Mathematics. — *A Note on Irreducible Star Bodies.* By C. A. ROGERS. (University College, London.) (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of September 27, 1947.)

In a recent paper MAHLER [1]) has developed the general theory of lattice points in n-dimensional star bodies. In one section of the second part of his paper (334—343) MAHLER developes a theory of irreducible star bodies. In this note I indicate a modified treatment of this portion of MAHLER's work. In particular I obtain a necessary and sufficient condition for a star body to be irreducible.

MAHLER's notation and terminology will be used; but instead of working with MAHLER's concept of a "free" lattice we introduce the concept of reducible and irreducible points. A point P of a star body S will be called reducible if there is a star body $T \prec S$ not containing P, i.e. a star body T, contained in S but not containing P, for which

$$\Delta(T) = \Delta(S).$$

A point of S which is not reducible will be called irreducible.

Theorem 1. *A star body S of the finite type is irreducible if and only if every point on its boundary is irreducible.*

Proof. (a) If S is irreducible then it is impossible to find a star body $T \prec S$. So every point of S is irreducible, and in particular every point on the boundary of S is irreducible.

(b) If S is reducible then there is a star body $T \prec S$. Every point in S but not in T is thus reducible. Thus we have only to show that there is a point, on the boundary of S, which is not in T; i.e. we have only to prove the following lemma.

Lemma. *Let T be a star body contained in but different from a star body S of the finite type. Then there is a point on the boundary of S which is not in T.*

Proof. Suppose, if possible, that every point on the boundary of S belongs to T. As T is contained in but different from S it follows that there is a point P in the interior of S which is not in T. Now the set of points outside T and interior to S is open and contains P. So we can choose a number $\varepsilon > 0$, so small that the sphere

$$|X - P| \leqslant \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is outside T but is interior to S.

1) K. MAHLER, Proc. Royal Soc., A **187**, 151—187 (1946) and Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam., **49**, 331—343, 444—454, 525—532, 622—631 (1946).

Suppose X is any point in the sphere (1); it is interior to S but outside T. Suppose that for some positive λ the point λX is outside S. Then as X is in the interior of the star body S there is μ with $1 < \mu < \lambda$ such that μX is on the boundary of S. But as X is not in the star body T, the point μX is not in T, contrary to our original supposition. Thus for every positive λ the point λX is in S and so S contains the sphere

$$|X - \lambda P| \leqslant \varepsilon \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If Λ is any lattice, λ can be chosen so large that there is a point of Λ, other than the origin O, in the sphere (2). Thus no lattice Λ is admissable for S and S is a star body of the infinite type. This contradiction proves the lemma.

Theorem 2. *If S is a star body of the finite type then the set Σ of irreducible points of S is closed* [2]).

Proof. Suppose, if possible, that P is a limit point of the set Σ, not in Σ. Then P is a reducible point of S and there is a star body $T \prec S$ not containing P. Then as the limit point P of Σ is outside the closed set T, there is a point Q of Σ outside T. But such a point Q of S outside T is reducible. This contradiction proves that P is in Σ and that Σ is closed.

Theorem 3. *A point P on the boundary of a star body S of the finite type is irreducible if and only if, for every $\varepsilon > 0$, there exists a lattice Λ, with*

$$d(\Lambda) < \Delta(S),$$

such that the only points of Λ in the interior of S are points O, Q and $-Q$ with

$$|P - Q| < \varepsilon.$$

Proof. (a) Suppose that P is an irreducible point of the boundary of S and ε is a positive number. Let S be the body given by $F(X) \leq 1$, where, as usual, we suppose that $F(X)$ is non-negative, bounded and continuous and, for all real t,

$$F(tX) = |t|\,F(X).$$

Write

$$\sigma = \varepsilon / (1 + |P|), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\left.\begin{aligned} &G(X) = 0, \text{ when } F(X) = 0, \\ &G(X) = F(X)\left[1 + \sigma - \min\left\{\sigma, \left|\frac{X}{F(X)} - \frac{P}{F(P)}\right|, \left|\frac{X}{F(X)} + \frac{P}{F(P)}\right|\right\}\right] \text{ otherwise,} \end{aligned}\right\} \quad (4)$$

Since $F(tX) = |t|\,F(X)$, we have $G(tX) = |t|\,G(X)$, and $G(X)$ is a

2) If S is bounded it is possible to prove that Σ is the closure of an open set containing the origin.

non-negative bounded continuous function of X. Thus the body T given by

$$G(X) \leqslant 1$$

is a star body. Further as $G(X) \geqq F(X)$ and $G(P) > F(P) = 1$, the star body T is contained in S but does not contain P. As P is an irreducible point of S this implies that

$$\Delta(T) < \Delta(S).$$

So there is a lattice Λ which is admissible for T and which has

$$d(\Lambda) < \Delta(S).$$

Suppose Q is any point of Λ other than O, in the interior of S. Then

$$F(Q) < 1 \leqslant G(Q).$$

But, by (4),

$$G(Q) \leqslant F(Q)\{1+\sigma\},$$

and

$$G(Q) = F(Q)$$

unless, for one of the signs,

$$\left|\frac{Q}{F(Q)} \pm \frac{P}{F(P)}\right| < \sigma.$$

Thus

$$\{1+\sigma\}^{-1} \leqslant F(Q) < 1, \qquad (5)$$

and (since $F(P) = 1$), for one of the signs

$$|Q \pm P\,F(Q)| < \sigma F(Q).$$

Consequently, for one of the signs,

$$|Q \pm P| \leqslant |Q \pm P\,F(Q)| + |P| \cdot |1 - F(Q)|$$
$$< \sigma\{1 + |P|\} = \varepsilon,$$

using (5) and (3).

We may without loss of generality assume that ε is so small that the sphere

$$|X| \leqslant 2\varepsilon$$

is within the star body T. Then, if Q' were any point of Λ other than O or Q or $-Q$ in the interior of S, we should have either

$$|Q - Q'| < 2\varepsilon \text{ or } |Q + Q'| < 2\varepsilon$$

and Λ would not be admissible for T. This establishes the existence of a lattice with the required properties.

(b) Suppose that for some point P on the boundary of S, for every $\varepsilon > 0$, there exists a lattice Λ, with

$$d(\Lambda) < \Delta(S),$$

such that the only points of Λ in the interior of S are points O, Q and $-Q$ with

$$|P - Q| < \varepsilon.$$

Suppose, if possible, that P is reducible. Then there exists a star body $T \prec S$ not containing P. Then as P is outside the closed symmetrical set T, we can choose $\varepsilon > 0$ so small that all points X, satisfying either

$$|X - P| < \varepsilon \text{ or } |X + P| < \varepsilon,$$

lie outside T. We can also choose a lattice Λ, with

$$d(\Lambda) < \Delta(S) = \Delta(T), \qquad (6)$$

such that the only points of Λ in the interior of S are points O, Q and $-Q$ with

$$|P - Q| < \varepsilon.$$

As this means that Q and $-Q$ are outside T, the lattice Λ is admissible for T and so

$$d(\Lambda) \geqslant \Delta(T),$$

contrary to (6). This proves that P is irreducible and completes the proof of the theorem.

Theorem 4. *If P is an irreducible point on the boundary of a star body S of the finite type, then P is a point of at least one of the critical lattices of S.*

Proof. As P is an irreducible point on the boundary of S, it follows from Theorem 3, that we can choose an infinite sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, with

$$d(\Lambda_r) < \Delta(S), \quad r = 1, 2, 3, \ldots,$$

and such that the only points of Λ_r in the interior of S are points O, P_r and $-P_r$, with

$$|P_r - P| < 1/r.$$

We have:

(a): $$\overline{\lim_{r\to\infty}}\, d(\Lambda_r) \leqslant \Delta(S);$$

(b): For every r, the lattice Λ_r is admissible for the body given by

$$F(X) \leqslant F(P_r);$$

(c): The points $P_1, P_2, P_3, \ldots$ tend to the limit point P on the boundary of S.

The proof can now be completed by the method used by MAHLER in proving his Theorem B. Briefly his argument shows that there is a subsequence of the sequence $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, which converges to a lattice Λ. The point P is a point of Λ. The lattice Λ is admissible for S. Further

$$\Delta(S) \leqslant d(\Lambda) \leqslant \overline{\lim_{r\to\infty}}\, d(\Lambda_r) \leqslant \Delta(S),$$

so that Λ is a critical lattice.

MAHLER's Theorem C is an immediate consequence of Theorems 1 and 4. It is also clear from Theorem 3 and MAHLER's definition B (reading $\pm P_k^*$ for $\pm P_k$ in condition (c)) that every point of the boundary of S which is a lattice point of a free lattice is irreducible in our sense. Using this, MAHLER's Theorem D follows at once from Theorems 1 and 2.

I am grateful to Professor DAVENPORT and Dr. MAHLER for reading the manuscript and making a number of useful suggestions.

University College, London.

Mathematics. — *Some examples concerning the relations between homology and homotopy groups.* By HSIEN-CHUNG WANG. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

1. The relations between homology and homotopy groups were firstly discussed by H. HOPF. To improve and generalize HOPF's results, ECKMANN, EILENBERG, FREUDENTHAL and MACLANE have made extensive researches. The investigation is still going on. Two of the main known results can be stated as follows:

A. $H^2(M, G)$ is determined by the groups $\pi_1(M)$, $\pi_2(M)$ and a certain homotopic invariant k^3 [4].

B. For a space M with

$$\pi_2(M) = \pi_3(M) = \ldots = \pi_{n-1}(M) = 0 \; , \quad \ldots \quad (1)$$

the factor group $H^n(M)/\Sigma^n(M)$ is determined by the fundamental group $\pi_1(M)$ [3, 5].

Here, as throughout this note, M denotes an arcwise connected space, $\pi_n(M)$ the nth homotopy group, $H^n(M)$ the nth Betti group, $H^n(M, G)$ the nth homology group with coefficient group G, and Σ^n the spherical subgroup of $H^n(M)$.

It is the aim of this note to show that all the homotopy groups together with the dimension of the space are not sufficient to determine H^n $(n = 2, 3, \ldots)$, nor the factor group H^{n+1}/Σ^{n+1} even when the space is an analytic closed manifold having the property (1). The proof consists of very simple arguments. However, this tells us that the above two results of EILENBERG and MACLANE cannot be further improved and the relation between homology and homotopy groups is quite loose.

2. Let S^k and P^k denote the k-sphere and the k-dimensional real projective space respectively. The homotopy groups of spheres have been well discussed. In what follows, we need the following results [2]:

$$\pi_r(S^k) = 0, \quad \pi_k(S^k) \approx I, \quad (r < k, \; k > 0). \quad \ldots \quad (2)$$

where I denotes the free cyclic group. The k-sphere is a covering space of P^k with two leaves. It follows that [6] 1)

$$\pi_1(P_k) \approx I_2, \quad \pi_r(P^k) \approx \pi_r(S^k), \quad (k > 1, \; r > 1). \quad \ldots \quad (3)$$

Let us consider the topological products

$$M_1 = S^n \times P^{n+2}, \quad M_2 = P^n \times S^{n+2}, \quad (n > 1). \quad \ldots \quad (4)$$

both of which are analytic closed manifolds of $2n + 2$ dimensions. The

1) Here, as well as throughout this note, I_2 denotes a cyclic group of order two.