§ 10. Die Trennungsaxiome in ihrer gewöhnlichen Form enthalten alle den Begriff des offenen Kernes. Duale Umsetzung liefert die Form, bei der in allen die abgeschlossene Hülle als Grundbegriff auftritt.

So sind in einer Struktur S_0, in welcher die topologischen Axiome I—VI erfüllt sind, gleichwertig die topologischen Axiome:

A^*. zu jedem Paar p_1, p_2 von Primsomen mit $p_1 \cdot p_2 = 0$ gibt es Somen a, b mit $ab = 0$, $p_1 \subset \underline{a}$ und $p_2 \subset \underline{b}$;

und:

A. zu jedem Paar p_1, p_2 von Primsomen mit $(1-p_1)+(1-p_2)=1$ gibt es Somen a, b mit $a+b=1$, $1-p_1 \supset \bar{a}$ und $1-p_2 \supset \bar{b}$.

Bei Benutzung des Randes als Grundbegriff hat man als aequivalentes Axiom:

A^r. zu jedem Paar von Primsomen p_1, p_2 mit $p_1 \cdot p_2 = 0$ gibt es Somen a, b mit $ab = 0$, $p_1 \subset a(a^r)'$ und $p_2 \subset b(b^r)'$.

Duale Umsetzung liefert als mit A^r gleichwertig:

A_r. zu jedem Paar p_1, p_2 von Primsomen mit $(1-p_1)+(1-p_2)=1$ gibt es Somen a, b mit $a+b=1$, $1-p_1 \supset a+(a_r)'$ und $1-p_2 \supset$ $\supset b+(b_r)'$.

Wir überlassen dem Leser die Formulierung der übrigen Trennungsaxiome mittels offenen Kernes oder Randes [10]), und der dualen Grundbegriffe.

[10]) Siehe ALBUQUERQUE, loc. cit. 6), S. 193—196; auch RIDDER, loc. cit. 1), § 28.

Mathematics. — *Non-homogeneous binary quadratic forms.* IV. By H. DAVENPORT. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 31, 1947.)

1. The present paper is a continuation of paper II of this series [1]). We are concerned with the minimum of the product

$$|(\xi - a)(\xi' - b)|,$$

where ξ is an arbitrary integer of the field $k(\theta)$, say

$$\xi = x + \theta y, \quad \theta = \tfrac{1}{2}(1+\sqrt{5}),$$

and a, b are given real numbers. It was proved in II that if a, b are not of the form

$$a = \tfrac{1}{2}\tau + \xi_0, \quad b = \tfrac{1}{2}\tau' + \xi_0' \quad \dots \dots \quad (1)$$

nor of the form

$$a = \frac{1}{\sqrt{5}}\tau + \xi_0, \quad b = -\frac{1}{\sqrt{5}}\tau' + \xi_0', \quad \dots \dots \quad (2)$$

where τ is a unit and ξ_0 an integer of $k(\theta)$, then there exists an integer ξ satisfying

$$|(\xi - a)(\xi' - b)| < \frac{1}{6.34}.$$

The question of the existence of a third minimum was left unsolved, though it was proved that if such a third minimum exists, it cannot be less than

$$\frac{1}{4\theta} = \frac{1}{6.472\ldots}. \quad \dots \dots \quad (3)$$

I have now established the existence of a third minimum, and indeed of an infinite sequence of minima having values greater than (3). These minima occur when

$$a = \frac{\tau}{\alpha_m} + \xi_0, \quad b = \frac{\tau'}{\alpha'_m} + \xi_0' \quad \text{(or vice versa)}, \quad \dots \quad (4)$$

where τ is a unit and ξ_0 an integer of $k(\theta)$, and where m is an odd positive integer, and α_m is defined by

$$\alpha_m = \frac{2(\theta^{n+1}-1)}{\theta^n+1}, \quad n = 3m. \quad \dots \dots \quad (5)$$

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 50 (1947) 378—389.

The case $m = 1$ gives

$$\alpha_1 = \frac{2(\theta^4 - 1)}{\theta^3 + 1} = 2\theta - 1 = \sqrt{5},$$

so that a, b are then of the form specified in (2). The case $m = 3$ gives

$$\alpha_3 = \frac{2(\theta^{10} - 1)}{\theta^9 + 1} = \frac{11}{4\theta - 3},$$

and the values of a, b defined by (4) are then those corresponding to the third minimum, the value of which is

$$\frac{1}{|\alpha_3 \alpha_3'|} = \frac{19}{121} = \frac{1}{6.368\ldots}.$$

The formal enunciation of the results is as follows.

Theorem 1. *Suppose that a, b are not of the form* (1), *nor of the form* (4), *where α_m is defined by* (5) *and $m = 1, 3, 5, \ldots$. Then the lower bound $M(a, b)$ of*

$$|(\xi - a)(\xi' - b)|$$

for integers ξ of $k(\theta)$ satisfies

$$M(a, b) \leqslant \frac{1}{4\theta}. \qquad (6)$$

Theorem 2. *If a, b are of the form* (1), *we have $M(a, b) = \frac{1}{4}$, and this minimum is then attained for an infinity of integers ξ. If a, b are of the form* (4), *we have*

$$M(a, b) = \frac{1}{|\alpha_m \alpha_m'|}, \qquad (7)$$

and this minimum is then also attained for an infinity of integers ξ.

We may note that the value of the minimum corresponding to any m, which is specified in (7), may also be expressed in terms of the Fibonacci numbers. If we define these by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \qquad (n = 1, 2, \ldots),$$

then $\theta^n = F_n\theta + F_{n-1}$, and we easily find that

$$\frac{1}{|\alpha_m \alpha_m'|} = \frac{F_{n+1} + F_{n-1}}{4(F_{n+2} + F_n - 2)}, \qquad n = 3m. \qquad (8)$$

The greater generality of the present arguments, as compared with those of II, has the effect that the present paper supersedes a great deal of the former one. In fact, the only results which will be quoted from II are the comparatively simple Lemmas 1, 2, 3.

2. The assertion of Theorem 1 is that if

$$M = M(a, b) > \frac{1}{4\theta}$$

then a, b must be either of the form (1) or of the form (4). We may therefore write

$$\frac{1}{M} = 4\theta(1 - \delta), \qquad (9)$$

where $\delta > 0$. Let ε_0 be an arbitrarily small positive number, which we can suppose to satisfy any desired inequality of the form $\varepsilon_0 < E(\delta)$, where $E(\delta)$ is any positive number depending only on δ. A finite number of such inequalities will be imposed on ε_0 in the course of the paper.

By the definition of M, there exists an integer ξ_0 of $k(\theta)$ such that

$$|(\xi_0 - a)(\xi_0' - b)| = \frac{M}{1 - \varepsilon}, \quad \text{where } 0 \leqslant \varepsilon < \varepsilon_0. \qquad (10)$$

We define α, β by

$$\alpha = (a - \xi_0)^{-1}, \quad \beta = (b - \xi_0')^{-1}, \qquad (11)$$

so that

$$|\alpha\beta| = \frac{1 - \varepsilon}{M} = 4\theta(1 - \varepsilon)(1 - \delta). \qquad (12)$$

By the definition of M, and by (10) and (11), we have

$$|(\alpha\xi - 1)(\beta\xi' - 1)| \geqslant 1 - \varepsilon \qquad (13)$$

for all integers ξ of $k(\theta)$.

By the operations of (i) replacing α, β by $\alpha\tau$, $\beta\tau'$, where τ is any unit, (ii) interchanging α, β, we can ensure, as in II, that

$$\alpha > 0, \quad \frac{\alpha}{\theta} \leqslant |\beta| \leqslant \alpha. \qquad (14)$$

It follows from (12) and (14) that

$$\alpha^2 \leqslant \alpha|\beta|\theta = 4\theta^2(1 - \varepsilon)(1 - \delta),$$
$$\alpha < 2\theta - \delta. \qquad (15)$$

Also

$$|\beta| \leqslant (\alpha|\beta|)^{\frac{1}{2}} < (4\theta)^{\frac{1}{2}},$$
$$|\beta| < 2.55. \qquad (16)$$

3. **Lemma 1.** *If $\alpha < \sqrt{5} - \varepsilon$ then $\alpha = \beta = 2$.*

Proof. Lemmas 1, 2, 3 of II.

Lemma 2. *If* $\sqrt{5}-\varepsilon \leqq \alpha < 2\theta - \delta$ *then* $\beta < 0$.

Proof. By (13) with $\xi = \theta^{-2}$, we have

$$|(\theta^{-2}\alpha - 1)(\theta^{2}\beta - 1)| \geqq 1 - \varepsilon.$$

Suppose $\beta > 0$, then $\beta \geqq \alpha/\theta \geqq (\sqrt{5}-\varepsilon)/\theta > 1$, by (14). Hence

$$|\alpha - \theta^{2}|(\beta - \theta^{-2}) \geqq 1 - \varepsilon.$$

If $\alpha < \theta^{2}$, this gives

$$\beta \geqq \theta^{-2} + (1-\varepsilon)(\theta^{2}-\alpha)^{-1} \geqq \theta^{-2} + (1-\varepsilon)(\theta^{2} - \sqrt{5} + \varepsilon)^{-1}$$
$$= \theta^{-2} + (1-\varepsilon)(\theta^{-2}+\varepsilon)^{-1}.$$

This last expression is almost 3, and so we have a contradiction to (16).

We may now suppose that $\alpha > \theta^{2}$. We have

$$\beta \geqq \theta^{-2} + (1-\varepsilon)(\alpha - \theta^{2})^{-1},$$
$$\alpha\beta \geqq (1-\varepsilon)\{\theta^{-2}\alpha + 1 + \theta^{2}(\alpha - \theta^{2})^{-1}\}.$$

This last expression decreases as α increases for $\theta^{2} < \alpha < 2\theta^{2}$. Since $\alpha < 2\theta - \delta$, it follows that

$$\alpha\beta \geqq (1-\varepsilon)\{2\theta^{-1} - \theta^{-2}\delta + 1 + \theta^{2}(2\theta - \theta^{2} - \delta)^{-1}\}$$
$$= (1-\varepsilon)\{2\theta^{-1} - \theta^{-2}\delta + 1 + \theta^{3}(1 - \delta\theta)^{-1}\}.$$

Since $2\theta^{-1} + 1 + \theta^{3} = 4\theta$, this gives $\alpha\beta > 4\theta$, provided ε is small compared with δ, and so gives a contradiction to (12). This proves Lemma 2.

Lemma 3. *If* $\sqrt{5} - \varepsilon \leqq \alpha < 2\theta - \delta$, *then*

$$2 + \tfrac{1}{7}\delta < -\beta < \theta^{2}.$$

Proof. By Lemma 2, we have $\beta < 0$, and we write $\beta = -\bar{\beta}$, where $\bar{\beta} > 0$. In view of (16), it suffices to prove that $\bar{\beta} > 2 + \frac{1}{7}\delta$.

By (13) with $\xi = \theta$, we have

$$(\alpha - \theta^{-1})|\bar{\beta} - \theta| \geqq 1 - \varepsilon.$$

Now $\alpha - \theta^{-1} < 2\theta - \delta - \theta^{-1} = \theta^{2} - \delta$. Hence

$$|\bar{\beta} - \theta| \geqq (1-\varepsilon)(\theta^{2} - \delta)^{-1} > (1-\varepsilon)\theta^{-2}(1 + \delta\theta^{-2}) > \theta^{-2} + \tfrac{1}{7}\delta,$$

if ε is small compared with δ. If $\bar{\beta} < \theta$, this would give

$$\bar{\beta} < \theta - \theta^{-2} - \tfrac{1}{7}\delta,$$
$$\frac{\bar{\beta}}{\alpha} < \frac{\theta - \theta^{-2} - \frac{1}{7}\delta}{\sqrt{5} - \varepsilon} < \frac{1.237}{2.236} < \frac{1}{\theta},$$

contrary to (14). Hence $\bar{\beta} > \theta$, and

$$\bar{\beta} > \theta + \theta^{-2} + \tfrac{1}{7}\delta = 2 + \tfrac{1}{7}\delta.$$

4. In virtue of (15), and Lemmas 1 and 3, we may suppose henceforth, in proving Theorem 1, that

$$\sqrt{5} - \varepsilon \leqq \alpha < 2\theta - \delta, \quad 2 + \tfrac{1}{7}\delta < \bar{\beta} < \theta^{2}, \quad \ldots \quad (17)$$

where $\bar{\beta} = -\beta$.

We define integers ξ_m, η_m of $k(\theta)$, for $m = 1, 2, 3, \ldots$, as follows:

$$\xi_m = \frac{1 + \theta^{-n+3}}{2\theta}, \quad \eta_m = \frac{1 - \theta^{-n}}{2}, \qquad \text{where } n = 3m. \quad (18)$$

That these are integers follows from the fact that $\theta^{3} = 2\theta + 1 \equiv 1 \pmod{2}$. Since

$$\frac{1}{\eta_1} = \frac{2}{1 - \theta^{-3}} = \frac{2\theta^{3}}{\theta^{3} - 1} = \theta^{2},$$

and $\dfrac{1}{\eta_m}$ decreases as m increases and has the limit 2, there will be exactly one value of m for which

$$\frac{1}{\eta_m} > \bar{\beta} > \frac{1}{\eta_{m+1}}. \quad \ldots \quad (19)$$

Moreover, by (17), m will have an upper bound depending only on δ. Hence, by a previous remark, we can suppose that ε_0 (and therefore also ε) is less than any positive number which depends only on m.

Lemma 4. *If* (19) *holds, then*

$$\frac{1}{\xi_m} < \alpha < \frac{1}{\xi_{m+2}}. \quad \ldots \quad (20)$$

Proof. One half of this is easily proved; by (12), (18), (19), we have

$$\alpha < \frac{4\theta}{\bar{\beta}} \leqq 4\theta\,\eta_{m+1} = 2\theta(1 - \theta^{-n-3}) < \frac{2\theta}{1 + \theta^{-n-3}} = \frac{1}{\xi_{m+2}}.$$

To obtain the other half, we first apply (13) with $\xi = \theta$. This gives

$$(\theta\alpha - 1)(\theta^{-1}\bar{\beta} - 1) \geqq 1 - \varepsilon, \quad \ldots \quad (21)$$

on noting that $\bar{\beta} > 2 > \theta$. By (19),

$$\theta^{-1}\bar{\beta} - 1 < \frac{1}{\theta\,\eta_m} - 1 = \frac{2 - \theta + \theta^{-n+1}}{\theta - \theta^{-n+1}}.$$

If we could neglect ε in (21), we should conclude that

$$\theta\alpha \geqq 1 + \frac{\theta - \theta^{-n+1}}{2 - \theta + \theta^{-n+1}} = \frac{2}{2 - \theta + \theta^{-n+1}} = \frac{2\theta^{2}}{1 + \theta^{-n+3}},$$

whence

$$\alpha \geqslant \frac{2\,\theta}{1+\theta^{-n+3}} = \frac{1}{\xi_m}.$$

It is clear, therefore, that the corresponding deduction from (21) when we do not neglect ε will be of the form

$$\alpha \geqslant \frac{1}{\xi_m} - \mu\,\varepsilon,$$

where μ is a positive number depending only on m. But from (13), with $\xi = \xi_m$, we have

$$|\alpha\,\xi_m - 1|\,(|\xi'_m|\,\bar{\beta} + 1) \geqslant 1 - \varepsilon,$$

and if $\alpha \leq 1/\xi_m$ the last two inequalities give a contradiction if ε is less than some positive number depending only on m. Thus we have $\alpha > 1/\xi_m$.

Lemma 5. *The value of m determined by (19) cannot be even.*

P r o o f. We observe first that if m is even, then from (18)

$$\eta'_m = -\frac{\theta^n - 1}{2}, \qquad \eta'_{m+1} = \frac{\theta^{n+3}+1}{2}.$$

The proof is based on the two inequalities derived from (13) by taking $\xi = -\eta'_m$ and $\xi = -\eta'_{m+1}$. These are:

$$(|\eta'_m|\,\alpha - 1)\,(1 - \eta_m\,\bar{\beta}) \geqslant 1 - \varepsilon,$$
$$(\eta'_{m+1}\,\alpha + 1)\,(\eta_{m+1}\,\bar{\beta} - 1) \geqslant 1 - \varepsilon.$$

The expressions in brackets are all positive, by (19) and the fact that

$$|\eta'_m| \geqslant \tfrac{1}{2}(\theta^3 - 1) = \theta, \quad \alpha \geqslant \sqrt{5} - \varepsilon.$$

We multiply these by η_{m+1} and η_m respectively and add, thus eliminating $\bar{\beta}$. The result may be written

$$\frac{\eta_{m+1} - \eta_m}{1-\varepsilon} \geqslant \frac{\eta_{m+1}}{|\eta'_m|\,\alpha - 1} + \frac{\eta_m}{\eta'_{m+1}\,\alpha + 1}. \quad . \quad . \quad . \quad . \quad (22)$$

This inequality determines a lower bound for α. Hence, if we prove that the inequality is violated when α is replaced by $1/\xi_{m+2}$, we shall have reached a contradiction, by (20).

Since

$$\eta_{m+1} - \eta_m = \tfrac{1}{2}(\theta^{-n} - \theta^{-n-3}) = \theta^{-n-2},$$

it will suffice to prove that the sum on the right of (22), with the above value of α, exceeds θ^{-n-2} by a positive amount deending only on m. This sum is

$$\frac{\eta_{m+1}\,\xi_{m+2}}{|\eta'_m| - \xi_{m+2}} + \frac{\eta_m\,\xi_{m+2}}{\eta'_{m+1} + \xi_{m+2}}$$
$$= \frac{(1-\theta^{-n-3})\,(1+\theta^{-n-3})}{2\,\{\theta\,(\theta^n - 1) - (1+\theta^{-n-3})\}} + \frac{(1-\theta^{-n})\,(1+\theta^{-n-3})}{2\,\{\theta\,(\theta^{n+3}+1) + 1 + \theta^{-n-3}\}}$$
$$= \frac{1-\theta^{-2n-6}}{2\,\{\theta^{n+1} - \theta^2 - \theta^{-n-3}\}} + \frac{1-\theta^{-n}}{2\,\{\theta^{n+4}+1\}}$$
$$> \tfrac{1}{2}\,\theta^{-n-1}\,(1-\theta^{-2n-6})\,(1-\theta^{-n+1})^{-1} + \tfrac{1}{2}\,\theta^{-n-4}\,(1-\theta^{-n})\,(1+\theta^{-n-4})^{-1}$$
$$> \tfrac{1}{2}\,\theta^{-n-1}\,(1-\theta^{-2n-6})\,(1+\theta^{-n+1}) + \tfrac{1}{2}\,\theta^{-n-4}\,(1-\theta^{-n})\,(1-\theta^{-n-4})$$
$$> \tfrac{1}{2}\,(\theta^{-n-1} - \theta^{-3n-7} + \theta^{-2n} - \theta^{-4n-6} + \theta^{-n-4} - \theta^{-2n-4} - \theta^{-2n-8})$$
$$= \theta^{-n-2} + \tfrac{1}{2}\,\theta^{-2n}\,(1 - \theta^{-4} - \theta^{-8} - \theta^{-n-7} - \theta^{-2n-6}).$$

Since $n = 3\,m \geq 6$, the sum in the bracket is greater than a positive absolute constant, and the result follows.

Lemma 6. *If m is odd, it is impossible that* (19) *holds and that*

$$\frac{1}{\xi_{m+1}} \leqslant \alpha < \frac{1}{\xi_{m+2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

P r o o f. The proof is based on two inequalities, derived from (13) by taking $\xi = \xi_{m+1}$ and $\xi = -\eta'_{m+1}$. Since m is odd, we have, from (18),

$$\xi'_{m+1} = \tfrac{1}{2}\,\theta\,(\theta^n - 1), \qquad \eta'_{m+1} = -\frac{\theta^{n+3}-1}{2}.$$

As we suppose that (19) and (23) hold, the inequalities, which are

$$|(\xi_{m+1}\,\alpha - 1)\,(\xi'_{m+1}\,\bar{\beta} + 1)| \geqslant 1 - \varepsilon,$$
$$|(\eta'_{m+1}\,\alpha + 1)\,(\eta_{m+1}\,\bar{\beta} - 1)| \geqslant 1 - \varepsilon,$$

take the form

$$(\xi_{m+1}\,\alpha - 1)\,(\xi'_{m+1}\,\bar{\beta} + 1) \geqslant 1 - \varepsilon, \quad . \quad . \quad . \quad . \quad . \quad (24)$$
$$(|\eta'_{m+1}|\,\alpha - 1)\,(\eta_{m+1}\,\bar{\beta} - 1) \geqslant 1 - \varepsilon, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where each factor is positive.

We use (24), together with the fact that $\bar{\beta} < 4\,\theta/\alpha$ by (12), to obtain a lower bound for α. It will be clear from the nature of the calculations and result that we can neglect ε. Thus (24) gives

$$(\xi_{m+1}\,\alpha - 1)\left(\frac{4\,\theta\,\xi'_{m+1}}{\alpha} + 1\right) \geqslant 1.$$

It will be convenient to write $P = \xi_{m+1}\,\xi'_{m+1}$, so that P is a positive integer. Substituting for ξ'_{m+1}, the last inequality gives

$$4\,\theta\,P + \xi_{m+1}\,\alpha - \frac{4\,\theta\,P}{\xi_{m+1}\,\alpha} - 1 \geqslant 1, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

whence

$$\{\xi_{m+1}\,\alpha + (2\,\theta P - 1)\}^2 \geqslant 4\,\theta^2 P^2 + 1,$$

$$\xi_{m+1}\,\alpha \geqslant \sqrt{4\,\theta^2 P^2 + 1} - 2\,\theta P + 1$$

$$> 2\,\theta P \left\{ 1 + \frac{1}{8\,\theta^2 P^2} - \frac{1}{128\,\theta^4 P^4} \right\} - 2\,\theta P + 1$$

$$= 1 + \frac{1}{4\,\theta P} - \frac{1}{64\,\theta^3 P^3}.$$

Now

$$P = N(\xi_{m+1}) = \tfrac{1}{4} N(1 + \theta^n) = \tfrac{1}{4}(\theta^n - \theta^{-n}).$$

Hence

$$\xi_{m+1}\,\alpha > 1 + \theta^{-n-1}(1 - \theta^{-2n})^{-1} - \theta^{-3n-3}(1 - \theta^{-2n})^{-3}$$

$$> 1 + \theta^{-n-1}(1 + \theta^{-2n}) \quad - \theta^{-3n-3}(1 - \theta^{-2n})^{-3}$$

$$= 1 + \theta^{-n-1} + \theta^{-3n-3}\{\theta^2 - (1 - \theta^{-2n})^{-3}\},$$

$$\xi_{m+1}\,\alpha > 1 + \theta^{-n-1}, \quad \ldots \ldots \quad (27)$$

since

$$(1 - \theta^{-2n})^{-3} < 1 - < \theta^2.$$

We now use (25) in a similar way to obtain a lower bound for $\overline{\beta}$. Writing $Q = |\eta_{m+1}\,\eta'_{m+1}|$, the inequality analogous to (26) is

$$4\,\theta Q - \eta_{m+1}\overline{\beta} - \frac{4\,\theta Q}{\eta_{m+1}\overline{\beta}} + 1 \geqslant 1,$$

whence

$$(2\,\theta Q - \eta_{m+1}\overline{\beta})^2 \leqslant 4\,\theta^2 Q^2 - 4\,\theta Q.$$

Now

$$\overline{\beta} < \theta^2 < 2\,\theta\,\tfrac{1}{2}(\theta^6 - 1) \leqslant 2\,\theta\,|\eta'_{m+1}| = 2\,\theta Q/\eta_{m+1}.$$

Hence

$$\eta_{m+1}\overline{\beta} \geqslant 2\,\theta Q - \sqrt{4\,\theta^2 Q^2 - 4\,\theta Q}$$

$$> 2\,\theta Q - 2\,\theta Q \left\{ 1 - \frac{1}{2\,\theta Q} - \frac{1}{8\,\theta^2 Q^2} \right\}$$

$$= 1 + \frac{1}{4\,\theta Q}.$$

We have

$$Q = |N(\eta_{m+1})| = \tfrac{1}{4}(1 - \theta^{-n-3})(\theta^{n+3} - 1) = \tfrac{1}{4}\,\theta^{n+3}(1 - \theta^{-n-3})^2.$$

Hence

$$\eta_{m+1}\overline{\beta} > 1 + \theta^{-n-4}. \quad \ldots \ldots \quad (28)$$

The inequalities (27) and (28) have been derived from (24) and (25) by neglecting ε. They should therefore be corrected by subtracting on the right terms depending only on m and ε, which tend to zero with ε for fixed m. We shall prove that (27) and (28) give a lower bound for $\alpha\overline{\beta}$ greater than $4\,\theta$. In view of (12), this gives a contradiction, since the correcting terms involving ε are negligible in comparison with δ, by a previous remark.

The product $\xi_{m+1}\,\eta_{m+1}$ is

$$\frac{1}{4\theta}(1 + \theta^{-n})(1 - \theta^{-n-3}) = \frac{1}{4\theta}(1 + 2\,\theta^{-n-2} - \theta^{-2n-3}).$$

The product of the expressions on the right of (27) and (28) is

$$(1 + \theta^{-n-1})(1 + \theta^{-n-4}) > 1 + \theta^{-n-4}(\theta^3 + 1) = 1 + 2\,\theta^{-n-2}.$$

Hence $\alpha\overline{\beta} > 4\,\theta$, as was to be proved.