Mathematics. — *The congruence order of the elliptic plane.* By J. HAANTJES and J. SEIDEL. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1947.)

A *metric space* is a set of abstract elements called points such that to each pair of points p, q there is attached a non-negative real number pq, called the distance of p and q, satisfying the conditions

1. $pq = 0$ implies $p = q$
2. $pq + qr \geq pr$ (triangle inequality)

Two metric spaces M and M' are said to be *congruent* if there exists a mapping of each into the other preserving the distances of the points.

A metric space M is called *imbeddable* in a metric space S if M is congruent with a subset of S.

In order to find a metric characterization of euclidean spaces MENGER showed in the "Zweite Untersuchung" [1]) that if every set of $n + 3$ points of a metric space M is congruent with $n + 3$ points of a euclidean n-dimensional space R_n, then M is congruent with a subset of R_n. The number $n + 3$ is a metric invariant of the R_n and is called the congruence order of the R_n. The notion congruence order is then defined as follows:

A space S' has *congruence order* k provided that any metric space M is congruent with a subset of S' whenever each set of k points of M is congruent with k points of S'.

It has been proved that the congruence order of the n-dimensional hyperbolical space and of the n-dimensional spherical space is also $n + 3$ [2]). It is however easily seen that the congruence order of the elliptic plane I_2 is at least seven [3]). For an I_2 of total curvature r^2 contains equilateral sextuples [3]) [4]) with edge $d = r \, arc \cos \frac{1}{5}\sqrt{5}$. Therefore, a metric space M with more than 6 points, for which the distance of every pair of different points equals d, has the property that every subset of 6 points is congruent with 6 points of I_2, whereas M is not congruent with a subset of I_2. This example shows that the congruence order of I_2 is at least 7. It is the purpose of this paper to show that this congruence order is 7. For the sake of brevity we have to confine ourselves to a general outline of the proof. The full proof will be published shortly in the thesis of J. SEIDEL.

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[1]) K. MENGER, Untersuchungen über allgemeine Metrik, Math. Annalen **100**, 113—141 (1928).

[2]) L. M. BLUMENTHAL, The geometry of a class of semimetric spaces. Tôhoku Math. Journal **43**, 205—214 (1937).

[3]) L. M. BLUMENTHAL, Metric characterization of elliptic space. Trans. Amer. Math. Soc. **59**, 381—400 (1946).

[4]) J. HAANTJES, Equilateral point-sets in elliptic two- and three-dimensional spaces. Nieuw Arch. v. Wiskunde XXII.

The methods used in order to find the congruence order of euclidean and hyperbolic spaces are wholly unsuitable for elliptic spaces. One of the main reasons is that the congruence of two subsets of an elliptic space I_2 does not imply *superposability* (that is the existence of a congruent transformation of the space on itself which carries one set into the other). A simple example is furnished by two triples a, b, c and a', b', c' with $ab = bc = ac = a'b' = b'c' = a'c' = \frac{1}{3}\pi r$, where the points a, b and c lie on a straight line and a', b' and c' form the vertices of a proper triangle.

As a first result the following theorem is obtained:

Theorem 1. *A metric space M consisting of exactly* 8 *points is imbeddable in I_2 whenever each seventuple is.*

In order to prove this theorem the following cases are distinguished:

A. M contains four points, which are congruent with four points of I_2 forming the vertices and the orthocenter of a proper triangle of which no side equals $\frac{1}{2}\pi r$. Let it be the points 1, 2, 3, 4. This property is denoted by O (1234).

B. M contains four points which are congruent with four points on a straight line in I_2. This property is denoted by L (1234).

C. M contains five points congruent with five points of I_2 three of which (1, 2, 3) are linear, whereas the line joining the two other points (4, 5) is perpendicular to the line (123). Moreover no four points of this fivetuple have the property O. This property is denoted by V (123,45).

D. M contains no point sets with the property O, L or V.

The point sets with one of the properties O, L and V play an important part in the proof because it can be shown that two congruent sets of this kind in I_2 are at the same time superposable.

In the case A [O(1234)] we consider the point sets in I_2 which are congruent with the sets 1234567, 1234568, 1234578, 1234678. The corresponding points in I_2 are denoted by 1234567, 12345'6'8, 12345''7'8', 12346''7''8'', which means that the elliptic representation of the first four points is the same in each set. This may be supposed because according to the above remark any two congruent representations of these four points are superposable. Now the triple 5, 5', 5'' may consist of only one point (the points are identical) or it may contain two different points or three different points. If $5 \neq 5'$ the perpendicular bisectors of the segment 55' must contain the points 1, 2, 3 and 4 because the points 5 and 5' have the same distances to the points 1234. If 5, 5' and 5'' are three different points the triple is wholly determined by the set 1234, the points of which being the four circumcenters of the triangle 55' 5''. The same can be said of the triples 66' 6'', 77' 7'' and 88' 8''. This leads to several cases which are treated seperately. In either possible case it can be shown that from each triple one point can be chosen in such a way that these points together with the points 1234 are congruent with the eight points of M. Thus M is imbeddable in I_2.

The cases mentioned under B and C are dealt with in much the same way. The treatment is somewhat simpler because as is easily seen at least two of the points 55′ 5″ (66′ 6″ etc.) must coincide.

If the metric space M contains no point sets with the property O, L or V (the case mentioned under D) it is proved that there exist three points (1, 2, 3) such that in the congruent representations of the sets 1234567, 1234568, 1234578 the representations of the points 1, 2, 3 are superposable. Then the corresponding points may be denoted by 1234567, 1234′5′6′8, 1234″5″7′8′. Again the points 5, 5′, 5″ etc. may be different or equal. This leads to several cases, which are all treated separately. Again it turns out that in either case eight points in I_2 can be found congruent with the metric space M. Therefore, M is imbeddable in I_2.

Then the proof that the congruence order is seven is completed by showing:

Theorem 2. *A metric space M is imbeddable in I_2 whenever each eighttuple is.*

This theorem may also be stated as follows:

The congruence order of the I_2 is ≤ 8.

The proof of this theorem is similar to that of the first theorem. From theorem 1 and 2 it is seen that the congruence order is ≤ 7, whereas an example shows, as we have seen, that it is ≥ 7. So the congruence order of the elliptic plane is seven.

Mathematics. — *On the zeros of composition-polynomials.* By N. G. DE BRUIJN and T. A. SPRINGER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1947.)

1. Introduction.

Recently [1]) we proved some inequalities, expressing that the zeros of the derivative of a polynomial lie, in the mean, closer to a given line or a given point in the complex plane than the zeros of the polynomial itself. One may expect that similar inequalities are valid if, instead of the derivative, a polynomial derived in a more general way from the given one is considered. Here we shall prove inequalities of this type for polynomials obtained by composition from two given polynomials.

This composition is defined in the following way [2]): if

$$A(z) \equiv \binom{n}{0} a_0 + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \dots + \binom{n}{n} a_n z^n$$

and

$$B(z) \equiv \binom{n}{0} b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \dots + \binom{n}{n} b_n z^n$$

are the two given polynomials, then the composition-polynomial is

$$AB(z) \equiv \binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 z + \binom{n}{2} a_2 b_2 z^2 + \dots + \binom{n}{n} a_n b_n z^n.$$

If $B(z) \equiv nz(1+z)^{n-1}$ we have $AB(z) \equiv zA'(z)$. For this special choice of $B(z)$ most of the theorems proved in this paper give rise to results already proved in I and II.

Throughout this paper the zeros of $A(z)$, $B(z)$ and $AB(z)$ will be denoted by $\alpha_1, \dots, \alpha_n$; $\beta_1, \dots, \beta_n$ and $\gamma_1, \dots, \gamma_n$, respectively. Furthermore we put

$$\{A, B\} = \binom{n}{0} a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \dots + (-1)^n \binom{n}{n} a_n b_0.$$

We shall often use the following well-known theorem of J. H. GRACE

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 1037—1044 (1946) = Indagationes Mathematicae **8**, 635—643; Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 458—464 (1947) = Indagationes Mathematicae **9**, 264—270. These papers are referred to as I and II.

[2]) This way of composition was introduced by G. SZEGÖ, Math. Zeitschrift, **13**, 28—55 (1922).