Mathematics. — *An outer limit of nonconformalness, for which* PICARD'S *theorem still holds.* By R. J. WILLE. (Communicated by Prof. W. VAN DER WOUDE.)

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1. Let $w = f(z)$ be a function, which involves a $1 - 1$ correspondence between the finite z-plane and a, possibly multiply sheeted, mapping surface on the w-plane. By projection of this mapping surface on a sphere with radius $\frac{1}{2}$, touching the w-plane at zero with its southpole and using its northpole as centre of projection, we obtain a mapping surface F on the sphere.

PICARD's theorem on exceptional points does not only hold in case of conformal mapping, in which infinitesimal circles on F correspond to infinitesimal circles in the z-plane, but also in case of quasi-conformal mapping, in which infinitesimal circles on F correspond to infinitesimal ellipses in the z-plane, with eccentricities uniformly bounded by $e < 1$, as L. AHLFORS has shown (Eindeutige Analytische Funktionen, by R. NEVANLINNA, chapter XIII, § 8).

Let $F(r)$ be the mapping surface of $|z| \leq r$; we suppose the eccentricities $\varepsilon(z)$ of the infinitesimal ellipses, corresponding to the infinitesimal circles at points on $F(r)$, to have the upper bound $e(r) < 1$. Then we propose to show the validity of PICARD's theorem, when $e(r)$ tends not too rapidly to 1 (r tending to infinity) and to indicate also the order of rapidity, for which the theorem just holds. The precise theorem, we shall prove, is the following:

If the integral

$$\int^{\infty} \frac{\sqrt{1-e(r)^2}}{r}\, dr$$

is divergent, there are two exceptional points at the most. On the contrary, if the integral is convergent, a mapping exists even with a whole circle of exceptional points.

2. In order to give the proof we memorise the following inequality, deduced by L. AHLFORS:

$$h\,L(r) + \sum_{i=1}^{3} n(a_i, r) \geq \frac{O(r)}{\pi}, \quad \ldots \ldots \quad (1)$$

where h is a constant, independent of the choice of $f(z)$; a_1, a_2, a_3 three fixed points on the sphere; $n(a_i, r)$ the number of times $F(r)$ covers a_i; $L(r)$ and $O(r)$ contourlength and area of $F(r)$.

If there are three exceptional points, we take them as a_1, a_2, a_3; the inequality (1) then reduces to:

$$\frac{L(r)}{O(r)} \geq \frac{1}{\pi h}. \quad \ldots \ldots \quad (2)$$

In the first place, supposing the integral

$$\int^{\infty} \frac{\sqrt{1-e(r)^2}}{r}\, dr \quad \ldots \ldots \quad (3)$$

divergent, we may find, as we shall prove, values

$$r_1 < r_2 < \ldots < r_\nu < \ldots \to \infty$$

so, that

$$\lim_{\nu \to \infty} \frac{L(r_\nu)}{O(r_\nu)} = 0, \quad \ldots \ldots \quad (4)$$

which is in contradiction with (2), so that PICARD's theorem then holds.

In order to prove the existence of the values r_ν we need a second inequality, deduced by AHLFORS:

$$L(r)^2 \frac{dr}{r} \leq 4\pi K(r)\, d\,O(r), \quad \ldots \ldots \quad (5)$$

where $K(r)$ means:

$$K(r) = \frac{1}{2}\left\{\sqrt{1-e(r)^2} + \frac{1}{\sqrt{1-e(r)^2}}\right\}. \quad \ldots \ldots \quad (6)$$

As

$$K(r) \leq \frac{1}{\sqrt{1-e(r)^2}}$$

inequality (5) reduces to:

$$L^2 \frac{dr}{r} \leq 4\pi \frac{1}{\sqrt{1-e^2}}\, dO. \quad \ldots \ldots \quad (7)$$

By multiplying both sides of (7) with $\frac{\sqrt{1-e^2}}{O^2}$ and integrating from r_μ to $r_{\mu+1}$, values to be defined further on, we obtain:

$$\int_{r_\mu}^{r_{\mu+1}} \frac{L^2}{O^2} \frac{\sqrt{1-e^2}}{r}\, dr \leq 4\pi \int_{r_\mu}^{r_{\mu+1}} \frac{dO}{O^2} = 4\pi \left\{\frac{1}{O(r_\mu)} - \frac{1}{O(r_{\mu+1})}\right\}. \quad . \quad (8)$$

As $\frac{\sqrt{1-e^2}}{r} > 0$ and $\frac{L^2}{O^2}$ continuous, there exist values r_ν, with

$$r_\mu \leq r_\nu \leq r_{\mu+1},$$

so that

$$\left\{\frac{L(r_\nu)}{O(r_\nu)}\right\}^2 \int_{r_\mu}^{r_{\mu+1}} \frac{\sqrt{1-e^2}}{r}\, dr = \int_{r_\mu}^{r_{\mu+1}} \frac{L^2}{O^2} \frac{\sqrt{1-e^2}}{r}\, dr.$$

Introducing this in (8) we find:

$$\left\{\frac{L(r_\nu)}{O(r_\nu)}\right\}^2 \leqslant \frac{4\pi}{\int\limits_{r_\mu}^{r_{\mu+1}} \frac{\sqrt{1-e^2}}{r}\,dr} \left\{\frac{1}{O(r_\mu)} - \frac{1}{O(r_{\mu+1})}\right\}. \quad . \quad . \quad . \quad (9)$$

As $\int^{\infty} \frac{\sqrt{1-e^2}}{r}\,dr$ is supposed to be divergent we may choose the values $r_\mu \rightarrow \infty$, so that

$$\lim_{\mu\to\infty} \int\limits_{r_\mu}^{r_{\mu+1}} \frac{\sqrt{1-e^2}}{r}\,dr = \infty. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From (9) and (10) follows (4).

E.g. in case of representations, for which the order of $\sqrt{1-e^2}$ is $\frac{1}{\log r}$, the integral (3) diverges and so PICARD's theorem holds.

In the second place we suppose $e(r)$ given in such a way that

$$\int^{\infty} \frac{\sqrt{1-e^2}}{r}\,dr$$

converges; then we shall show, that there exists a corresponding mapping with a whole circle of exceptional points.

Before proving this, we give an example of a mapping for which the above integral converges (see fig. 1).

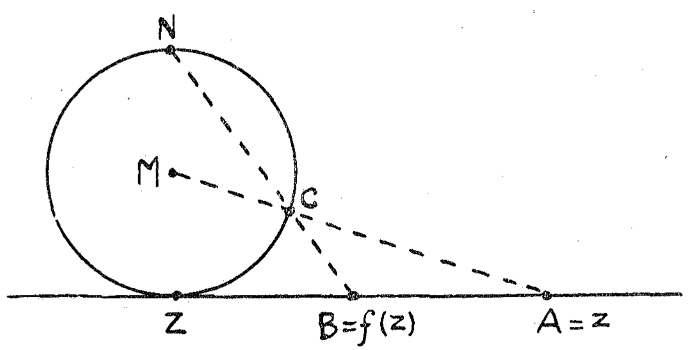

Fig. 1.

We project A to C on the sphere (rad. $\frac{1}{2}$), with its centre M as centre of projection. Then we reproject C to B on the plane, but now with the northpole N as centre. The function $w = f(z)$ will be defined as the 1 — 1 correspondence A to B. It is clear that the points C form the mapping surface called above F. An infinitesimal circle at C, with radius ds, corresponds to an infinitesimal ellipse at A, with major axis a:

$$a = (1 + 4r^2)\,ds, \qquad (|z| = r)$$

and with minor axis b:

$$b = \sqrt{1 + 4r^2}\,ds.$$

We find

$$\sqrt{1-e^2} = \frac{b}{a} = \frac{1}{\sqrt{1+4r^2}},$$

hence

$$\int^{\infty} \frac{\sqrt{1-e^2}}{r}\,dr = \int^{\infty} \frac{dr}{r\sqrt{1+4r^2}},$$

which integral is convergent.

We now examine a more general mapping, defined by an arbitrary function $f(r)$, but again depending only on r (see fig. 2).

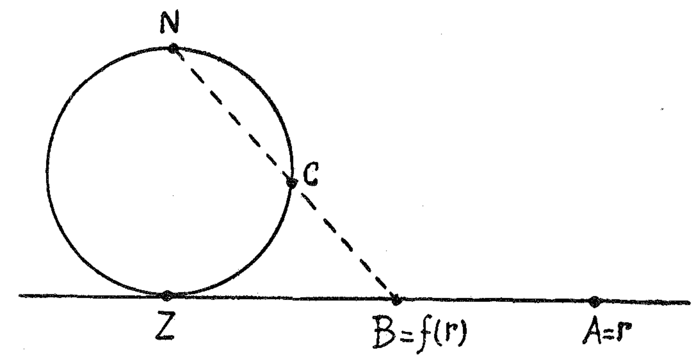

Fig. 2.

An infinitesimal circle at C, with radius ds, corresponds to an infinitesimal ellipse at A with one axis a_1, along the radius r:

$$a_1 = \frac{1+f^2}{f'}\,ds$$

and the other axis a_2:

$$a_2 = \frac{1+f^2}{f}\,r\,ds.$$

a_1 is major or minor axis according to $\frac{r f'}{f} < 1$ or $\frac{r f'}{f} > 1$, so that:

$$\sqrt{1-e^2} \text{ or } \frac{1}{\sqrt{1-e^2}} = \frac{a_2}{a_1} = r\,\frac{f'}{f}.$$

After these preliminaries we prove the second part of our theorem. Let $e(r) < 1$ be a given function, with $e(r) \to 1$ for $r \to \infty$, so that

$$\int^{\infty} \frac{\sqrt{1-e^2}}{r}\, dr$$

is convergent.

Putting

$$r\,\frac{f'}{f} = \sqrt{1-e^2}, \quad \ldots \ldots \ldots \quad (11)$$

we are sure of finding a mapping to which corresponds the given $e(r)$.

From (11) follows at once ($r_0 > 0$):

$$\log f(r) = C + \int_{r_0}^{r} \frac{\sqrt{1-e^2}}{r}\, dr,$$

from which it follows, that $f(r)$ is a bounded function. Hence the mapping, only depending on r, which we may construct by means of this function, will have a whole circle of exceptional points round infinity.

(Evidently the mapping has no exceptional points near zero, for $\frac{\sqrt{1-e^2}}{r}$ being aequivalent to $\frac{1}{r}$ near zero, $\log f(0) = -\infty$ and $f(0) = 0$.)

If, instead of (11), we had put $r\,\frac{f'}{f} = \frac{1}{\sqrt{1-e^2}}$, we should have found a mapping for which a_1, is the minor axis of the ellipse, and for which infinity is the only exceptional point.

Mathematics. — *Non-homogeneous binary quadratic forms.* IV (continued). By H. DAVENPORT. (Communicated by Prof. J. G. VAN DER CORPUT.)

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5. By the remarks at the beginning of § 4, and by Lemmas 4, 5, 6, there remains for consideration only the possibility

$$\frac{1}{\eta_m} > \bar{\beta} > \frac{1}{\eta_{m+1}}, \quad \frac{1}{\xi_m} < \alpha < \frac{1}{\xi_{m+1}}, \quad \ldots \ldots \quad (29)$$

where m is an odd positive integer. Our aim will be to show that α, $\bar{\beta}$ necessarily have the values

$$\alpha = \alpha_m, \qquad \bar{\beta} = -\alpha'_m,$$

where α_m is defined by (5). That these values do in fact satisfy (29) follows from the simple inequalities

$$\frac{\theta}{1+\theta^{-n+3}} < \frac{\theta^{n+1}-1}{\theta^n+1} < \frac{\theta}{1+\theta^{-n}},$$

$$\frac{1}{1-\theta^{-n}} > \frac{1-\theta^{-n-1}}{1-\theta^{-n}} > \frac{1}{1-\theta^{-n-3}}.$$

Lemma 7. *If $\alpha = \alpha_m$, $\beta = \alpha'_m$, where m is an odd positive integer, then*

$$|(\alpha\xi - 1)(\beta\xi' - 1)| = 1$$

for the following values of ξ:

$$\theta, \quad \xi_m, \quad \xi_{m+1}, \quad -\eta'_m, \quad -\eta'_{m+1}.$$

Proof. We have to show that $N(\alpha_m\xi - 1) = \pm 1$ for the above values of ξ. In fact

$$\alpha_m\theta - 1 = \frac{2\theta(\theta^{n+1}-1)}{\theta^n+1} - 1 = \frac{(2\theta^2-1)\theta^n - (2\theta+1)}{\theta^n+1} = \theta^3\left(\frac{\theta^n-1}{\theta^n+1}\right),$$

and since n is odd, this has norm 1. Similarly we find that

$$\alpha_m\xi_m - 1 = \theta^{-n+2}\left(\frac{\theta^n-1}{\theta^n+1}\right),$$

$$\alpha_m\xi_{m+1} - 1 = -\theta^{-n-1},$$

$$\alpha_m(-\eta'_m) - 1 = -\theta^{n+1},$$

$$\alpha_m(-\eta'_{m+1}) - 1 = \theta^{n+4}\left(\frac{\theta^n-1}{\theta^n+1}\right),$$

and all these have norm 1.