TABEL 1.

Conc. KCl C_1	2.10^{-1}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Aantal metingen	5	30	30	30	30	30
E gemiddeld	+16.0	+12.47	−7.87	−29.37	−44.27	−51.43
V berekend	—	0.802	0.356	0.151	0.0832	0.0627

Het geringe aantal metingen bij de hoogste concentraties is het gevolg van de onbetrouwbaarheid dezer metingen, vermoedelijk ten gevolge van beschadiging van de wortel door de hoge KCl concentratie.

Door lineaire vereffening van (2) werd berekend:

$$K' = 2{,}36 \cdot 10^3$$

$$A = 0.0653$$

Met deze waarden werden vervolgens de waarden van V berekend met (1), en vervolgens de potentialen volgens:

$$E = 0.577 \log V.$$

In tabel 2 zijn de aldus berekende potentialen verzameld met de bij de verschillende KCl-concentraties gemeten waarden.

TABEL 2.

Conc. KCl C_1	$>10^0$	2.10^{-1}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
E gev.	—	+16.0	+12.47	− 7.87	−29.37	−44.27	−51.43
E ber.	+18.0	+15.8	+13.7	− 5.7	−32.6	−47.7	−51.0

Samenvatting.

Volgens een reeds eerder gegeven theorie blijkt, dat Sinapis alba gekarakteriseerd wordt door

$$K' = 2{,}36 \cdot 10^3 \text{ en } A = 0.0653.$$

Laboratorium voor Physische- en Kolloidchemie.

Wageningen, Maart–April 1947.

Applied Mechanics. — *On the plastic stability of thin plates and shells.* By P. P. BIJLAARD. (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of May 31, 1947.)

About nine years ago we published in these Proceedings our theory on the plastic stability of thin plates [1]). Some time ago KOLLBRUNNER communicated very extensive and systematic tests on the same subject [2]). It appeared that these tests confirm our theory completely.

The tests were effected with thin plates of avional, an aluminium alloy, whilst the stress-strain graph of the material was determined by compression of short plates.

We will compare here the tests as given by KOLLBRUNNER in his figures 33, 34 and 35 with the results of our theory. All these tests relate to plates, compressed in longitudinal direction and supported at the unloaded sides.

Fig. 33 refers to plates of which the unloaded sides are simply supported. According to our theory [3]) the buckling force of such plates is, if the entire plate deforms plastically

$$h\sigma_x = (2\pi^2 EI/b^2)(\sqrt{AD} + B + 2F) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The modulus of elasticity E of avional is 715000 kg/cm^2. The thickness h and the breadth b of the plates were 0,2 cm and 6,2 cm respectively, so that $\pi^2 EI/b^2 = \pi^2 Eh^3/12\,b^2 = 122{,}8$ kg/cm. Furthermore, as in this case the second principal stress is zero, so that $\beta = \varrho_2/\varrho_1 = 0$ and $\eta^2 = \beta^2 - \beta + 1 = 1$, we have [4])

$$A = \varphi_1/\varphi_4,\ B = \varphi_2/\varphi_4,\ D = \varphi_3/\varphi_4,\ F = m/(2m + 2 + 3em),$$

in which

$$\left.\begin{aligned} \varphi_1 &= m^2\{E + (4 + 3e)\tan\varphi\} \\ \varphi_2 &= 2m(mE + 2\tan\varphi) \\ \varphi_3 &= 4m^2(E + \tan\varphi) \\ \varphi_4 &= m(5m - 4 + 3em)E + \{4(m^2 - 1) + 3em^2\}\tan\varphi \end{aligned}\right\} \quad (2)$$

in which $m = 1/\nu = 10/3$, value ν being POISSON's ratio, $e = E\varepsilon_p/\sigma$ and $\tan\varphi = d\sigma/d\varepsilon_p$, the latter two values having to be measured from the stress-strain graph with pure compression at a stress σ being equal to the

1) BIJLAARD, Proc. Kon. Akad. v. Wetensch., Amsterdam, Nrs. 5 and 7 (1938).
2) KOLLBRUNNER, Mitt. a. d. Institut f. Baustatik, Zürich, Nr. 17 (1946).
3) BIJLAARD, loc. cit. eq. (53).
4) BIJLAARD, loc. cit. eqs. (22)—(24).

buckling stress σ_x. The plastic strain ε_p is equal to the total strain ε minus the elastic strain $\varepsilon_e = \sigma/E$, so that

$$\tan\varphi = \frac{d\sigma}{d\varepsilon_p} = \frac{d\sigma}{d\varepsilon - d\varepsilon_e} = \frac{1}{1/E_t - 1/E} = \frac{E E_t}{E - E_t} \quad . \quad . \quad . \quad (3)$$

$E_t = d\sigma/d\varepsilon$ being called the tangent modulus.

With a stress $\sigma_x = 2200$ kg/cm² we find from the stress-strain graph of avional $E_t = 350000$ kg/cm² $= 0{,}49\,E$, e $= 0{,}065$, $\tan\varphi = 0{,}96\,E$, by which eqs. (2) yield $A = 0{,}655$, $B = 0{,}41$, $D = 1{,}02$, $F = 0{,}36$, so that it follows from eq. (1)

$$\sigma_x = 3{,}886\,\pi^2 EI/b^2h = 2385 \text{ kg/cm}^2$$

being more than the stress 2200 kg/cm² we started from.

Assuming now a stress $\sigma_x = 2300$ kg/cm² we find in the same way $E_t = 300000$ kg/cm² $= 0{,}42\,E$, e $= 0{,}11$, $\tan\varphi = 0{,}725\,E$, $A = 0{,}59$, $B = 0{,}41$, $D = 0{,}985$, $F = 0{,}34$, $\sigma_x = 2276$ kg/cm².

Interpolating linearly between assumed as well as between resulting values σ_x we finally find a buckling stress $\sigma_x =$ **2288** kg/cm².

With these tests the eccentricity of the load was certainly such, that with buckling practically no discharge occurred, so that we will have to assume that the plates showed no elastic region. Hence the buckling stress is indeed determined by eq. (1) only, in the same way as also in practice, in connection with small eccentricities, the critical stress with a plate, assumed to be plastic all over, is determinant for the strength of the plate, as we stated already previously [5]).

According to our theory [6]) the plate should buckle in waves with a half wave length $a/p = (A/D)^{\frac{1}{4}}\, b$. With a buckling stress 2288 kg/cm² we find $A = 0{,}60$ and $D = 0{,}99$, so that the half wave length is **0,882 b.** This wave length will occur if the plate is free in selecting its most favourable wave length, i.e. if it is infinitely long. With finite length and if the loaded edges of the plate are simply supported, the half wave length will have to be an integer part of the total length, so that it will deviate to both sides from the most favourable wave length. With the tests, however, the loaded edges were not simply supported, but somewhat clamped, so that the number of half waves will have been the same as with a plate with simply supported extremities, that may be assumed to be about one third of a half wave length shorter. Hence the half wave length, calculated by dividing the total length a of the plates by the number of half waves p, will give a somewhat too high value for the real wave length, whilst moreover these lengths will deviate to both sides from this too high value. The longer the plates, however, the better the optimum wave length will be approximated. According to fig. 33 of KOLLBRUNNER's publication

[5]) BIJLAARD, Publications Int. Association for Bridge and Structural Engineering. Zürich, Vol. 6 (1940/1941), p. 54, footnote 10.

[6]) BIJLAARD, lit. footnote 1, p. 739.

we give in the following table I the lengths a of the plates and the calculated half wave lengths a/p, the numbers in brackets indicating the number of tests.

TABLE I.

a	3,2 b	4,85 b	6,45 b	8,1 b
a/p	1,07 b (3)	0,97 b (3)	0,92 b (3)	0,90 b (3)

It is clear that with longer plates the half wave length converges indeed towards value 0,882 b according to our theory. That for example value 1,07 b, with a plate length 3,2 b, is also in accordance with our theory, may be seen as follows. With a free supported plate length of 3 times 0,882 b = 2,646 b we could expect a buckling in 3 half waves of the optimum wave length 0,882 b, yielding the minimum buckling stress. But from eq. (52), given in our paper mentioned in footnote 1, it may be computed easily, that the same number of waves will occur with freely supported plate lengths between $0{,}882\,b\,\sqrt{6} = 2{,}16\,b$ and $0{,}882\,b\,\sqrt{12} = 3{,}06\,b$. According to our statement above here, the freely supported plate length of the plate with $a = 3{,}2\,b$ will be somewhat less than 3,0 b, so that according to our theory it should indeed buckle in 3 waves, yielding the calculated wave length $a/p = 3{,}2\,b/3 = 1{,}07\,b$, in accordance with the tests.

The buckling stresses of the 12 testplates vary between **2040** and **2190** kg/cm², except one, that yields the low value 1950 kg/cm². Hence, disregarding the latter value, the lowest buckling stress 2040 kg/cm² is only 11 % below the theoretical value 2288 kg/cm² according to our theory. As this discrepancy is not more than the percentage the experimental values in the elastic domain remain underneath the theoretical values, owing to unavoidable inaccuracies, and the theoretical values in the elastic domain being undoubtedly right, we may conclude also to the exactness of our buckling stresses for the plastic domain.

We now consider the buckling of plates of which the unloaded sides are fixed, the test results of which are given in KOLLBRUNNER's fig. 34. The plates have a thickness 0,2 cm and a breadth $b = 4{,}4$ cm. According to our theory the buckling condition for these plates in the plastic domain is [7])

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which

$$\left.\begin{aligned} \alpha_{1,2} &= \sqrt{\pm G\lambda^2 + \lambda\sqrt{H\lambda^2 + K\varphi^2}} \\ G &= \frac{B+2F}{D},\quad H = \frac{(B+2F)^2 - AD}{D^2},\quad K = 1/D \\ \varphi^2 &= \frac{h\,\sigma_x}{EI},\quad \lambda = \pi p/a \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

[7]) BIJLAARD, lit. footnote 5, p. 57.

After assuming σ_x to be 3100 and 3150 kg/cm², it appears that its real value is about 3140 kg/cm². With $\sigma_x = 3150$ kg/cm² we found, in the same way as before: $E_t = 0{,}04\,E$, $e = 1{,}49$, $\tan\varphi = E/24$, $A = 0{,}157$, $B = 0{,}238$, $D = 0{,}485$, $F = 0{,}141$, and by using eqs. (5): $G = 1{,}07$, $H = 0{,}825$, $K = 2{,}06$, $\varphi^2 = 1{,}32$. Assuming now $\lambda = 1{,}55$ we obtain $\alpha_1 = 2{,}435$, $\alpha_2 = 0{,}89$, so that eq. (4) yields

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) = 0{,}27$$

instead of zero.

Assuming now $\sigma_x = 3140$ kg/cm² we find $G = 1{,}072$, $H = 0{,}827$, $K = 2{,}01$, $\varphi^2 = 1{,}315$, yielding, with $\lambda = 1{,}55$, values $\alpha_1 = 2{,}43$ and $\alpha_2 = 0{,}872$, by which eq. (4) yields

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) = 0{,}02$$

so that σ_x is round **3140** kg/cm² with $\lambda = \pi p/a = 1{,}55$ and a half wave length $a/p = \pi/1{,}55 = 2{,}02$ cm $=$ **0,46 b.**

Calculating in the same way the buckling stresses with other values λ it appeared that with $\lambda = 1{,}55$ the critical stress is about a minimum. In the following table the plate lengths a and the half wave lengths a/p according to the tests are given.

TABLE II.

a	$4{,}55\ b$	$6{,}80\ b$	$9{,}10\ b$	$11{,}40\ b$
a/p	$0{,}495\ b$ (3)	$0{,}52\ b$ (2) $0{,}485\ b$ (1)	$0{,}535\ b$ (1) $0{,}505\ b$ (2)	$0{,}475\ b$ (3)

It is clear that with longer plates the half wave length converges indeed to our theoretical value 0,46 b. The latter plate, of which the free supported length may be assumed as 11,20 b, buckled in 24 half waves, whilst with the theoretical buckling length 0,46 b this number of waves should occur between freely supported plate lengths of about 23,5 and 24,5 times 0,46 b, or 10,81 b and 11,27 b, so that the 24 half waves are indeed in accordance with our theory.

One extraordinary low value, being 2580 kg/cm², excepted, the buckling stresses for the other 11 tests were between **2830** and **3115** kg/cm², the lowest value 2830 kg/cm² being only 10 % underneath our theoretical value.

Finally considering fig. 35, referring to plates that are simply supported at one unloaded side and fixed at the other, with $h = 0{,}2$ cm and $b = 5{,}3$ cm, the buckling condition is given by [7])

$$\alpha_1 \coth \alpha_1 b - \alpha_2 \cot \alpha_2 b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whilst α_1 and α_2 follow from eqs. (5).

In the same way as before we find here that the buckling stress σ_x acquires a minimum value with about $\lambda = 0{,}865$, hence with a half wave length $a/p = \pi/\lambda = 3{,}63$ cm $=$ **0,685 b**, whilst $\sigma_x =$ **2882** kg/cm².

The following table gives the experimental values a/p, converging again in an excellent way to our theoretical value 0,685 b. The buckling stresses of all thirteen tests are between **2660** and **2850** kg/cm², the lowest value being only 8 % underneath our theoretical value.

TABLE III.

a	$3{,}8\ b$	$5{,}65\ b$	$7{,}55\ b$	$9{,}45\ b$
a/p	$0{,}635\ b$ (3) $0{,}760\ b$ (1)	$0{,}705\ b$ (3)	$0{,}755\ b$ (1) $0{,}685\ b$ (2)	$0{,}727\ b$ (1) $0{,}675\ b$ (2)

Hence we may conclude, that KOLLBRUNNER's tests have proved the applicability of our theory, as the number of waves shows, that the anisotropic behaviour of the material is exactly such as predicted by our theory, whilst the discrepancies of the buckling stresses are not more than in the elastic domain.

Under these circumstances it may be of interest to give here a short indication of the application of our theory to the buckling of shells.

Our fundamental equations are those giving the relation between the excess stresses and the excess strains with buckling, being [8])

$$\left.\begin{aligned} \sigma'_x &= E\,(A\,\varepsilon'_x + B\,\varepsilon'_y) \\ \sigma'_y &= E\,(B\,\varepsilon'_x + D\,\varepsilon'_y) \\ \tau'_{xy} &= E\,F\,\gamma'_{xy} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

values A, B, D and F being given by eqs. (22) and (23) of our publication mentioned in footnote 1.

Using the same notations as TIMOSHENKO [9]), except our primes, indicating infinitely small stresses and strains occurring with buckling, we have

$$\left.\begin{aligned} \varepsilon'_x &= \varepsilon'_1 - \chi'_x\,z \\ \varepsilon'_y &= \varepsilon'_2 - \chi'_y\,z \\ \gamma'_{xy} &= \gamma' - 2\,\chi'_{xy}\,z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which ε'_1, ε'_2 and γ' are the excess strains of the middle surface in X- and Y-directions, χ'_x and χ'_y are the changes of curvature and χ'_{xy} is the twist, whilst z is the distance from the middle surface. Substituting eqs. (8) in eqs. (7) we get

$$\left.\begin{aligned} \sigma'_x &= E\,\{A\,\varepsilon'_1 + B\,\varepsilon'_2 - z\,(A\,\chi'_x + B\,\chi'_y)\} \\ \sigma'_y &= E\,\{B\,\varepsilon'_1 + D\,\varepsilon'_2 - z\,(B\,\chi'_x + D\,\chi'_y)\} \\ \tau'_{xy} &= E\,F\,(\gamma' - 2\,\chi'_{xy}\,z) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

[8]) BIJLAARD, lit. footnote 1, eqs. (21)—(24).

[9]) TIMOSHENKO, Theory of elastic stability. Chapters VIII and IX

Hence we find [10])

$$\left.\begin{aligned} N'_x &= \int_{-h/2}^{+h/2} \sigma'_x \, dz = E h \,(A\, \varepsilon'_1 + B\, \varepsilon'_2) \\ N'_y &= \int_{-h/2}^{+h/2} \sigma'_y \, dz = E h \,(B\, \varepsilon'_1 + D\, \varepsilon'_2) \\ N'_{xy} &= N'_{yx} = \int_{-h/2}^{+h/2} \tau'_{xy} \, dz = E F h \gamma' \\ M'_x &= \int_{-h/2}^{+h/2} \sigma'_x \, z \, dz = -E I \,(A \chi'_x + B \chi'_y) \\ M'_y &= \int_{-h/2}^{+h/2} \sigma'_y \, z \, dz = -E I \,(B \chi'_x + D \chi'_y) \\ M'_{xy} &= -M'_{yx} = -\int_{-h/2}^{+h/2} \tau'_{xy} \, z \, dz = 2 E I F \chi'_{xy} \end{aligned}\right\} \quad . \; . \; . \; . \; (10)$$

As an example we will consider the buckling of a cylindrical shell under the action of uniform axial pressure $h\sigma_x$ per unit breadth. If buckling symmetrical to the axis of the cylinder occurs, the equilibrium of an element hdx of a strip OP of unit width (fig. 1) requires, if compressive stresses

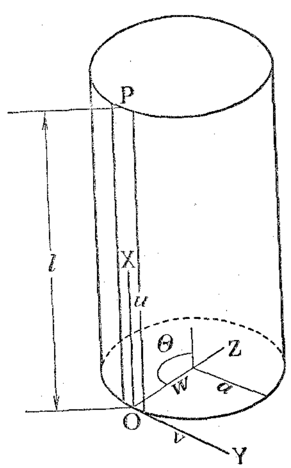

Fig. 1.

are denoted as positive and denoting displacements with buckling in Z-direction by w

$$\frac{d\,Q'_x}{dx} - h\,\sigma_x \frac{d^2 w}{dx^2} - \frac{N'_y}{a} = 0. \quad . \; . \; . \; . \; . \; . \; (11)$$

Value $Q'_x = dM'_x/dx$. Owing to impediment of distortion of the cross section, value χ'_y may be equated to zero, so that eqs. (10) yield

$$M'_x = -E I A\, \chi'_x = -E I A\, d^2 w / d x^2$$

10) Cf. same values in elastic domain in TIMOSHENKO, l.c. p. 421, 422.

and, as $\varepsilon'_2 = -w/a$

$$N'_x = E h\,(A\,\varepsilon'_1 - B\, w/a).$$

As, however, $h\sigma_x$ does not increase with buckling, N'_x must be zero, yielding $\varepsilon'_1 = (B/A)\,(w/a)$, by which we obtain

$$N'_y = E h\,(B^2/A - D)\, w/a.$$

Substitution of these values in eq. (11) yields, as in (11) N'_y is a compression, the differential equation

$$E I A \frac{d^4 w}{dx^4} + h\,\sigma_x \frac{d^2 w}{dx^2} - \left(\frac{B^2}{A} - D\right) E h\, w/a^2 = 0. \quad . \; . \; . \; (12)$$

With $w = w_0 \sin p\pi x/l$ eq. (12) yields, after ranging

$$h\,\sigma_x = E I A\, p^2 \pi^2/l^2 + (D - B^2/A)\,(E h/a^2)\, l^2/p^2\pi^2. \quad . \; . \; . \; (13)$$

With sufficiently long cylinders the wave length can establish itself in such a way as to make $h\sigma_x$ a minimum. Differentiation shows that then

$$\frac{p\pi}{l} = \sqrt[4]{\frac{AD - B^2}{A^2} \frac{h}{a^2 I}}$$

insertion of which in eq. (13) yields the critical stress

$$\sigma_x = \frac{E h}{a} \sqrt{\frac{AD - B^2}{3}} \quad . \; . \; . \; . \; . \; . \; . \; . \; (14)$$

whilst the length of the half waves in X-direction is

$$\frac{l}{p} = \pi \sqrt[4]{\frac{A^2}{12(AD - B^2)}} \sqrt{a h}. \quad . \; . \; . \; . \; . \; (15)$$

In the elastic domain $e = 0$ and $\tan \varphi = \infty$, so that eqs. (2), which also apply to this case, yield $A = D = m^2/(m^2 - 1)$ and $B = m/(m^2 - 1)$, by which eqs. (14) and (15) transform in

$$\sigma_x = \frac{E h}{a} \sqrt{\frac{m^2}{3(m^2 - 1)}} \quad \text{and} \quad \frac{l}{p} = \pi \sqrt[4]{\frac{m^2 a^2 h^2}{12(m^2 - 1)}}$$

in accordance with the values obtained directly for this case [11]).

Considering the more general case of buckling of a cylindrical shell under axial compression and denoting the displacements with buckling in X-, Y- and Z-direction by u, v and w respectively, our equations (10) yield, after

11) TIMOSHENKO, loc. cit. p. 440—441.

expression of values $\varepsilon_1', \varepsilon_2', \gamma', \chi_x', \chi_y'$ and χ_{xy}' in terms of the displacements [12]),

$$\left.\begin{aligned}
N_x' &= E h \left\{ A \frac{\partial u}{\partial x} + B \left(\frac{\partial v}{a \partial \theta} - \frac{w}{a} \right) \right\} \\
N_y' &= E h \left\{ B \frac{\partial u}{\partial x} + D \left(\frac{\partial v}{a \partial \theta} - \frac{w}{a} \right) \right\} \\
N_{xy}' &= N_{yx}' = E F h \left(\frac{\partial u}{a \partial \theta} + \frac{\partial v}{\partial x} \right) \\
M_x' &= - E I \left\{ A \frac{\partial^2 w}{\partial x^2} + \frac{B}{a^2} \left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right) \right\} \\
M_y' &= - E I \left\{ B \frac{\partial^2 w}{\partial x^2} + \frac{D}{a^2} \left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right) \right\} \\
M_{xy}' &= - M_{yx}' = 2 E F I \frac{1}{a} \left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial \theta} \right)
\end{aligned}\right\} \quad . \ . \ . \ . \quad (16)$$

value θ being indicated in fig. 1.

The conditions of equilibrium are in our notations and neglecting second order terms [13])

$$\left.\begin{aligned}
& a \frac{\partial N_x'}{\partial x} + \frac{\partial N_{yx}'}{\partial \theta} = 0 \\
& \frac{\partial N_y'}{\partial \theta} + a \frac{\partial N_{xy}'}{\partial x} - a h \sigma_x \frac{\partial^2 v}{\partial x^2} + \frac{\partial M_{xy}'}{\partial x} - \frac{\partial M_y'}{a \partial \theta} = 0 \\
& - a h \sigma_x \frac{\partial^2 w}{\partial x^2} + N_y' + a \frac{\partial^2 M_x'}{\partial x^2} + \frac{\partial^2 M_{yx}'}{\partial x \partial \theta} + \frac{\partial^2 M_y'}{a \partial \theta^2} - \frac{\partial^2 M_{xy}'}{\partial x \partial \theta} = 0
\end{aligned}\right\} \quad . \ . \quad (17)$$

Substitution of eqs. (16) in eqs. (17) yields

$$\left.\begin{aligned}
& A \frac{\partial^2 u}{\partial x^2} + \frac{B+F}{a} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{B}{a} \frac{\partial w}{\partial x} + \frac{F}{a^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \\
& (B+F) \frac{\partial^2 u}{\partial x \partial \theta} + a F \frac{\partial^2 v}{\partial x^2} + \frac{D}{a} \left(\frac{\partial^2 v}{\partial \theta^2} - \frac{\partial w}{\partial \theta} \right) + \\
& \alpha \left[\frac{D}{a} \left(\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3} \right) + a (B + 2F) \frac{\partial^3 w}{\partial x^2 \partial \theta} + 2 a F \frac{\partial^2 v}{\partial x^2} \right] - \frac{a h \sigma_x}{E h} \frac{\partial^2 v}{\partial x^2} = 0 \\
& - \frac{a h \sigma_x}{E h} \frac{\partial^2 w}{\partial x^2} + B \frac{\partial u}{\partial x} + \frac{D}{a} \frac{\partial v}{\partial \theta} - \frac{D}{a} w - \alpha \left[a (B + 4F) \frac{\partial^3 v}{\partial x^2 \partial \theta} + \frac{D}{a} \frac{\partial^3 v}{\partial \theta^3} + \right. \\
& \left. a^3 A \frac{\partial^4 w}{\partial x^4} + 2 a (B + 2F) \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{D}{a} \frac{\partial^4 w}{\partial \theta^4} \right] = 0
\end{aligned}\right\} \quad (18)$$

in which $\alpha = h^2/12\, a^2$.

[12]) TIMOSHENKO, loc. cit. p. 434.
[13]) Cf. TIMOSHENKO, loc. cit. eqs (c), p. 454.

Assuming

$$\left.\begin{aligned}
u &= u_0 \cos n \theta \cos p \pi x / l \\
v &= v_0 \sin n \theta \sin p \pi x / l \\
w &= w_0 \cos n \theta \sin p \pi x / l
\end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (19)$$

eqs. (18) yield

$$\left.\begin{aligned}
& (A \lambda^2 + F n^2) u_0 - (B + F) n \lambda v_0 + B \lambda w_0 = 0 \\
& - (B + F) n \lambda u_0 + [D n^2 (1 + \alpha) + \\
& \qquad F \lambda^2 (1 + 2 \alpha) - \sigma_x \lambda^2 / E] v_0 - n [D (1 + \alpha n^2) + (B + 2F) \alpha \lambda^2] w_0 = 0 \\
& B \lambda u_0 - n [D (1 + \alpha n^2) + (B + 4F) \alpha \lambda^2] v_0 + \\
& \qquad [A \alpha \lambda^4 + 2 (B + 2F) \alpha n^2 \lambda^2 + D (1 + \alpha n^4) - \sigma_x \lambda^2 / E] w_0 = 0
\end{aligned}\right\} \quad (20)$$

in which $\lambda = p \pi a / l$.

Hence the critical stress σ_x follows by equating the determinant of these equations to zero. Further computations may be effected along the same lines as in the elastic domain.

In order to check eqs. (20) we assume again buckling symmetrical to the axis of the cylinder, so that in eqs. (19) we have to equal n to zero, so that also v becomes zero. Hence the second equation (20) vanishes and eqs. (20) transform in

$$\left.\begin{aligned}
& A \lambda u_0 + B w_0 = 0 \\
& B \lambda u_0 + (A \alpha \lambda^4 + D - \sigma_x \lambda^2 / E) w_0 = 0
\end{aligned}\right\} \quad . \ . \ . \ . \quad (21)$$

yielding $\qquad \sigma_x = E\, [A \alpha \lambda^2 + (AD - B^2) / A \lambda^2]$

or $\qquad h \sigma_x = E I A p^2 \pi^2 / l^2 + (D - B^2 / A)\, (E h / a^2)\, l^2 / p^2 \pi^2$

in accordance with eq. (13).

As a matter of fact thick tubes, buckling in the plastic domain, do this usually in a symmetrical way, whilst with thin tubes buckling which is non-symmetrical with respect to the axis usually occurs [14]). This behaviour is in good accordance with our theory, because non-symmetrical buckling causes twisting stresses, against which, if e is small, as with steel, the resistance is only slightly diminished according to our theory. We can proof this directly with our eqs. (20).

For the elastic domain TIMOSHENKO proofs, that, if λ^2 is a large number, the critical stress with non-symmetrical buckling is equal to that with symmetrical deformation [15]). Taking into account the same terms as he

[14]) TIMOSHENKO, loc. cit. p. 443.
[15]) TIMOSHENKO, loc. cit. p. 456.

does, we find for the plastic domain, by equating the determinant of eqs. (20) to zero, the following equation

$$[(A\lambda^2+Fn^2)(F\lambda^2+Dn^2)-(B+F)^2n^2\lambda^2]\,\lambda^2\,\sigma_x/E=$$
$$(AD-B^2)F\lambda^4+$$
$$\alpha[A\lambda^4+2(B+2F)n^2\lambda^2+Dn^4]\,[(A\lambda^2+Fn^2)(F\lambda^2+Dn^2)-(B+F)^2n^2\lambda^2]$$

or, after some transformation

$$\frac{\sigma_x}{E}=\frac{(AD-B^2)\lambda^2}{A\lambda^4+Dn^4+[(AD-B^2)/F-2B]n^2\lambda^2}+\frac{\alpha[A\lambda^4+Dn^4+2(B+2F)n^2\lambda^2]}{\lambda^2}. \quad (22)$$

In the elastic domain we have[16] $A=D=m^2/(m^2-1)$, $B=m/(m^2-1)$, $F=m/2(m+1)$, so that $(AD-B^2)/F-2B= =2\,(B+2F)=2m^2/(m^2-1)$, by which the denominator of the first fraction of the second member of eq. (22) is equal to the term in brackets of the numerator of the second fraction. If in the plastic domain this would be so too, we could write eq. (22) as follows

$$\sigma_x/E=(AD-B^2)\,\psi+\alpha/\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

value ψ being a function of values λ and n, that determine the number of waves in axial and circumferential direction. In order to make σ_x a minimum, we then would have the condition

$$\psi=\sqrt{\frac{\alpha}{AD-B^2}}$$

by which eq. (23) would yield

$$\sigma_x=2E\sqrt{\alpha(AD-B^2)} \quad \text{or} \quad \sigma_x=(Eh/a)\sqrt{(AD-B^2)/3}$$

in accordance with the buckling stress given by eq. (14) for symmetrical buckling. In the plastic domain, however, considering for example buckling at the yield stress with mild steel, value $(AD-B^2)/F-2B$ will be much less than value $2(B+2F)$. In this case we have[17] $A=0{,}421$, $B=0{,}426$, $D=0{,}938$, $F=0{,}322$, by which $(AD-B^2)/F-2B=\mathbf{-0{,}19}$, whilst $2(B+2F)=\mathbf{2{,}14}$. Hence the first fraction of the second member of eq. (22) has a much higher value than with our assumption that gave eq. (23). Hence we may conclude that with higher values λ^2 in the plastic domain the critical stress with non-symmetrical buckling is higher than with symmetrical buckling, which may explain why short and thick tubes usually buckle symmetrically. In another paper we will consider these questions more in detail.

As follows also from the good agreement, obtained in this way with the tests of KOLLBRUNNER, with a given stress-strain graph of the material,

[16]) BIJLAARD, lit. footnote 1, p. 739.
[17]) BIJLAARD, lit. footnote 1, p. 736.

the buckling stresses of thin plates will have to be computed under the assumption that the entire plate deforms plastically. If, however, not the stress-strain graph is given, but for example the relation between slenderness ratio and buckling stress of columns, it evidently makes little difference in the resulting buckling stresses whether both columns and plates are assumed to show elastic regions or not, the latter way being, however, the most simple one. Although with shells conditions are in several respects different from those with plates, we think that here too, with a given stress-strain graph, the most logical way is to assume that no elastic regions occur with buckling.