is identical with the set of all power series

$$\sum_{k=1}^{\infty} a_k\, x^{n_k}$$

convergent on $a \leq x < b$ *for which*

$$\lim_{x \to b} \sum_{k=1}^{\infty} a_k\, x^{n_k}$$

exists.

Another corollary to theorem 5 is

Theorem 7. *Let the increasing sequence of non-negative integers* $\{n_k\}$ *satisfy the condition*

$$\sum_{n_k > 0} \frac{1}{n_k} < \infty .$$

Let the sequence of linear aggregates

$$P_j(x) = \sum_k a_{jk}\, x^{n_k} \qquad (j = 1, 2, \ldots)$$

converge uniformly to $f(x)$ *on* $a \leq x \leq b,\ a \geq 0$. *Then the sequence* $\{P_j(x)\}$ *is uniformly convergent in every circle* $|x| \leq b - \delta,\ \delta > 0$. *Its limit is the analytic extension of* $f(x)$.

In particular, let $\{P_j(x)\}$ *converge uniformly to zero on* $a \leq x \leq b$, $a \geq 0$. *Then it will do so in every circle* $|x| \leq b - \delta,\ \delta > 0$.

Proof. See (4.4).

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Mathematics. — *Inequalities for the coefficients of trigonometric polynomials.* II. By R. P. BOAS Jr. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 31, 1947.)

1. Let $F(t) = \sum_{-n}^{n} a_j\, e^{ijt}$ be a real trigonometric polynomial. The inequality

$$|a_0| + \tfrac{2}{3}\,|a_k| \leq \tfrac{1}{4} \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n \quad . \quad . \quad . \quad . \quad (1)$$

was given by VAN DER CORPUT and VISSER [1]. The constant $\frac{1}{4}$ in (1) was improved [2] to $\frac{1}{2}(1 + \frac{1}{3}\sqrt{2})/\pi = .234 \ldots$. Here I shall obtain the best possible result

$$|a_0| + \tfrac{2}{3}\,|a_k| \leq C \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n \quad . \quad . \quad . \quad . \quad (2)$$

with

$$C = 1/(2\pi - 4\delta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\sin \delta + \tfrac{1}{3}\delta = \tfrac{1}{6}\pi, \quad 0 < \delta < \pi/2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have $.2136 < C < .2137$.

More generally, for any positive γ,

$$|a_0| + 2\gamma\, |a_k| \leq C_\gamma \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n \quad . \quad . \quad . \quad . \quad (5)$$

where C_γ is given by (3) and δ is the smallest positive root of $\sin \delta = \frac{1}{2}\gamma(\pi - 2\delta)$; equality occurs in (5) for some $F(t) \not\equiv 0$. For example, $C_1 = .338$; the value given before [2] was $\frac{1}{2}(1 + \sqrt{2})/\pi = .384 \ldots$. Thus we have

$$|a_0| + 2\,|a_k| < 2.126 \cdot \frac{1}{2\pi} \int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{2} n,\ F(t) \not\equiv 0,$$

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[1]) J. G. VAN DER CORPUT and C. VISSER, Inequalities concerning polynomials and trigonometric polynomials, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 383—392 (1946).

[2]) R. P. BOAS Jr., Inequalities for the coefficients of trigonometric polynomials, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 492 (1947).

which it is interesting to compare with the inequality [1])

$$|a_0| + 2|a_k| \leqq \max|F(t)|, \qquad k > \tfrac{1}{2} n.$$

2. To prove (5), let

$$h(x) = 1 + 2\gamma\cos x + \sum_{m=2}^{\infty} c_m \cos m x, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the coefficients c_m are to be determined later; if

$$g(x) = \frac{1}{2\pi}\int_0^{2\pi} F(x-t)\, h(kt)\, dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

then for $k > n/2$

$$|g(x)| = |a_0 + \gamma(a_k e^{ikx} + \bar{a}_k e^{-ikx})| \leqq \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\, dt. \quad . \quad (8)$$

By choosing x appropriately, we have

$$|a_0| + 2\gamma|a_k| \leqq \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\, dt \quad . \quad . \quad . \quad . \quad (9)$$

and it remains to choose $h(x)$ to make $\max|h(x)|$ as small as possible. We shall take $h(x)$ to be an even step function of period 2π, of the form

$$h(x) = a, \qquad 0 < x < \pi - \delta,$$
$$h(x) = -a, \qquad \pi - \delta < x < \pi,$$

and try to determine δ so that (6) is true. Then we must have

$$\frac{1}{\pi}(\pi - 2\delta)\, a = 1, \quad \frac{2}{\pi} \cdot 2a\sin\delta = 2\gamma,$$

and so $\gamma = \dfrac{2\sin\delta}{\pi - 2\delta}$. With this choice of δ, $\max|h(x)| = a = \pi/(\pi - 2\delta)$. Using this in (9), we have (5).

To prove that (5) is best possible, consider the polynomial

$$F(x) = 1 + \sec\delta\cos x,$$

with $n = 1$. Then by (8) with $x = 0$

$$1 + 2\gamma\sec\delta = \frac{1}{2\pi}\int_0^{2\pi} h(t)\, F(-t)\, dt;$$

since $F(x)$ changes sign at $x = \pi/\delta$, $F(x)$ has the sign of $h(x)$ and so,

since $|h(x)|$ is constant,

$$1 + 2\gamma\sec\delta = \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\, dt.$$

This shows that equality in (5) is attained for this $F(x)$.

3. In the same way we can prove

$$|a_k| \leqq \tfrac{1}{8}\int_0^{2\pi} |F(t)|\, dt, \qquad k > \tfrac{1}{3} n \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which was given by FEJES [3]) and VAN DER CORPUT and VISSER [1]), and is best possible. Here we take $h(x)$ even, $h(x) = \pi/4$ in $0 < x < \pi/2$, $h(x) = -\pi/4$ in $\pi/2 < x < \pi$. Then

$$h(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cos(2m+1)x}{2m+1}$$

and $g(x)$, defined by (7), satisfies, for $k > n/3$,

$$|g(x)| = \tfrac{1}{2}|a_k e^{ikx} + \bar{a}_k e^{-ikx}| \leqq \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\, dt = \tfrac{1}{8}\int_0^{2\pi} |F(t)|\, dt.$$

Choosing x appropriately, we have (10).

4. Inequalities for $k \leq n/2$ or $k \leq n/3$ can be obtained by the method used previously [2]) for inequalities involving $\max|F(x)|$. To generalize (5), we take $h(x)$ as in (6) and consider

$$H(x) = \frac{1}{2\pi k}\int_0^{2\pi} h(k(x-t))\, d\alpha(kt),$$

where $\alpha(t)$ is nondecreasing and has FOURIER–STIELTJES coefficients $b_0 = 1$, $b_1 = b_{-1} = \lambda$, $b_j = b_{-j} = 0$, $j = 2, 3, \ldots, p$, $p = [n/k]$, $0 < \lambda < \tfrac{1}{2}\sec\pi/(p+2)$. Then

$$H(x) = 1 + 2\lambda\gamma\cos kx + \sum_{m=p+1}^{\infty} b_m c_m \cos mkx,$$
$$|H(t)| \leq \max|h(t)|.$$

Now replace (7) by

$$g(x) = \frac{1}{2\pi}\int_0^{2\pi} F(x-t)\, H(t)\, dt;$$

[3]) L. FEJES, Two inequalities concerning trigonometric polynomials, J. London Math. Soc., 14, 44—46 (1939).

then (8) is replaced by

$$|a_0 + \lambda\gamma(a_k e^{ikx} + \bar{a}_k e^{-ikx}| \leqq \max |h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\, dt.$$

Hence, letting $\lambda \to \frac{1}{2} \sec \pi/(p+2)$, we obtain

$$|a_0| + |a_k| \gamma \sec \frac{\pi}{[n/k]+2} \leqq C_\gamma \int_0^{2\pi} |F(t)|\, dt$$

with C_γ as in (5).

The corresponding generalization of (10) is

$$|a_k| \leqq \tfrac{1}{4} \cos \frac{\pi}{p+2} \cdot \int_0^{2\pi} |F(t)|\, dt,$$

where $p = 2[\frac{1}{2}(n-k)/k] + 1$ is the largest odd integer with $pk \leq n$.

Chemistry. — *Elektrochemisch gedrag van ionen-wisselende stoffen. Potentiaalmetingen aan plantenwortels.* III. *Metingen aan Sinapis alba.* By W. H. VAN DER MOLEN and H. J. C. TENDELOO. (Communicated by Prof. J. W. BIJVOET.)

(Communicated at the meeting of April 26, 1947.)

Door de theorie van het Donnan-evenwicht uit te breiden en toe te passen op systemen met niet-diffusabele anionen van een zwak zuur, kon in een vorige mededeling [1]) worden aangetoond, dat het mogelijk was de potentiaal van een planten-wortel in verdunde oplossingen van KCl te beschrijven. Uit metingen van de potentiaal is de verhouding, V, der concentraties van de diffusabele ionen in en buiten de wortel te berekenen; uit de formule:

$$V^3 + \left(K' C_2 + \frac{A}{C_2 + C_1}\right) V^2 - V - K' C_2 = 0 \quad . \quad . \quad . \quad (1)$$

waarin:

$K' = \frac{1}{K}$, K = dissociatie constant

C_2 = waterstofionenconcentratie der buitenoplossing

C_1 = kaliumionenconcentratie der buiten oplossing

A = concentratie van het zwakke zuur in de wortel

volgt:

$$K' C_2 + A \frac{V^2}{(V^2-1)(C_2+C_1)} = -V \quad . \quad . \quad . \quad . \quad . \quad (2)$$

waaruit te zien is, dat er een lineair verband moet zijn tussen

$$\frac{V^2}{(V^2-1)(C_2+C_1)} \text{ en } -V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Voor C_2 werd tevoren gebruik gemaakt van de pH van de buitenoplossing. Het is echter waarschijnlijk, dat veel meer bepalend is de pH van het milieu in de onmiddellijke omgeving van de wortel, die niet te bepalen is. Daarom werd bij metingen aan wortels van Sinapis alba voor verschillende waarden van C_2 de betrekking (3) grafisch nagegaan. Hierbij bleek, dat de rechte lijn het best benaderd werd met $C_2 = 3{,}2 \cdot 10^{-4}$.

De wortel-potentialen bij verschillende KCl-concentraties zijn vermeld in tabel 1, als gemiddelden van daarin genoemde aantallen metingen. Voorts zijn in deze tabel de berekende waarden van V vermeld.

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1) Versl. Ned. Akad. v. Wetensch., Afd. Natuurkunde, Vol. 53, 169 (1944).