Mathematics. — *A characterization of the sub-manifold of* $C[a, b]$ *spanned by the sequence* $\{x^{n_k}\}$. By J. KOREVAAR. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 1. *Introduction and results.* As usual, $C[a, b]$ denotes the space of all continuous functions of x, $a \leq x \leq b$. A set $\{\varphi_k(x)\} \in C[a, b]$ is said to be *closed* in $C[a, b]$, or to *span* this space, if it is possible to approximate uniformly over $[a, b]$ to every function $\in C[a, b]$ by linear aggregates $\Sigma a_k \varphi_k(x)$.

This note is concerned with closure-properties in $C[a, b]$ of sets

(1.1) $$\{x^{n_k}\}, \qquad (k = 1, 2, \ldots)$$

where $\{n_k\}$ is an increasing sequence of non-negative integers. The classical result in this field is the theorem of MÜNTZ [3]) and SZÁSZ [5]) *which implies that the set* (1.1) *is closed in* $C[0, 1]$ *if, and only if,*

(1.2) $$n_1 = 0, \sum_{k=2}^{\infty} \frac{1}{n_k} = \infty.$$

It was shown recently by CLARKSON and ERDÖS [1]), and independently by L. SCHWARTZ [4]), that the sub-manifold of $C[0, 1]$ spanned by a sequence

(1.3) $$\{x^{n_k}\}, \qquad \left(\sum_{n_k > 0} \frac{1}{n_k} < \infty \right)$$

is rather small: *a continuous function which is the uniform limit on* $[0, 1]$ *of a sequence of linear aggregates of functions* (1.3) *can be extended to be analytic in the interior of the unit-circle. The power series for this analytic extension contains only powers* x^{n_k}.

Conversely, the question arises whether it is possible to approximate uniformly over $[0, 1]$ by linear aggregates of functions (1.3) to every function

(1.4) $$f(x) = \sum_{k=1}^{\infty} a_k x^{n_k}$$

where the series converges for $0 \leq x < 1$ and where

(1.5) $$\lim_{x \to 1} f(x)$$

exists. This question was answered in part by CLARKSON and ERDÖS, viz. for the lacunary case $n_{k+1}/n_k > c > 1$. Here the existence of the limit (1.5) is known to imply the convergence of Σa_k and hence the uniform

convergence on $[0,1]$ of the series (1.4). (See HARDY and LITTLEWOOD [2]).) It is shown below (theorem 2) that $f(x)$ can be uniformly approximated by linear aggregates of functions (1.3) in the general case also. This yields a complete characterization of the sub-manifold of $C[0, 1]$ spanned by the sequence (1.3).

In § 4 a similar result is reached for the interval $[a, b]$, $0 \leq a < b$. The establishment of this theorem 6 requires some more information than was known until now about the best approximation to powers x^m by linear aggregates $\Sigma a_k x^{n_k}$ on $[a, b]$. This information is obtained in § 3 by the method used by CLARKSON and ERDÖS to prove the extension to $[a, b]$ of the theorem of MÜNTZ and SZÁSZ for $[0, 1]$. (See [1]), § 3].)

§ 2. *The interval* $[0, 1]$. The characterization-theorem for $[0, 1]$ is

Theorem 1. *The set of continuous functions of* x, $0 \leq x \leq 1$, *spanned by the sequence*

$$\{x^{n_k}\}, \qquad \left(\sum_{n_k > 0} \frac{1}{n_k} < \infty \right)$$

is identical with the set of all power series

$$\sum_{k=1}^{\infty} a_k x^{n_k}$$

convergent on $0 \leq x < 1$ *for which*

$$\lim_{x \to 1} \sum_{k=1}^{\infty} a_k x^{n_k}$$

exists.

The proof follows from the result of CLARKSON, ERDÖS and L. SCHWARTZ mentioned in § 1 combined with the following theorem, which may be of some interest in itself.

Theorem 2. *If the series*

$$g(x) = \Sigma b_k x^k$$

converges for $0 \leq x < 1$, *and if*

$$\lim_{x \to 1} g(x)$$

exists — name it $g(1)$ *— then it will be possible to approximate uniformly to* $g(x)$ *on* $[0, 1]$ *by linear aggregates of the partial sums* $s_n(x)$ *of* $\Sigma b_k x^k$.

The proof of theorem 2 is exceedingly simple. It follows from the uniform continuity of $g(x)$ on $0 \leq x \leq 1$ that if ε is an arbitrary positive number, then

(2.1) $$|g(x) - g(\theta x)| < \varepsilon/2 \qquad (0 \leq x \leq 1)$$

for θ sufficiently near to 1 $(\theta < 1)$. But the series for $g(\theta x)$ is convergent on $0 \leq x < \theta^{-1}$. Hence if N is sufficiently large,

(2.2) $$\left| g(\theta x) - \sum_{1}^{N} b_k \theta^k x^k \right| < \varepsilon/2, \qquad (0 \leq x \leq 1).$$

A combination of (2.1) and (2.2) yields the theorem:

$$|g(x)-\sum_{1}^{N}\theta^k\{s_k(x)-s_{k-1}(x)\}|<\varepsilon, \qquad (0\leqq x\leqq 1).$$

The interval $[0, b]$. By the substitution $bx=x'$ theorem 1 yields a corresponding result for the interval $[0, b]$.

§ 3. *Approximation to powers* x^m *on* $[a, b]$. Starting from MÜNTZ's formula (see [3], [5])

$$\underset{\{a_k\}}{\text{l. b.}}\int_0^1 |x^m-\Sigma a_k x^{n_k}|^2\,dx=\frac{1}{2m+1}\prod_{k=1}^{\infty}\left(1-\frac{2m+1}{n_k+m+1}\right)^2$$

CLARKSON and ERDÖS proved an estimate (see [1], theorem 2]) which implies the following fundamental

Theorem 3. *Let S be an increasing sequence of non-negative integers $\{n_k\}$ satisfying the condition*

$$\sum_{n_k>0}\frac{1}{n_k}<\infty.$$

Let $0<\varepsilon<1$. Then there exists an integer $m_0(\varepsilon, S)$ such that $m>m_0(\varepsilon, S)$, m not in S, implies

$$\underset{\substack{\{a_k\}\\ \{n_k\}=S}}{\text{l. b.}}\ \max_{0\leq x\leq 1}|x^m-\Sigma a_k x^{n_k}|>(1-\varepsilon)^m.$$

Furthermore this m_0 can be chosen independently of the particular sequence S for every family F of sequences S for which the functions

$$\Phi(m, S)=\sum_{m<n_k}\frac{1}{n_k},\quad \Psi(m, S)=\frac{1}{m}\sum_{n_k\leq m}1$$

tend to zero uniformly as $m\to\infty$ for $S\in F$.

Remark 1. Let the family F of sequences S satisfy the conditions of theorem 3. Now replace every sequence $S=\{n_k\}_{k=1,2,\ldots}\in F$ by the set of sequences $S_j=\{n_k+j\}_{k=1,2,\ldots}$, $j=0, 1, 2, \ldots$. Then the family F' formed by all sequences S_j will also satisfy the conditions of theorem 3. For

$$\Phi(m, S_j)=\sum_{m<n_k+j}\frac{1}{n_k+j}=\sum_{m<n_k}\frac{1}{n_k+j}+\sum_{m-j<n_k\leq m}\frac{1}{n_k+j}\leqq$$

$$\leqq\sum_{m<n_k}\frac{1}{n_k}+\frac{1}{m}\sum_{n_k\leq m}1=\Phi(m, S)+\Psi(m, S),$$

and

$$\Psi(m, S_j)=\frac{1}{m}\sum_{n_k+j\leq m}1\leqq\frac{1}{m}\sum_{n_k\leq m}1=\Psi(m, S).$$

Remark 2. Let the family F again satisfy the conditions of theorem 3. Now replace every sequence $S=\{n_k\}_{k=1,2,\ldots}\in F$ by the set of sequences

$S'_j=\{n_k\}_{k\neq j}$, $j=1, 2, \ldots$. Then the family F'' formed by all sequences S'_j will again satisfy the conditions of theorem 3. For clearly

$$\Phi(m, S'_j)\leqq\Phi(m, S),\quad \Psi(m, S'_j)\leqq\Psi(m, S).$$

From theorem 3 a similar result will be derived for the interval $[a, b]$, $a\geqq 0$.

Theorem 4. *Let S denote a sequence of integers $\{n_k\}$ satisfying the conditions*

$$0\leqq n_1<n_2<\ldots<n_k<\ldots,\qquad \sum_{n_k>0}\frac{1}{n_k}<\infty.$$

Let F be a family of sequences S satisfying the conditions that

$$\Phi(m, S)=\sum_{m<n_k}\frac{1}{n_k}\quad\text{and}\quad\Psi(m, S)=\frac{1}{m}\sum_{n_k\leq m}1$$

tend to zero as $m\to\infty$ uniformly for $S\in F$. Let $0<\varepsilon<b$. Then there exists an integer $m_0(\varepsilon, F)$ such that $m>m_0(\varepsilon, F)$ implies

$$\underset{\substack{S\in F\\ S\text{ not}\ni m}}{\text{l. b.}}\ \underset{\substack{\{a_k\}\\ \{n_k\}=S}}{\text{l. b.}}\ \max_{a\leq x\leq b}|x^m-\sum_k a_k x^{n_k}|\geqq(b-\varepsilon)^m.$$

There is no restriction in taking $b=1$. (Substitution $x=bx'$.) Further let $a>0$. Finally, let $1-\varepsilon>a$. Now if there were no integer $m_0(\varepsilon, F)$ satisfying the above condition then there would be a sequence of integers $m_k\to\infty$ $(k\to\infty)$ and a corresponding sequence of linear aggregates

$$P_k(x)=\sum_j b_{kj}x^{n_{kj}}, \tag{3.1}$$

m_k not in $S_k=\{n_{kj}\}_{j=1,2,\ldots}$, $S_k\in F$, such that

$$\max_{a\leq x\leq 1}|x^{m_k}-P_k(x)|<(1-\varepsilon)^{m_k}. \tag{3.2}$$

Now let $\|f\|$, $\|f\|'$, $\|f\|''$ denote the maximum of $|f|$ on $[a, 1]$, $[0, a]$ and $[0, 1]$ respectively. Let $0<\theta<1$. Then it will be possible to choose $k_1=k_1(\theta)$ so large that

$$\|x^{2m_k}-x^{m_k}P_k(x)\|''>\theta^{2m_k},\qquad (k>k_1(\theta)). \tag{3.3}$$

This is a consequence of theorem 3, remark 1. For the set F_1 of all sequences $S'_k=\{n_{kj}+m_k\}_{j=1,2,\ldots}$ is a sub-set of the set F' considered there. Taking $\theta^2=1-\varepsilon$ in (3.3) and comparing with (3.2) one sees that

$$\|x^{2m_k}-x^{m_k}P_k(x)\|'>(1-\varepsilon)^{m_k},\qquad (k>k_1).$$

Hence

$$\|x^{m_k}-P_k(x)\|'>\{(1-\varepsilon)/a\}^{m_k}\qquad (k>k_1).$$

As $(1-\varepsilon)/a>1$, $\|P_k\|'$ must increase exponentially:

$$A_k=\|P_k\|''>c^{m_k},\qquad (k>k_2) \tag{3.4}$$

where $c>1$.

Now let r_0 be so large that

$$\sum_{r_0<r} \{a(2-a)\}^r < \tfrac{1}{2},$$

and let n_0 be such that

(3. 5) $$\underset{\substack{|a_{ki}| \\ |n_{ki}|=S_k}}{\text{l. b.}} \| x^{n_{kj}} - \sum_{i \neq j} a_{ki} x^{n_{ki}} \|'' > (2-a)^{-n_{kj}}$$

for all k and j for which $n_{kj} > n_0$. This is possible by theorem 3, remark 2. By (3.5), if $b_{kj} \neq 0$, $n_{kj} > n_0$,

$$\left\| \frac{P_k(x)}{b_{kj}} \right\|'' = \| x^{n_{kj}} + \ldots \|'' > (2-a)^{-n_{kj}}.$$

Hence

$$|b_{kj}| < A_k (2-a)^{n_{kj}} \qquad (n_{kj} > n_0).$$

It follows that if $p_0 = \max(r_0, n_0)$,

$$\| \sum_{p_0<n_{kj}} b_{kj} x^{n_{kj}} \|' < A_k \sum_{p_0<r} \{a(2-a)\}^r < \tfrac{1}{2} A_k.$$

Hence if k_3 is so large that for $k > k_3$ $\| P_k \|'' = \| P_k \|'$, then $k > k_3$ must imply that

$$\sum_{n_{kj} \leq p_0} |b_{kj}| \geq \| \sum_{n_{kj} \leq p_0} b_{kj} x^{n_{kj}} \|' > \tfrac{1}{2} A_k.$$

But then there exists, to every $k > k_3$, a $j(k)$ such that

(3. 6) $$n_{k, j(k)} \leq p_0$$

(3. 7) $$|b_{k, j(k)}| > \frac{A_k}{2(p_0+1)}.$$

By (3.4) it is then possible to take k_4 so large that $k > k_4$ implies

(3. 8) $$|b_{k, j(k)}| > 1.$$

Now for $k > k_{1,2,3,4}$ consider the expression

(3. 9) $$|b_{k, j(k)}|^{-1} | x^{2m_k} - x^{m_k} P_k(x) | = | x^{\lambda_k} + Q_k(x) |.$$

Here $\lambda_k = n_{k, j(k)} + m_k$, and $Q_k(x)$ represents a linear aggregate of x^{2m_k} and $x^{n_{ki}+m_k}, i \neq j(k)$. If k is sufficiently large, the expression (3.9) will be less than α^{m_k}, where $0 < \alpha < 1$, everywhere on $[0, 1]$. This follows from (3.2) and (3.8) for $[a, 1]$. For $[0, a]$, by (3.8) and (3.7),

$$|b_{k, j(k)}|^{-1} | x^{2m_k} - x^{m_k} P_k(x) | \leq a^{2m_k} + 2(p_0+1) a^{m_k}.$$

Thus

(3. 10) $$\| x^{\lambda_k} + Q_k(x) \|'' = \| x^{\lambda_k} - b_{k, j(k)}^{-1} x^{2m_k} + R_k(x) \|'' < \alpha^{m_k}, \qquad (k > k_5).$$

The inequalities (3.10), (3.7) and (3.4) imply

$$\| x^{\lambda_k} + R_k(x) \|'' < \beta^{m_k}, \qquad (k > k_6)$$

where $0 < \beta < 1$. Finally, by (3.6),

(3. 11) $$\| x^{\lambda_k} + R_k(x) \|'' < \gamma^{\lambda_k}, \qquad (k > k_7),$$

$0 < \gamma < 1$. Here $R_k(x)$ contains only powers $x^{n_{ki}+m_k}$, $i \neq j(k)$. As the family of sequences $\{n_{ki} + m_k\}_{i \neq j(k)}$ $(k = 1, 2, \ldots)$ satisfies the conditions of theorem 3 the inequality (3.11) must be false. This completes the proof of theorem 4.

Corollary 1. (CLARKSON *and* ERDÖS) *The set*

(3. 12) $$\{x^{n_k}\}$$

of $C[a, b]$, *where* $0 \leq n_1 < n_2 < \ldots < n_k < \ldots \to \infty$, *is closed in* $C[a, b]$ $(0 < a < b)$ *if and only if the series*

(3. 13) $$\sum_{n_k>0} \frac{1}{n_k}$$

diverges.

Proof. If the series (3.13) diverges the set (3.12) is closed in $C_0[0, b]$ by the theorem of MÜNTZ and SZÁSZ. Here $C_0[0, b]$ denotes the space of all continuous functions of x, $0 \leq x \leq b$, vanishing at $x = 0$. The set (3.12) is then a fortiori closed in $C[a, b]$, $a > 0$.

If the series (3.13) converges on the other hand, then it follows from theorem 4 by taking $F = S = \{n_k\}$ that it is impossible to approximate uniformly on $[a, b]$ to any function x^m, m not in S, as soon as $m > m_0$. However, this is true for $m \leq m_0$ also:

Corollary 2. *If m does not belong to the increasing sequence of non-negative integers $\{n_k\}$ satisfying*

$$\sum_{n_k>0} \frac{1}{n_k} < \infty$$

then it is impossible to approximate uniformly to x^m on $[a, b]$ by linear aggregates

$$\sum a_k x^{n_k}.$$

Proof. Take in theorem 4

$$F = \{S_j\}_{j=0,1,2,\ldots} \qquad S_j = \{n_k + j\}_{k=1,2,\ldots}.$$

It follows that for $m + j > m_0$, m not in $\{n_k\}$,

$$\underset{|a_k|}{\text{l. b.}} \max_{a \leq x \leq b} | x^{m+j} - x^j \sum a_k x^{n_k} | \geq (b-\varepsilon)^{m+j}.$$

Hence, taking $j = m_0 + 1$,

$$\underset{|a_k|}{\text{l. b.}} \max_{a \leq x \leq b} | x^m - \sum a_k x^{n_k} | \geq (b-\varepsilon)^{m+m_0+1} b^{-m_0-1}$$

whenever m is not an n_k.

The following corollary will also be used in § 4.

Corollary 3. *Let $\{n_k\}$ be an increasing sequence of non-negative integers satisfying the condition*

$$\sum_{n_k>0} \frac{1}{n_k} < \infty .$$

Let $0 < \varepsilon < b$. Then there exists an integer $k_0 = k_0(\varepsilon)$ such that $k > k_0$ implies

$$\text{l. b.}_{|a_j|} \max_{a \leq x \leq b} |x^{n_k} - \sum_{j \neq k} a_j x^{n_j}| \geq (b-\varepsilon)^{n_k}.$$

Proof. See theorem 4, and theorem 3, remark 2.

§ 4. *The principal theorem.* The above corollaries 2, 3 yield the analogue for $[a, b]$ of the result of CLARKSON, ERDÖS and L. SCHWARTZ for $[0, 1]$ mentioned in § 1.

Theorem 5. *Let the increasing sequence of non-negative integers $\{n_k\}$ satisfy the condition*

$$\sum_{n_k>0} \frac{1}{n_k} < \infty .$$

Let the sequence of linear aggregates

$$P_j(x) = \sum_k a_{jk} x^{n_k} \qquad (j = 1, 2, \ldots)$$

converge uniformly to $f(x)$ on $a \leq x \leq b$, $a \geq 0$, as $j \to \infty$. Then $f(x)$ can be extended to be analytic in the interior of the circle $|x| < b$. The power series for this analytic extension is

$$\sum_{k=1}^{\infty} A_k x^{n_k},$$

where

$$A_k = \lim_{j\to\infty} a_{jk} \qquad (k = 1, 2, \ldots).$$

The proof is essentially the same as that given by CLARKSON and ERDÖS for the interval $[0, 1]$. It follows here for the sake of completeness, however.

(i) a_{jk} tends to a limit as $j \to \infty$. For to any $\varepsilon > 0$ there exists a j_0 such that $j, j' > j_0$ implies

$$\varepsilon > \max_{a \leq x \leq b} |P_j(x) - P_{j'}(x)| = |a_{jk} - a_{j'k}| \max_{a \leq x \leq b} |x^{n_k} - Q(x)|,$$

where $Q(x)$ is a linear aggregate of powers x^{n_i}, $i \neq k$. Now by theorem 4, corollary 2,

$$\max_{a \leq x \leq b} |x^{n_k} - Q(x)| \geq c_k > 0.$$

Hence if $j, j' > j_0$,

$$|a_{jk} - a_{j'k}| < \varepsilon c_k^{-1}.$$

Let

$$\lim_{j\to\infty} a_{jk} = A_k \qquad (k = 1, 2, \ldots). \tag{4.1}$$

(ii) The next step is to find a proper estimate for the A_k. Let $\varepsilon > 0$. Let $Q_{jk}(x)$ be defined by the equation

$$P_j(x) = a_{jk}\{x^{n_k} + Q_{jk}(x)\}$$

whenever $a_{jk} \neq 0$. By theorem 4, corollary 3,

$$\max_{a \leq x \leq b} |x^{n_k} + Q_{jk}(x)| \geq (b-\varepsilon)^{n_k}$$

for $k > k_0(\varepsilon)$, $j = 1, 2, \ldots$. The $P_j(x)$ will be uniformly bounded:

$$|P_j(x)| < M \qquad (a \leq x \leq b;\ j = 1, 2, \ldots).$$

Hence for all j

$$M > |a_{jk}| (b-\varepsilon)^{n_k}, \qquad (k > k_0) \tag{4.2}$$

and by (4.1),

$$|A_k| \leq M (b-\varepsilon)^{-n_k}, \qquad (k > k_0). \tag{4.3}$$

The power series $\sum A_k x^{n_k}$ will thus at least converge for $|x| < b$. Let the sum of the series be $g(x)$.

(iii) It remains to be proved that $f(x) = g(x)$ on $a \leq x < b$. This will follow from the relation

$$\lim_{j\to\infty} |P_j(x) - g(x)| = 0, \qquad (a \leq x < b).$$

To prove it, let x be fixed on $a \leq x < b$.

$$\begin{aligned} |P_j(x) - g(x)| &= |\sum_{k=1}^{\infty} A_k x^{n_k} - \sum_{k=1}^{\infty} a_{jk} x^{n_k}| \leq \\ &\leq \sum_{1}^{N} |A_k - a_{jk}| x^{n_k} + \sum_{N+1}^{\infty} (|A_k| + |a_{jk}|) x^{n_k}. \end{aligned} \tag{4.4}$$

Now first choose N so large that

$$|a_{jk}|,\ |A_k| \leq M \left(\frac{b+x}{2}\right)^{-n_k}$$

for $k > N$. (See (4.2) and (4.3).) Next let N increase until the last term of (4.4) is sufficiently small. Finally take j so large that the first term of the third member of (4.4) is also small enough.

A combination of theorems 5 and 2 now yields the principal theorem.

Theorem 6. *The set of continuous functions of x, $a \leq x \leq b$, $a \geq 0$, spanned by the sequence*

$$\{x^{n_k}\} \qquad \left(\sum_{n_k>0} \frac{1}{n_k} < \infty\right)$$

is identical with the set of all power series

$$\sum_{k=1}^{\infty} a_k\, x^{n_k}$$

convergent on $a \leqq x < b$ *for which*

$$\lim_{x \to b} \sum_{k=1}^{\infty} a_k\, x^{n_k}$$

exists.

Another corollary to theorem 5 is

Theorem 7. *Let the increasing sequence of non-negative integers* $\{n_k\}$ *satisfy the condition*

$$\sum_{n_k > 0} \frac{1}{n_k} < \infty .$$

Let the sequence of linear aggregates

$$P_j(x) = \sum_k a_{jk}\, x^{n_k} \qquad (j = 1, 2, \ldots)$$

converge uniformly to $f(x)$ *on* $a \leqq x \leqq b$, $a \geqq 0$. *Then the sequence* $\{P_j(x)\}$ *is uniformly convergent in every circle* $|x| \leqq b - \delta$, $\delta > 0$. *Its limit is the analytic extension of* $f(x)$.

In particular, let $\{P_j(x)\}$ *converge uniformly to zero on* $a \leqq x \leqq b$, $a \geqq 0$. *Then it will do so in every circle* $|x| \leqq b - \delta$, $\delta > 0$.

Proof. See (4.4).

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Mathematics. — *Inequalities for the coefficients of trigonometric polynomials.* II. By R. P. BOAS Jr. (Communicated by Prof. J. G. VAN DER CORPUT.)

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1. Let $F(t) = \sum_{-n}^{n} a_j\, e^{ijt}$ be a real trigonometric polynomial. The inequality

$$|a_0| + \tfrac{2}{3}|a_k| \leqq \tfrac{1}{4} \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n \quad \ldots \quad (1)$$

was given by VAN DER CORPUT and VISSER [1]. The constant $\frac{1}{4}$ in (1) was improved [2] to $\frac{1}{2}(1 + \frac{1}{3}\sqrt{2})/\pi = .234 \ldots$ Here I shall obtain the best possible result

$$|a_0| + \tfrac{2}{3}|a_k| \leqq C \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n \quad \ldots \quad (2)$$

with

$$C = 1/(2\pi - 4\delta), \quad \ldots \quad (3)$$

$$\sin\delta + \tfrac{1}{3}\delta = \tfrac{1}{6}\pi, \quad 0 < \delta < \pi/2 \quad \ldots \quad (4)$$

We have $.2136 < C < .2137$.

More generally, for any positive γ,

$$|a_0| + 2\gamma|a_k| \leqq C_\gamma \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n \quad \ldots \quad (5)$$

where C_γ is given by (3) and δ is the smallest positive root of $\sin\delta = \frac{1}{2}\gamma(\pi - 2\delta)$; equality occurs in (5) for some $F(t) \not\equiv 0$. For example, $C_1 = .338$; the value given before [2] was $\frac{1}{2}(1 + \sqrt{2})/\pi = .384 \ldots$ Thus we have

$$|a_0| + 2|a_k| < 2.126 \cdot \frac{1}{2\pi} \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n, \; F(t) \not\equiv 0.$$

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