmetrical in O, viz.

$$\Delta(\Pi_3) = \tfrac{1}{4} V(\Pi_2), \quad Q(\Pi_3) > 4;$$

here Π_2 is the inscribed parallelogram of maximum area. There are an infinity of critical lattices, and every critical lattice has points only on four of the sides of Π_3.

where p is the parallelogram $OP_1P_2P_3$. Since in (I) the lower bound is attained, it is allowed to replace "*fin inf*" by the sign "*min*".

The following theorem gives a formula analogous to (I) but involving the circumscribed hexagons.

Theorem 1: *Let K be an convex domain symmetrical in O, and let U_K be the set of all hexagons H bounded by any three pairs of tac-lines $\mp L_1, \mp L_2, \mp L_3$ of K* [6]. *Then*

$$\text{(II)}: \qquad \triangle(K) = \tfrac{1}{4} \operatorname*{fin\,inf}_{H \varepsilon U_K} V(H).$$

Proof: By the Lemmas 4 and 5, since K is a subset of every hexagon H,

$$\triangle(K) \leqq \triangle(H) = \tfrac{1}{4} V(H),$$

hence

$$\triangle(K) \leqq \tfrac{1}{4} \operatorname*{fin\,inf}_{H \varepsilon U_K} V(H). \quad \dots\dots\dots \quad (1)$$

Next choose any critical lattice Λ of K, and denote by $\mp P_1, \mp P_2, \mp P_3$, where $P_1 + P_3 = P_2$, its points on C (Lemma 2), and by $\mp L_1, \mp L_2, \mp L_3$ three pairs of symmetrical tac-lines of K at these points. The hexagon H bounded by these tac-lines is convex; hence, by Lemma 1, Λ is H-admissible, and so by Lemmas 4 and 5,

$$\triangle(K) = d(\Lambda) \geqq \triangle(H) = \tfrac{1}{4} V(H). \quad \dots\dots \quad (2)$$

Since H belongs to U_K, the assertion follows from (1) and (2).

By this proof, the lower bound is attained also in (II); hence the sign "*fin inf*" may also in this formula be replaced by the sign "*min*".

§ 4. Properties of critical lattices.

The two formulae (1) and (2) of the last paragraph imply that

$$V(H) = 4\,\triangle(K) \quad \dots\dots\dots \quad (3)$$

for every hexagon H belonging to a critical lattice. Hence we find:

Theorem 2: *Let K be a convex domain symmetrical in O which is not a parallelogram; let Λ be any critical lattice of K; and let $\mp P_1, \mp P_2, \mp P_3$, where $P_1 + P_3 = P_2$, be the points of Λ on C. Then,* (i) *there are unique tac-lines $\mp L_1, \mp L_2, \mp L_3$ of K at these points* [7]; (ii) *no two of these tac-lines coincide;* (iii) *the hexagon H bounded by the tac-lines is of area $V(H) = 4\triangle(K)$;* (iv) *each side $\mp L_k$ of H is bisected at the lattice point $\mp P_k$ where it meets and touches C.*

Proof. The notation can be chosen such that when C is described in positive direction, then the six lattice points follow one another in the sequence

$$P_1, P_2, P_3, P_4 = -P_1, P_5 = -P_2, P_6 = -P_3.$$

[6]) Parallelograms are considered as limiting cases of hexagons and must be included in U_K.

[7]) These tac-lines are therefore tangents of C.

Since K is not a parallelogram, none of the six arcs

$$\widehat{P_1P_2},\ \widehat{P_2P_3},\ \widehat{P_3P_4},\ \widehat{P_4P_5},\ \widehat{P_5P_6},\ \widehat{P_6P_1}$$

of C is a line segment [8]), and so (ii) is true. Hence H is a proper hexagon, and the tac-lines L_1 at P_1 and L_3 at P_3 are not parallel or coincident. Assume there is more than one tac-line L_2 at P_2; then this tac-line can vary over a whole angle, and so $V(H)$ is also variable and not constant, contrary to (3). Therefore the assumption is false and (i) is true. The assertion (iii) is identical to (3); from it, Λ must be a critical lattice of H, and so (iv) follows at once from Lemma 5.

One consequence of Theorem 2 is of particular interest:

Theorem 3: *Let K be an irreducible convex domain symmetrical in O which is not a parallelogram. Then the boundary C of K has everywhere a continuous tangent.*

Proof: Obvious from Lemma 3 and the last theorem.

This theorem is rather surprising, since the boundary of non-convex irreducible star domains may have angular points.

§ 5. An inequality property of convex domains.

Theorem 4: *To every convex domain K symmetrical in O, there exist an inscribed hexagon h and a circumscribed hexagon H both symmetrical in O such that*

$$4V(h) = 3V(H).$$

Proof: Obvious from (I) and (II), since the bounds are attained.

We deduce that if h runs over all inscribed symmetrical hexagons and H over all circumscribed symmetrical hexagons, then

$$4 \text{ fin sup } V(h) \geqq 3 \text{ fin inf } V(H);$$

and here the ratio $^4/_3$ of the constants can not be replaced by a smaller one, as the example of the ellipse shows [8a]).

§ 6. The case of a polygon.

Let Π_n be a convex polygon of $2n$ sides $\mp L_1, \mp L_2, \dots, \mp L_n$ where $n \geqq 3$, and let $H_{\alpha\beta\gamma}$ be the proper hexagon bounded by $\mp L_\alpha, \mp L_\beta, \mp L$ where α, β, γ run over all systems of three different indices $1, 2, \dots, n$. The number of such hexagons is thus

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

Theorem 5: *If Π_n is a polygon of $2n \geqq 6$ sides symmetrical in O, then*

$$\text{(III)}: \qquad \triangle(\Pi_n) = \tfrac{1}{4} \min_{\alpha,\beta,\gamma} V(H_{\alpha\beta\gamma}).$$

Every critical lattice of Π_n is also a critical lattice of at least one hexagon $H_{\alpha\beta\gamma}$; hence Π_n has at most $\binom{n}{3}$ different critical lattices.

[8]) See paper A, Lemma 5.

[8a]) Theorem 4 is a special case of a more general result of L. FEJES, Compositio Mathematica **6**, 456—467 (1939), § 3.

Proof: Analogous to that of Theorem 1, except that U_K is replaced by the set of all hexagons $H_{\alpha\beta\gamma}$.

The upper bound $\binom{n}{3}$ for the number of critical lattices of Π_n is attained for $n = 3$ and $n = 4$, but not for larger n; it would therefore be of interest to find then the exact upper bound for this number.

§ 7. **The constants $\mathbf{Q}$ and $\mathbf{Q}_n$.**

The lower bound

$$\mathbf{Q} = \text{fin inf } Q(K)$$

extended over all convex domains symmetrical in O exists and satisfies the inequalities [9])

$$\sqrt{12} < \mathbf{Q} < \frac{2\pi}{\sqrt{3}}. \quad \dots \dots \dots \quad (4)$$

Moreover, there exist convex domains for which this bound is attained; they are called extreme domains.

Let, similarly, $\mathbf{Q}_n$ denote the lower bound

$$\mathbf{Q}_n = \text{fin inf } Q(\Pi_n)$$

extended over all convex polygons Π_n of $2n \geq 4$ sides. It is evident that this limit exists and that $\mathbf{Q}_n \geq \mathbf{Q}$. From Lemmas 4 and 5,

$$\mathbf{Q}_2 = \mathbf{Q}_3 = 4.$$

We call Π_n extreme if

$$Q(\Pi_n) = \mathbf{Q}_n.$$

§ 8. **The existence of extreme polygons Π_n.**

Theorem 6: *If $n \geq 3$, then there exists to every given polygon Π_n of $2n$ sides a polygon Π_{n+1} of $2(n+1)$ sides such that*

$$Q(\Pi_{n+1}) < Q(\Pi_n).$$

Proof: From Lemma 3 and any one of the Theorems 1, 3, or 5, every polygon not a parallelogram is reducible. Hence Π_n contains a convex domain K symmetrical in O and satisfying

$$V(K) < V(\Pi_n), \quad \Delta(K) = \Delta(\Pi_n).$$

At least one pair of vertices of Π_n, say the vertices $\mp Q_1$, lie outside K. Therefore there exist a pair of symmetrical tac-lines $\mp L$ of K such that L separates Q_1 and $-L$ separates $-Q_1$ from O, while all the other vertices of Π_n lie between these two lines. Denote by Π_{n+1} the set of all points of Π_n lying between L and $-L$. Then Π_{n+1} is a proper polygon of $2(n+1)$ sides, and from the construction

$$V(\Pi_{n+1}) < V(\Pi_n), \qquad \Delta(\Pi_{n+1}) \geq \Delta(\Pi_n),$$

hence

$$Q(\Pi_{n+1}) = \frac{V(\Pi_{n+1})}{\Delta(\Pi_{n+1})} < \frac{V(\Pi_n)}{\Delta(\Pi_n)} = Q(\Pi_n),$$

as asserted.

[9]) See paper B, §§ 1 and 5.

Theorem 7: *For every $n \geq 2$, there exists a polygon Π_n such that*

$$Q(\Pi_n) = \mathbf{Q}_n,$$

and this polygon is a proper $2n$-side.

Proof: There exists an infinite sequence of polygons

$$\Pi_n^{(1)}, \quad \Pi_n^{(2)}, \quad \Pi_n^{(3)}, \quad \dots \dots \dots \quad (5)$$

satisfying

$$\lim_{r\to\infty} Q(\Pi_n^{(r)}) = \mathbf{Q}_n.$$

By affine invariance, these polygons may be assumed to satisfy the two conditions,

(*a*): $$Q(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2} \qquad (r = 1, 2, 3, \dots).$$

(*b*): The six fixed points

$$p_1 = (1, 0), \qquad p_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad p_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$p_4 = -p_1, \qquad p_5 = -p_2, \qquad p_6 = -p_3$$

lie on the boundary of each polygon $\Pi_n^{(r)}$.

Denote by H the regular hexagon of vertices $p_1, \dots, p_6$, and by S the figure consisting of six equilateral triangles of unit side, where each such triangle has its base on one of the sides of H, while its opposite vertex lies outside H. From (*b*) and from the assumed convexity, all $2n$ vertices of each polygon $\Pi_n^{(r)}$ belong to the finite set S. It is therefore possible to select an infinite subsequence

$$\Pi_{n,1} = \Pi_n^{(r_1)}, \quad \Pi_{n,2} = \Pi_n^{(r_2)}, \quad \Pi_{n,3} = \Pi_n^{(r_3)}, \dots \qquad (r_1 < r_2 < r_3 < \dots)$$

of (5) such that the vertices of these polygons tend to $2n$ limiting points,

$$\mp Q_1, \pm Q_2, \dots, \mp Q_n, \quad \text{say}.$$

Let Π_n be the polygon which has these points as its vertices. Then by the continuity of V and Δ,

$$\Delta(\Pi_n) = \lim_{r\to\infty} \Delta(\Pi_{n,r}) = \lim_{r\to\infty} \Delta(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2},$$

hence

$$V(\Pi_n) = \lim_{r\to\infty} V(\Pi_{n,r}) = \lim_{r\to\infty} V(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2} \lim_{r\to\infty} Q(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2}\mathbf{Q}_n,$$

whence

$$Q(\Pi_n) = \mathbf{Q}_n,$$

so that Π_n is an extreme polygon. This implies that Π_n is a proper $2n$-side, since it would otherwise be possible, by Theorem 6, to inscribe a polygon Π_n^* of at most $2n$-sides for which

$$Q(\Pi_n^*) < Q(\Pi_n) = \mathbf{Q}_n,$$

contrary to the definition of $\mathbf{Q}_n$.

§ 9. **Properties of the constants Q and Q_n.**

Theorem 8: *The constants Q_n and Q satisfy the relations,*

$$4 = Q_2 = Q_3 > Q_4 > Q_5 > \ldots > Q,$$
$$\lim_{n \to \infty} Q_n = Q.$$

Proof: The inequalities $Q_n > Q_{n+1}$ for $n \geq 3$ follow at once from the last two theorems. The further inequality $Q_n > Q$ holds since every polygon which is not a parallelogram is reducible. Finally, for the proof of the limit formula, denote by K any extreme convex domain, so that

$$Q(K) = Q.$$

Given $\varepsilon > 0$, it is possible to approximate to K by a polygon Π_n of sufficiently large n such that

$$V(\Pi_n) < (1+\varepsilon)\, V(K), \; \Delta(\Pi_n) \geq \Delta(K),$$

hence

$$Q(\Pi_n) < (1+\varepsilon)\, Q(K) = (1+\varepsilon)\, Q.$$

On allowing ε to end to zero, the assertion becomes obvious.

§ 10. **The triangles T_k belonging to an extreme octagon.**

The preceding results enable us to determine the extreme octagons Π_4 and to evaluate the constant Q_4, as follows.

Let Π_4 be a fixed extreme octagon; for its vertices and sides, we use the notation of § 1, and we denote by k one of the four indices 1, 2, 3, 4.

On omitting the pair of sides $\mp L_k$ of Π_4, the remaining sides

$$\mp L_h, \text{ where } h \neq k, \; 1 \leq h \leq 4,$$

form the boundary of a hexagon, H_k say. This hexagon contains Π_4 as a subset and is, in fact, the sumset of Π_4 and two triangles T_k and $-T_k$ symmetrical to one another in O. Let T_k be that triangle with its base on L_k, and $-T_k$ the triangle with its base on $-L_k$. Then

$$V(H_k) = V(\Pi_4) + 2V(T_k),$$

whence by Theorem 5,

$$\Delta(\Pi_4) = \tfrac{1}{4} V(\Pi_k) + \tfrac{1}{2} \min_{1 \leq k \leq 4} V(T_k).$$

Therefore,

$$Q(\Pi_4)^{-1} = \tfrac{1}{4} + \tfrac{1}{2} M(\Pi_4), \text{ where } M(\Pi_4) = \min_{1 \leq k \leq 4} \frac{V(T_k)}{V(\Pi_4)}. \quad . \quad (6)$$

For an extreme octagon, $M(\Pi_4)$ evidently assumes its largest value.

Theorem 9: *If Π_4 is an extreme octagon, then*

$$V(T_1) = V(T_2) = V(T_3) = V(T_4).$$

Proof: It suffices to show that if these equations are not all satisfied, then there exists an octagon Π_4^* satisfying

$$M(\Pi_4^*) > M(\Pi_4). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

We may assume, without loss of generality, that T_2 is the triangle of smallest area and that, say,

$$V(T_1) \geq V(T_2), \qquad V(T_3) > V(T_2). \; . \; . \; . \; . \; . \quad (8)$$

The line L_2 intersects L_1 at the vertex Q_2 of Π_4, and it intersects $-L_4$ at a point R_1 which is a vertex of T_1. Denote by Q_2^* an *inner* point of the line segment Q_1Q_2, and by R_1^* the point on $-L_4$ near to R_1 for which the triangle $T_1^* = Q_1R_1^*Q_2^*$ is of equal area to T_1:

$$V(T_1^*) = V(T_1). \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Let further L_2^* be the line through Q_2^* and R_1^*, and let Π_4^* be the octagon bounded by the sides $\mp L_1$, $\mp L_2^*$, $\mp L_3$, $\mp L_4$. Then, firstly,

$$V(\Pi_4^*) < V(\Pi_4), \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

since Π_4^* is contained in Π_4. Next let T_1^*, T_2^*, T_3^*, T_4^* be the triangles analogous to T_1, T_2, T_3, T_4 which belong to Π_4^*, and assume that Q_2^* is chosen sufficiently near to Q_2. Then $V(T_3^*)$ differs arbitrarily little from $V(T_3)$; further, from the construction,

$$V(T_2^*) > (T_2), \qquad V(T_3^*) < V(T_3), \qquad V(T_4^*) = V(T_4), \quad . \quad (11)$$

the last formulae holding since T_4^* and T_4 are the same triangle. On combining (8), (9), and (11), secondly,

$$\min_{1 \leq k \leq 4} V(T_k^*) \geq \min_{1 \leq k \leq 4} V(T_k). \; . \; . \; . \; . \; . \; . \quad (12)$$

The assertion (7) follows now immediately from (6), (10), and (12).

§ 11. **Determination of the extreme octagons.**

We determine now the octagons Π_4 for which

$$V(T_1) = V(T_2) = V(T_3) = V(T_4), \; . \; . \; . \; . \; . \quad (13)$$

and select from among these the extreme ones. Since $M(\Pi_4)$ is an affine invariant, it suffices to consider octagons which are normed in the following way:

Denote by R_1, R_2, R_3, R_4 the points of intersection of $-L_4$ and L_2, L_1 and L_3, L_2 and L_4, and L_3 and $-L_1$, respectively, and by $\Pi_2^{(1)}$ the parallelogram of vertices $\mp R_1$, $\mp R_3$, and by $\Pi_2^{(2)}$ the parallelogram of vertices $\mp R_2$, $\mp R_4$. Hence $\Pi_2^{(1)}$ has the sides $\mp L_2$, $\mp L_4$, and $\Pi_2^{(2)}$ has the sides $\mp L_1$, $\mp L_3$, and Π_4 is the intersection of $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$. Apply an affine transformation such that $\Pi_2^{(1)}$ becomes the square of vertices

$$R_1 = (1, -1), \; R_3 = (1, 1), -R_1, -R_3.$$

The second parallelogram $\Pi_2^{(2)}$ is then subject only to the conditions that its sides intersect those of $\Pi_2^{(1)}$ so as to form together a convex octagon Π_4. Let the sides of $\Pi_2^{(2)}$ be, say,

$$L_1:\; x_2 = t x_1 - \tau; \quad L_3:\; x_2 = -s x_1 + \sigma;$$
$$-L_1:\; x_2 = t x_1 + \tau; \quad -L_3:\; x_2 = -s x_1 - \sigma;$$

its vertices are therefore

$$R_2=\left(\frac{\sigma+\tau}{s+t},\ \frac{\sigma t-s\tau}{s+t}\right),\quad R_4=\left(\frac{\sigma-\tau}{s+t},\ \frac{\sigma t+s\tau}{s+t}\right),\ -R_2,\ -R_4.$$

On intersecting the sides of $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$, the vertices of Π_4 become,

$$Q_1=\left(\frac{\tau-1}{t},\ -1\right);\quad Q_2=(1,\ t-\tau);\quad Q_3=(1,\ -s+\sigma);\quad Q_4=\left(\frac{\sigma-1}{s},\ 1\right);$$
$$-Q_1,\ -Q_2,\ -Q_3,\ -Q_4.$$

From the construction, L_1 is of positive and L_3 of negative gradient, and these lines meet the coordinate axes outside $\Pi_2^{(1)}$; hence

$$s>0,\ t>0,\ \sigma>1,\ \tau>1. \qquad (14)$$

The conditions that the four points R_1, Q_2, Q_3, R_3 on L_2, and the four points R_3, Q_4, $-Q_1$, $-R_1$ on L_4, follow one another in this order, give the further inequalities,

$$\xi>0,\quad \eta>0,\quad \xi+\eta<2,\quad 2st-t\xi-s\eta>0, \qquad (15)$$

where ξ and η are defined by

$$\xi=s-\sigma+1,\qquad \eta=t-\tau+1.$$

The areas of the triangles T_k are easily obtained; on substituting in (13), these equations take the form,

$$2\,V(T_k)=\frac{\xi^2}{s}=\frac{\eta^2}{t}=\frac{(2-\xi-\eta)^2}{s+t}=\frac{(2st-t\xi-s\eta)^2}{st(s+t)},\qquad =\frac{1}{\lambda}\text{ say,}$$

where, from (14) and (15), λ is positive; hence

$$s=\lambda\xi^2,\ t=\lambda\eta^2,\ s+t=\lambda(2-\xi-\eta)^2,\ st(s+t)=\lambda(2st-t\xi-s\eta)^2.$$

From these equations, firstly

$$\xi^2+\eta^2=(2-\xi-\eta)^2,\quad\text{hence}\quad 2-\xi-\eta=\xi+\eta-\xi\eta, \qquad (16)$$

and secondly,

$$\lambda^3\xi^2\eta^2(2-\xi-\eta)^2=\lambda(2st-t\xi-s\eta)^2=\lambda^3\xi^2\eta^2(2\lambda\xi\eta-\xi-\eta)^2,$$

whence, from (15),

$$2-\xi-\eta=\mp(2\lambda\xi\eta-\xi-\eta),$$

and so, either

(A): $\qquad 2-\xi-\eta=+(2\lambda\xi\eta-\xi-\eta),\qquad \lambda=\dfrac{1}{\xi\eta},$

or

(B): $\qquad 2-\xi-\eta=\xi+\eta-\xi\eta=-(2\lambda\xi\eta-\xi-\eta),\qquad \lambda=\tfrac{1}{2}.$

In case (A),

$$s=\frac{\xi}{\eta},\quad t=\frac{\eta}{\xi},\quad st=1,$$

so that adjacent sides of $\Pi_2^{(2)}$ are perpendicular; hence $\Pi_2^{(2)}$ is a rectangle. It is even a square congruent to $\Pi_2^{(1)}$, since the distances

$$\delta_1=+\tau(1+t^2)^{-\frac12}=\left|\left(\frac{\eta}{\xi}-\eta+1\right)\left(1+\frac{\eta^2}{\xi^2}\right)^{-\frac12}\right|=\left|\frac{\xi\eta-\xi-\eta}{\sqrt{\xi^2+\eta^2}}\right|,$$

$$\delta_3=+\sigma(1+s^2)^{-\frac12}=\left|\left(\frac{\xi}{\eta}-\xi+1\right)\left(1+\frac{\xi^2}{\eta^2}\right)^{-\frac12}\right|=\left|\frac{\xi\eta-\xi-\eta}{\sqrt{\xi^2+\eta^2}}\right|,$$

of L_1 and L_3 from O are both equal to unity, as follows from (16). The four triangles T_k are therefore congruent and of area

$$V(T_k)=\frac{\xi^2}{2s}=\frac{\xi\eta}{2}.$$

Further

$$V(\Pi_4)=V(\Pi_2^{(1)})-4\,V(T_k)=4-2\xi\eta;$$

hence

$$M(\Pi_4)=\frac{\xi\eta}{4(2-\xi\eta)}$$

is an increasing function of $\xi\eta$. By (15) and (16),

$$\xi>0,\quad \eta>0,\quad \xi+\eta<2,\quad (2-\xi)(2-\eta)=2,$$

and so $M(\Pi_4)$ attains its maximum when

$$\xi=\eta=2-\sqrt{2},\quad \xi\eta=6-4\sqrt{2},\quad s=t=1,\quad \sigma=\tau=\sqrt{2},$$

that is, when Π_4 is a regular octagon. For such an octagon,

$$M(\Pi_4)=\frac{6-4\sqrt{2}}{4(4\sqrt{2}-4)}=\frac{\sqrt{2}-1}{8},\qquad Q(\Pi_4)=\left\{\frac{1+2M(\Pi_4)}{4}\right\}^{-1}=\frac{16}{7}(3-\sqrt{2}).$$

Next, in case (B),

$$s=\frac{\xi^2}{2},\quad t=\frac{\eta^2}{2},$$

whence from (15) and (16),

$$\xi\eta>0,\ \xi+\eta<2,\ 2st-t\xi-s\eta=\frac{\xi\eta}{2}(\xi\eta-\xi-\eta)=\frac{\xi\eta}{2}(\xi+\eta-2)>0,$$

which is impossible; this case therefore cannot arise.

We have thus proved [10])

Theorem 10: *For every convex octagon Π_4 symmetrical in O,*

$$Q(\Pi_4)\geqslant\frac{16}{7}(3-\sqrt{2}),$$

with equality if and only if Π_4 is affine-equivalent to the regular octagon.

[10]) Dr. LEDERMANN, to whom I showed this paper, has since found a much simpler proof of Theorem 10.

§ 12. An upper bound for Q.

The last theorem implies that

$$Q_4 = \frac{16}{7}(3-\sqrt{2}) = 3.624654715\ldots \text{ [11]}.$$

This result is rather surprising, since in the case of an ellipse E [12]

$$Q(E) = \frac{2\pi}{\sqrt{3}} = 3.627598727\ldots > Q_4.$$

As we show now, one can construct an irreducible convex domain K for which $Q(K)$ is even smaller.

Let again Π_4 be the regular octagon which is the intersection of the square $\Pi_2^{(1)}$ of vertices

$$R_1 = (1, -1),\ R_3 = (1, 1),\ -R_1,\ -R_3,$$

and the square $\Pi_2^{(2)}$ of vertices

$$R_2 = (\sqrt{2}, 0),\quad R_4 = (0, \sqrt{2}),\ -R_2,\ -R_4.$$

The vertices of Π_4 itself are

$$Q_1 = (\sqrt{2}-1, -1),\quad Q_2 = (1, 1-\sqrt{2}),\quad Q_3 = (1, \sqrt{2}-1),\quad Q_4 = (\sqrt{2}-1, 1),$$
$$-Q_1,\ -Q_2,\ -Q_3,\ -Q_4,$$

and further

$$V(\Pi_4) = 8(\sqrt{2}-1),\quad \triangle(\Pi_4) = \sqrt{2}-\frac{1}{2},\quad Q(\Pi_4) = \frac{16}{7}(3-\sqrt{2}). \quad (17)$$

There are four hexagons H_k circumscribed to Π_4, namely,

the hexagon H_1 of vertices R_1, Q_3, Q_4, $-R_1$, $-Q_3$, $-Q_4$;
the hexagon H_2 of vertices R_2, Q_4, $-Q_1$, $-R_2$, $-Q_4$, Q_1;
the hexagon H_3 of vertices R_3, $-Q_1$, $-Q_2$, $-R_3$, Q_1, Q_2;
the hexagon H_4 of vertices R_4, $-Q_2$, $-Q_3$, $-R_4$, Q_2, Q_3.

Each hexagon H_k possesses just one critical lattice Λ_k, and this is also a critical lattice of Π_4. On the boundary of Π_4, Λ_k has exactly six points, say the points

$$\mp U_k,\quad \mp V_k,\quad \mp W_k,$$

namely the midpoints of the sides of H_h. The coordinates of these points are given in the following table:

$$U_1 = (\sqrt{\tfrac{1}{2}}-1, -1),\qquad V_1 = (\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}),\qquad W_1 = (1, 1-\sqrt{\tfrac{1}{2}}),$$
$$U_2 = (\tfrac{1}{2}, \tfrac{1}{2}-\sqrt{2}),\qquad V_2 = (1, 0),\qquad W_2 = (\tfrac{1}{2}, \sqrt{2}-\tfrac{1}{2}),$$
$$U_3 = (1, \sqrt{\tfrac{1}{2}}-1),\qquad V_3 = (\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}),\qquad W_3 = (\sqrt{\tfrac{1}{2}}-1, 1),$$
$$U_4 = (\sqrt{2}-\tfrac{1}{2}, \tfrac{1}{2}),\qquad V_4 = (0, 1),\qquad W_4 = (\tfrac{1}{2}-\sqrt{2}, \tfrac{1}{2}).$$

Evidently,

$$U_k + W_k = V_k,\qquad \{U_k, W_k\} = \triangle(\Pi_4)\qquad (k = 1, 2, 3, 4). \quad (18)$$

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[11] I am in great debt to Mr. D. F. FERGUSON, M. A., for the evaluation of this constant and the two other ones.

[12] See paper B, § 1.

Consider now two variable points

$$P_1 = (1, \alpha),\qquad P_3 = (\beta, \beta + \sqrt{2})$$

on the line segments joining V_2 to W_1 and $-U_2$ to $-V_1$, respectively, and assume that the determinant of these two points has the value,

$$\{P_1, P_3\} = \triangle(\Pi_4). \quad (19)$$

Then the point

$$P_2 = (x_1, x_2) = P_1 + P_3 \quad (20)$$

describes a hyperbola arc A_4 connecting W_2 with $-U_1$. Since by (19),

$$\beta - \alpha\beta + \sqrt{2} = \sqrt{2} - \tfrac{1}{2},$$

and by (20),

$$x_1 = 1 + \beta,\qquad x_2 = \alpha + \beta + \sqrt{2},$$

this hyperbola has the equation,

$$x_2 = x_1 + \frac{1}{2(x_1-1)} + \sqrt{2}. \quad (21)$$

The arc A_4 touches the boundary Γ_4 of Π_4 at the two points $-U_1$ and W_2, and together with this boundary encloses a curvilinear triangle, τ_4 say, which is of area,

$$\left.\begin{aligned} V(\tau_4) &= \{[\tfrac{1}{2}-(1-\sqrt{\tfrac{1}{2}})]\cdot 1 - \tfrac{1}{2}[\tfrac{1}{2}-(\sqrt{2}-1)]^2\} - \\ &\qquad - \int_{1-\sqrt{\frac{1}{2}}}^{\frac{1}{2}} \left(x_1 + \frac{1}{2(x_1-1)} + \sqrt{2}\right) dx_1 \\ &= (-\tfrac{21}{8} + 2\sqrt{2}) - (\tfrac{3}{8} - \tfrac{1}{4}\log 2) = 2\sqrt{2} - 3 + \tfrac{1}{4}\log 2. \end{aligned}\right\} \quad (22)$$

In just the same way, each vertex $\mp Q_k$ of Π_4 can be separated from O by means of a hyperbola arc $\mp A_k$; this arc is congruent to A_4 and touches Γ_4, and it encloses, together with Γ_4, a triangle τ_k congruent to τ_4.

Let now K be the convex domain obtained from Π_4 by cutting off all eight triangles $\mp \tau_k$. Then every point on the boundary C of K belongs to a lattice of determinant $\triangle(\Pi_4)$ which has on C just six points $\mp P_1, \mp P_2$, $\mp P_3$ satisfying $P_1 + P_3 = P_2$, and is therefore K-admissible (Lemma 1). Hence K is irreducible and of determinant

$$\triangle(K) = \triangle(\Pi_4) = \sqrt{2} - \tfrac{1}{2}.$$

(Lemmas 2 and 3). On the other hand, from (17) and (22),

$$V(K) = V(\Pi_4) - 8\,V(\tau_4) = 16 - 8\sqrt{2} - \log 4.$$

By combining these two equations, we find that

$$Q(K) = \frac{32 - 16\sqrt{2} - 4\log 2}{2\sqrt{2} - 1} = 3.609656737\ldots$$

This is an upper bound for **Q**, and possibly even its exact value.

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