Mathematics. — *On the principles of intuitionistic and affirmative mathematics* [1]). I. By D. VAN DANTZIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 28, 1947.)

Ch. 1. BROUWER's *intuitionism*.

In 1907 in his thesis [1] [2]) L. E. J. BROUWER developed the principles of intuitionistic mathematics. Some of the most important among the many original ideas he defended there and worked out in several later papers on the subject may be circumscribed about as follows.

Mathematics has to be regarded as a part of human activity, rather than as a system of books, theorems, words or symbols. It is (not a *result of* human experience, but) a method of *dealing with* human experience. In order to grasp its characteristic features we have to abstract from all qualitative properties of particular experiences, in the most elementary description of which the method usually has been applied already a good many times.

In its most elementary form it consists of fixing our attention upon a single one out of the totality of our perceptions, and of distinguishing this one from the rest of them. As the distinguishing of a single perception lies at the basis of the mathematical idea of discreteness, and the totality of perceptions at the basis of the notion of continuity, it presupposes both these notions, though, of course, in an unanalysed form. This mental process BROUWER calls the continuum-intuition, or the primordial intuition ("Urintuition") of mathematics, or also the time-intuition, as also the possibility of ordering our perceptions according to time is not reducible to a more elementary mental process [3]).

[1]) The present paper has been written in February 1942 on bequest of the redaction of the Revista Mathematica Hispano Americana and was sent to that journal through an official Spanish instance in the (then occupied) Netherlands in March 1942. For some unknown reason the redaction of the journal did not publish the paper. Although the paper is not anymore entirely up to the present situation in mathematical logic nor to my own present state of mind, there might be some use in publishing it nevertheless without other than a few alterations, mostly of style. I hope to have leasure for publishing soon a closer examination of some questions raised here. Footnotes [1]), [4]), [16])—[19]), [35]), [38]) have been added 1947.

[2]) The numbers in square brackets refer to the bibliography at the end of the paper.

[3]) "...... the primordial intuition of mathematics (and of any intellectual activity) is the substratum of all observations of change, stripped of all qualitative properties; a unity of continiuty and discreteness, a possibility of mentally joining several units, connected by a "between" which never is exhausted by intercalation of new units." (L. E. J. BROUWER [1] p 8.)

"This neo-intuitionism considers as the elementary occurrence of muman intellect: the splitting up of moments of life into qualitatively different parts, which can be reunited only if separated in the time, and it sees as the elementary occurrence of mathematical thinking: the process of stripping this splitting up from every emotional content until the intuition of abstract di-unity remains." (L. E. J. BROUWER [2] III p. 12.)

Here and in the other quotations from BROUWER [1] and [2] an entirely literal translation could not be reached. Although I have tried to paraphrase in Ch. 1 BROUWER's own ideas without letting my sometimes somewhat deviating opinions interfere with them unless where they seem to me to be in accordance with his, I am not quite certain that I always have succeeded in representing BROUWER's ideas correctly. Anyhow, the responsibility for the representation rests with me alone.

The term "mathematics" is used by BROUWER for every mental process which can be conceived as being built up out of such elementary processes, hence in a much more general sense than it is done usually, which includes, not only what HILBERT calls "metamathematics", but also logic, and science in general. The principal result obtained by BROUWER is, that for the foundation of mathematics (in the ordinary, restricted sense) it is sufficient as well as necessary to consider systems composed of sequences obtained by applying an unlimited number of times (limited only for finite arithmetics, etc.) such a single mental process, without paying attention to the qualitative differences which in such processes effectively may occur.

The sufficiency BROUWER proved by undertaking the laborious work, which was later continued by his disciples, M. J. BELINFANTE [4]) and A. HEYTING, to derive explicitly the fundamental parts of arithmetics, set-theory, analytic geometry, analysis, etc. The necessity he showed by giving critical analyses of the most prominent rival theories, and proving that they always presuppose explicitly or implicitly indefinite repetitions of a single process, in particular the idea of order and (as far as infinite systems are concerned) complete induction.

Once one is convinced of the necessity and sufficiency, it becomes natural to demand that mathematical considerations shall be restricted to such "constructions", i.e. complexes, consisting of indefinite repetitions of a single act, and BROUWER reserves the word "mathematics" to developments satisfying this condition.

With regard to the form in which mathematical statements are given as well as in the way they are proved, this standpoint has several peculiar consequences. First, a demonstration in this sense is not a method of "convincing" a reader or auditor in a more or less indirect way of the "truth" of a statement, viz. by the application of certain aprioristic "logical principles", but it is (or ought to be) the construction itself, the possibility of which is stated in the theorem. The only form of demonstration admitted here is "showing by doing". As, however, the process, according to BROUWER, is a mental one, it cannot be directly observed by a reader or auditor, so that the teacher has to describe it in words. But, as a description of an experience never is "adequate" [5]), never determines this

[4]) Since this was written Dr. M. J. BELINFANTE was killed with his family in a german concentration-camp in Poland.

[5]) This should not be misunderstood: of course we can not mention any part of an

experience completely and uniquely, also the construction under consideration cannot be described "exactly", in the sense in which this was formerly meant to be possible, nor is there any "certainty" in an absolute sense, that misunderstanding is excluded [6]).

A demonstration therefore rests essentially on "suggestion", not of the "truth" of the statement, but of the method of construction itself. An able mathematician may "suggest" to a gifted student, how to built his constructions, like an able musician may suggest a gifted scolar, how to compose a sonata; in both cases by showing and describing as well as he can, how de does it, by telling the principles in accordance with which (as he detects *a posteriori*) he has built them, and by criticising the student's exercises. Hence the logics "underlying" a construction, like the harmonics "underlying" a symphony, is (or ought to be!): not a system of rules, given a priori and followed dogmatically, but a system of regularities, observed a posteriori in one's own or somebody else's constructions [7]).

The question whether a logical principle is "trustworthy" or not, then means: if a mathematician, in order to save time, skips over some part of his construction on account of such a regularity, observed in previous constructions, may he then "reasonably expect" that he (or some one else) can later make it complete? Of course this question can only be answered by experience. The answer of such mathematicians as have much experience with this kind of work is: sometimes yes, sometimes no. In particular it is often "no", if the regularity under consideration is the so-called principium tertii exclusi, applied to infinite systems. This is not astonishing at all,

experience which can *not* be described in words. This would imply a direct contradiction. But — even apart from the emotion of "insufficiency" we have with regard to many descriptions —, with respect to every given description we can mention *afterwards*, numerous elements of the experience, *not* involved in the description which show the given description to be incomplete.

6) "To the question where mathematical exactness *does* exist, both parties give different answers: the intuitionist says: in the mind of men, the formalist: on paper." (L. E. J. BROUWER [2] III p. 7.)

"Nun gibt es aber für Willensübertragung, insbesondere für durch die Sprache vermittelte Willensübertragung, weder Exaktheid, noch Sicherheit. Und diese Sachlage bleibt unvermittelt bestehen, wenn die Willensübertragung sich auf die Konstruktion reinmathematischer Systeme bezieht. *Es gibt also für die reine Mathematik keine sichere Sprache*, d.h. keine Sprache welche in der Unterhaltung Missverständnisse ausschliesst, und bei der Gedächtnisunterstützung vor Fehlern schützt." (L. E. J. BROUWER [3] p. 157.)

7) "Theoretical logic as well as logistics therefore are *empirical sciences*, and *applications* of mathematics, which never can teach us anything about the organisation of human intellect, and must be regarded to belong to *ethnography* rather than to *psychology*.

And the language of logical arguments is no more an *application of theoretical logic* than the human body is an application of anatomy". (L. E. J. BROUWER [1] p. 130.)

"Moreover, in arguments concerning empirical facts spanned upon mathematical systems, the logical principles are not directories, but regularities discovered afterwards in the accompanying language ......" (L. E. J. BROUWER [2] I p. 7.)

since by more general and less specific experience we know that, if we have ascertained that a certain construction can not be impossible, we need not have the slightest guarantee that we — or someone else — may succeed in carrying it out.

We pass here over the further consequences of these ideas, in particular with regard to set theory, where especially BROUWER's criticism of the comprehension-axiom should be mentioned, and refer the reader to BROUWER's original publications and to HEYTING's [2] monography, where also further litterature can be found.

An abstract scheme for a way of dealing with human experience, we called mathematics before. In dealing with our experiences we always single out a finite number of them, disregarding all other ones as belonging to the "continuous background" of our perceptions. Among them we see certain sequences, ordered in time, which to a certain degree are identified with each other, and give rise to a common substratum, called a "causal sequence" by BROUWER.

More or less characteristic for the behaviour of men is, as BROUWER calls it, the "mathematical action" or the "replacement of aim by means". "And the behaviour of men shows a tendency to observe as many as possible of those mathematical sequences, in order to choose the earlier element as a directory for their actions, always when interference with reality seems to be more succesfull at an earlier element than at a later one, even if instinct is only affected by the latter." (L. E. J. BROUWER [1] p. 81). The construction "in advance" of mathematical systems by abstraction from the qualitative differences of the systems of causal sequences on which they may be applied, is itself an example of such a "jump from aim to means" [8]).

We shall not go here further into BROUWER's ideas concerning physical science. We also end herewith our reference of a part of BROUWER's thesis and ulterior papers on intuitionism, although we could mention only a small portion of the ideas contained in them.

In the beginning BROUWER's ideas met with great resistance. In 1919 [2] he still had to admit that "the ideas defended here still have found only few adherents". In fact, during a long time H. WEYL was one of the few under the leading mathematicians who, to a large extent, accepted BROUWER's principles. Since about ten or twelve years however the situation has greatly changed.

In Amsterdam Prof. G. MANNOURY following a train of thought partly deviating from, to a large extent however in accordance with BROUWER's lines, had often expressed the opinion that a formal description of the regularities occurring in the intuitionistic way of reasoning must be possible.

8) "Selbstverständlich besitzt eine kausale Folge keine weitere *Existenz* ausser als Korrelat einer mathematische Handlungen hervorrufenden Einstellung des menschlichen Willens, und kann von der Existenz eines kausalen Zusammenhangs der Welt unabhängig vom Menschen keine Rede sein." (L. E. J. BROUWER [3] p. 154.)

A prize-question to this purpose, published by the mathematical society of Amsterdam, was answered in an excellent way by BROUWER's disciple A. HEYTING, under the characteristic motto "stones instead of bread". His difficult and laborious, but successfull work [1], and later his clear exposition of BROUWER's ideas [2] have greatly contributed not only to the interest in, but also to the understanding of intuitionism, in particularly among logicians. Moreover GÖDEL [1] showed that every sufficiently extended non-contradictory logistic system, satisfying certain simple conditions, allowes the formulation of problems, unsolvable *within the system*. Among them occurs the question after the formal non-contradictority of the system itself. Finally the "multi-valued logics" of the Polish school of LUKASIEWICZ and TARSKI contained important contributions to a better understanding of the logical structure of intuitionism. Several papers of HERBRAND, SKOLEM, CHURCH and his disciples, GENTZEN, I. JOHANNSSEN, etc. worked in the same direction, and showed that nowadays at least intuitionistic *logic* is completely recognised by the great majority of leading logicians.

Among mathematicians, however, there still is a certain resistance or indifference with regard to the intuitionistic way of reasoning.

There are some possible causes, which may, at least partly, explain this attitude.

1⁰. Many mathematicians are not particularly interested in philosophy, nor even in logic. They desire to carry their science further instead of uprooting the very fundamentals. They often don't quite understand what should be wrong with their customary way of reasoning, and they don't see any but philosophical reasons (to which they often don't attribute great importance) that anything at all should be wrong. Like so many important discoveries however, intuitionism is independent of the particular philosophy of its maker; its good sense can be demonstrated on a purely technical base. In ch. 3 I shall try to show, that the very desire of every genuine mathematician, viz. to prove his theorems as rigorously and as "economically" as he can, if consequently followed, leads almost automatically to a special form of intuitionistic mathematics.

2⁰. Many mathematicians are of opinion that they have no need of a "constructive" mathematics. I shall deal with their standpoint in the beginning of ch. 3 and in ch. 4.

3⁰. There is a general dislike under mathematicians of the great number of "almost equivalent" notions which occur in intuitionistic mathematics, and the great number of cases which often have to be considered wounds their sense of elegance. The fact that there are e.g. no less than 10 countability-relations [9]) for a set conflicts with their desire "to kill n birds with one stone" [10]). This fact, which of course is of a less principal nature, can be avoided by restricting the number of notions and theorems by one of the two ways shown in Ch. 2 and 3.

4⁰. The same is true for some other incommodities of a still more accidental nature, e.g. concerning terminology and notations. One can for instance use the word "set" for what BROUWER calls a "Spezies", i.e. in a sense not very different from the ordinary notion of set, whereas BROUWER's entirely different notion of "Menge" may (and will here) be described by the term "Brouwerian set". The fact that then the notion of "Brouwerian set" is prior to "set" in general will hardly be considered as an inconvenient.

The purpose of the present paper is to show, how perhaps a better unterstanding between intuitionists and "ordinary mathematicians" could be reached. This might be possible in one of two ways. In Ch. 2 we shall mention a method by which the intuitionist might try to meet the ordinary mathematician half way. In this chapter we shall therefore start from the intuitionist's point of view. In Ch. 3 on the contrary we shall take the standpoint of the classical ("formal") mathematician and see which way leads him to intuitionistic (i.e. "affirmative") mathematics. In Ch. 4 finally some remarks of a more general nature concerning formalism and intuitionism will be made.

Ch. 2. *The weak interpretation. Stable mathematics.* (*Classical mathematics from an intuitionistic standpoint.*)

We here take the intuitionist's standpoint and ask, in how far we can meet the ordinary mathematician's demands. The latter may remark that many misunderstandings arise from the fact that BROUWER interprets the ordinary mathematical theorems much "stronger" than he (the ordinary mathematician) intends them to be. If BROUWER rejects the theorem that every function $f(x)$, continuous for $0 \leqq x \leqq 1$, has at least one maximum in this interval, he does so, because he can not construct the abscissis of this maximum. But the mathematician might already be content if the supposition that $f(x)$ "has" (whatever this may mean!) a maximum, were exempt of contradiction. He may take POINCARÉ's point of view, according to which a mathematical entity "exists" if it is free from contradiction.

The intuitionist might meet this remark as follows. If A denotes any statement in the intuistionistic sense, then $\neg A$ (non-A) is its negation, i.e. $\neg A$ denotes that A would imply a contradiction (that A is "absurd", according to BROUWER's terminology). The classical mathematician therefore demands only $\neg\neg A$ instead of A to be proven. The intuitionist might therefore help the classical mathematician by replacing every statement A by $\neg\neg A$. This might seem hopeless (as $\neg\neg A$ would have to be replaced by $\neg\neg\neg\neg A$ for the same reason), were it not that BROUWER has proved [6] that $\neg\neg\neg A$ always implies and therefore is equivalent with $\neg A$, though $\neg\neg A$ generally not with A). Calling

9) L. E. J. BROUWER [5] I p. 255, A. HEYTING [3].

10) G. MANNOURY [2] p. 69.

a statement *stable* if it is equivalent with its double negation, we see that every *negative* statement (i.e. the negation of any statement) and only such a one is stable.

If A and B are stable, then also $\neg A$ and $A \wedge B$ (i.e. "A and B") are. But $A \vee B$ (i.e. "A or B") and $A \supset B$ (i.e. "A implies B"), interpreted in BROUWER's "strong" sense are not. We can however interpret these relations in a weaker sense if we *define* them by

$$A \vee B \overset{\mathrm{df}}{\equiv} \neg(\neg A \wedge \neg B)$$

$$A \supset B \overset{\mathrm{df}}{\equiv} \neg(A \wedge \neg B)$$

with these definitions $A \vee B$ and $A \supset B$ become stable. If A depends upon a variable x and for every x is stable, then also $\forall_x A(x)$ (i.e. $A(x)$ holds for all x) is stable (as generally $\neg\neg \forall_x A(x)$ implies $\forall_x \neg\neg A(x)$), but $\exists_x A(x)$ (i.e. an x with $A(x)$ exists), if interpreted in the strong sense, in general is not. If, however, we *define* the latter symbol by

$$\exists_x A(x) \overset{\mathrm{df}}{\equiv} \neg \forall_x \neg A(x)$$

then the statement becomes stable. With these definitions we obtain a system of formulae, closed with respect to the elementary logical operations, forming a part of intuitionistic logic and satisfying the formal rules of classical logic, including the principium tertii exclusi. In fact, with the above definitions we have not only $\neg\neg A \supset A$, but even $A \vee \neg A$ for every statement A. All this was essentially found by K. GÖDEL [2].

Passing from logic to mathematics, we have, of course, to take care that all definitions of mathematical objects are given in a stable form. At first, viz as long as we are concerned with natural (or rational) numbers only, no difficulties arise, as the fundamental relations between these numbers, equality and inequality, are stable.

The introduction of real numbers, however, leads to difficulties. Of course we have to avoid here BROUWER's definition, which certainly is too strong for our present purpose. We try CANTOR's definition by means of fundamental sequences of rational numbers, CAUCHY's criterium $\forall_\varepsilon \exists_n \forall_m |a_{n+m} - a_n| \leqq \varepsilon$[11])[12]) of course has to be interpreted in the weak sense. After elimination of the defined symbols $\vee$, $\supset$ and $\exists$ this reads:

$$\forall_\varepsilon \neg \forall_n \neg \forall_m |a_{n+m} - a_n| \leqq \varepsilon$$

---

[11]) We omit for abbreviation the condition that n, m etc. are natural and ε positive rational numbers, etc. Zero is considered as a natural number.

[12]) For real numbers the relations $x \leqq y$ and $x = y$ are stable; $x < y$ and $x \neq y$, if interpreted in the strong sense, are not. They become stable if they are interpreted in the weak sense:

$$x < y \overset{\mathrm{df}}{\equiv} \neg(y \leqq x); \qquad x \neq y \overset{\mathrm{df}}{\equiv} \neg(x = y).$$

A sequence $\{a_n\}$ of rational numbers a_n satisfying this condition may be called a *weak fundamental sequence*. In the same way a *weak null sequence* may be defined by

$$\forall_\varepsilon \neg \forall_n \neg \forall_m |a_{n+m}| \leqq \varepsilon.$$

If, however, we wish to define e.g. the quotient-sequence of two weak fundamental sequences $\{a_n\}$, $\{b_n\}$, the latter not being a weak null sequence, we must take account of the fact that the a_n/b_n need not be rational numbers, as some of the b_n may be zero. We can avoid this difficulty by working with integers instead of rational numbers only, viz with the numerators p_n, r_n and denominators q_n, s_n of the rational numbers a_n and b_n. Then the relation $|a_{n+m} - a_n| < \varepsilon$ becomes $|p_{n+m}\, q_n - p_n\, q_{n+m}| < \varepsilon\, |q_n\, q_{n+m}|$. We take here $< \varepsilon$ instead of $\leqq \varepsilon$ in order to exclude the trivial solution $q_n = q_{n+m} = 0$ of the inequality. Now it implies $|q_n| \geqq 1$, $|q_{n+m}| \geqq 1$. Now let $\{r_n, s_n\}$ be a weak fundamental non-null sequence, determined by the same relations as those for $\{p_n, q_n\}$, together with inequalities $\neg(|p_{n+m}| < \varepsilon\, |q_{n+m}|)$. Then the quotient-sequence $\{p_n\, s_n,\ q_n\, r_n\}$ is a weak fundamental sequence.

This is proved by showing that the conjunction of the four statements $\Omega_1, \Omega_2, \Omega_3, \Omega_4$, (which are easily seen to be stable) leads to a contradiction[13]), where the following abbreviations have been introduced:

$$\begin{array}{llll}
\Omega_1 \equiv \quad \forall_\varepsilon \quad \Omega_1(\varepsilon) & \Omega_1(\varepsilon) \equiv \neg \forall_n \neg \Omega_1(\varepsilon,n) & \Omega_1(\varepsilon,n) \equiv \quad \forall_m \quad \Omega_1(\varepsilon,n,m) \\
\Omega_2 \equiv \quad \forall_\varepsilon \quad \Omega_2(\varepsilon) & \Omega_2(\varepsilon) \equiv \neg \forall_n \neg \Omega_2(\varepsilon,n) & \Omega_2(\varepsilon,n) \equiv \quad \forall_m \quad \Omega_2(\varepsilon,n,m) \\
\Omega_3 \equiv \neg \forall_\varepsilon \neg \Omega_3(\varepsilon) & \Omega_3(\varepsilon) \equiv \quad \forall_n \quad \Omega_3(\varepsilon,n) & \Omega_3(\varepsilon,n) \equiv \neg \forall_m \neg \Omega_3(\varepsilon,n,m) \\
\Omega_4 \equiv \neg \forall_\varepsilon \neg \Omega_4(\varepsilon) & \Omega_4(\varepsilon) \equiv \quad \forall_n \quad \Omega_4(\varepsilon,n) & \Omega_4(\varepsilon,n) \equiv \neg \forall_m \neg \Omega_4(\varepsilon,n.m)
\end{array}$$

$$\Omega_1(\varepsilon, n, m) \equiv |p_n\, q_{n+m} - p_{n+m}\, q_n| < \varepsilon\, |q_n\, q_{n+m}|$$

$$\Omega_2(\varepsilon, n, m) \equiv |r_n\, s_{n+m} - r_{n+m}\, s_n| < \varepsilon\, |s_n\, s_{n+m}|$$

$$\Omega_3(\varepsilon, n, m) \equiv |r_{n+m}| \geqq \varepsilon\, |s_{n+m}|$$

$$\Omega_4(\varepsilon, n, m) \equiv |p_n\, q_{n+m}\, r_{n+m}\, s_n - p_{n+m}\, q_n\, r_n\, s_{n+m}| \geqq \varepsilon\, |q_n\, q_{n+m}\, r_n\, r_{n+m}|.$$

In fact, the system of relations

$$\Omega_1(\varepsilon_0, k_0, n-k_0),\ \Omega_1(\varepsilon_1, k, n-k),\ \Omega_1(\varepsilon_1, k, n+m-k),\ \Omega_2(\varepsilon_0, l_0, n+m-l_0),$$
$$\Omega_2(\varepsilon_2, l, n-l),\ \Omega_2(\varepsilon_2, l, n+m-l),\ \Omega_3(\varepsilon_3\, j, i),\ \Omega_4(\varepsilon_4, n, m)$$

with

$$j = k_0 + l_0 + k + l, \quad n = j + i,$$

implies a relation of the form

$$\varepsilon_3^2\, \varepsilon_4 < 2\,(A_1\, \varepsilon_1 + A_2\, \varepsilon_2)$$

with coefficients

$$A_1 = \varepsilon_0 + |r_{l_0}/s_{l_0}|,\ A_2 = \varepsilon_0 + |p_{k_0}/q_{k_0}| + \varepsilon_3\, \varepsilon_4,$$

---

[13]) If statements of the form $\neg \forall_\varepsilon \neg A(\varepsilon)$ are used, we must of course take care, not to use demonstrations of the type "Let ε be a number satisfying $A(\varepsilon)$" as this goes beyond the weak interpretation.

independent of m, i, k, l. The same remains the case if these variables are successively dropped by applying; first $\neg \forall_m \neg$ to the last relation and $\forall_m$ to the other ones; then $\neg \forall_i \neg$ to the second last relation and $\forall_i$ to the other ones, and finally $\neg \forall_k \neg$ and $\neg \forall_l \neg$ to the second (= third) and fifth (= sixth) relation respectively, and $\forall_j$ to the seventh one [14]. This however can not be true for all positive ε_1 and ε_2. Hence the system

$$\Omega_1(\varepsilon_0, k_0) \wedge \Omega_1 \wedge \Omega_2(\varepsilon_0, l_0) \wedge \Omega_2 \wedge \Omega_3(\varepsilon_3) \wedge \Omega_4(\varepsilon_4) \wedge \varepsilon_0 > 0 \wedge \varepsilon_3 > 0 \wedge \varepsilon_4 > 0$$

(which implies the relations obtained in this way) implies a contradiction viz $\forall_{\varepsilon_1} \forall_{\varepsilon_2}\ \varepsilon_3{}^2 \varepsilon_4 < 2(A_1 \varepsilon_1 + A_2 \varepsilon_2)$ with $A_1, A_2, \varepsilon_3, \varepsilon_4$ all positive and independent of $\varepsilon_1, \varepsilon_2$. This remains so after successive application of the operators

$$\neg \forall_{k_0} \neg,\ \neg \forall_{l_0} \neg,\ \neg \forall_{\varepsilon_3} \neg,\ \neg \forall_{\varepsilon_4} \neg,\ \forall_{\varepsilon_0},$$

so that the system $\Omega_1 \wedge \Omega_2 \wedge \Omega_3 \wedge \Omega_4$ is contradictory, which proves the theorem.

We have worked out this example, in order to show that proofs of this type do *not run* entirely along the customary lines, as the ordinary proofs usually consists of rather inconsequent mixtures of weak and strong interpretations.

There is a further question we have to consider. Is the relation $x \in N$ stable? Apparently this question is meaningless, as long as the negation of the statement $x \in N$ has no definite meaning (i.e. we can not conclude anything from it). This however becomes different after the real numbers have been introduced. Let us write $x \in N^*$ if x is a set of weak fundamental sequences $\{p_n, q_n\}$, weakly concurrent with a constant sequence $\{m, 1\}$, hence defined by

$$\neg \forall_m \neg \forall_\varepsilon \neg \forall_n \neg \forall_k \,|\, p_{n+k} - m\, q_{n+k} \,| < \varepsilon \,|\, q_{n+k} \,|.$$

Evidently $x \in N^*$ has a definite negation and is stable.

It should also be noted that we may meet the relation $x \in N^*$ where we would superficially expect $x \in N$. We show this by an example. Let $F(n)$ be a "fugitive property" [15] of natural numbers $n \in N$, i.e. let it be decidable for every $n \in N$ whether $F(n)$ or $\neg F(n)$ holds. Let for those numbers which have been investigated $\neg F(n)$ have been proved, though no proof of $\forall_n \neg F(n)$ be known.

On the contrary, we suppose that $\neg \forall_n \neg F(n)$ has been proved. Let

[14] This follows from the properties that $A(x)$ implies $\neg \forall_x \neg A(x)$ that $\forall_x (A(x) \supset B)$, where B is independent of x, implies $\neg \forall_x \neg (A(x) \supset B)$, (provided $\neg \forall_x \neg A(x)$), and that for natural h and k $\forall_h A(h)$ implies $\forall_h A(h+k)$.

[15] L. E. J. BROUWER [3] p. 161.

us then define $\mu[F]$ as the smallest "natural" number n for which $F(n)$ holds. Then we can not state $\mu[F] \in N$. In fact, defining the characteristic function $\iota_n[F]$ of F by

$$\iota_n[F] = \begin{cases} 0 & \text{if } F(n) \\ 1 & \text{if } \neg F(n), \end{cases}$$

then $n \in N \supset \iota_n[F] \in N$. Moreover, if $p_n = \mu_n[F]$, $q_n = 1$, where

$$\mu_n[F] = \sum_{i=0}^{n} \prod_{k=0}^{i} \iota_k[F]$$

then $\mu_n[F]$ is the smallest number $k \leq n$ with $F(k)$ if such a one exists, and otherwise $\mu_n[F] = n+1$. Evidently $n \in N \supset \mu_n[F] \in N$. Then $\{p_n, q_n\}$ is a weak fundamental sequence because of $\neg \forall_n \neg F(n)$, and therefore determines a weak real number $\mu[F]$. If $\{p_n, q_n\}$ were not concurrent with any natural number $m \in N$, then for every n p_n would be $> n$

(as $|\mu_{n+m}[F] - \mu_n[F]| \geqq \varepsilon > 0 \supset \mu_{n+m}[F] > \mu_n[F] \supset \mu_n[F] > n$),

hence $\forall_n\ \iota_n[F] = 1$ contradicting $\neg \forall_n \neg F(n)$ and the definition of $\iota_n[F]$. Formally we may of course write

$$\mu[F] = \lim \mu_n[F] = \sum_{i=0}^{\infty} \prod_{k=0}^{i} \iota_k[F]. \qquad \text{Hence } \mu[F] \in N^*.$$

All these questions have to be considered with greater care and precision than I could give to them here [16].

BROUWER's "strong real numbers" form a Brouwerian set [17] which is

[16] The following lines (till the end of the chapter) replace some rather hesitating remarks in the original MS.

[17] In terms somewhat different from BROUWER's [51] ones, a Brouwerian set ("Menge") may be defined as a law which 1° allows to distinguish certain finite sequences $a_1, \ldots\ldots, a_n$ of natural numbers as "allowed ("ungehemmt") sequences" from other ones, such that 1*a*) for every n if $a_1, \ldots\ldots, a_{n-1}$ is an allowed sequence, then for every natural x it can be decided whether or not $a_1, \ldots\ldots a_{n-1}, x$ is an allowed sequence, 1*b*) for every n and every allowed $a_1, \ldots\ldots, a_{n-1}$ at least one natural x can be found such that $a_1, \ldots\ldots, a_{n-1}, x$ is allowed, and which 2° for every n and every allowed sequence $a_1, \ldots\ldots, a_n$ determines an n^{th} symbol (or sequence of symbols, "Zeichenreihe") σ_n.

An infinite sequence $\sigma_1, \sigma_2, \ldots$ obtained in this way is called an *element* of the Brouwerian set.

The Brouwerian set will be called *special* if for every n a natural k_n is determined, such that $a_1, \ldots\ldots, a_n$ can only be admitted if for every $i \leq n$ $a_i \leq k_n$. (BROUWER's term "finite Menge" is not very well chosen, as the set itself need neither be finite nor even enumerable. Example: if the rational numbers are enumerated in a definite way, then all sequences of rational numbers form a Brouwerian set, where for every n all natural numbers a_n are allowed, and $\sigma(a_n)$ is the $a_n{}^{th}$ rational number. All sequences of rational numbers $r_1, r_2 \ldots$ with $r_n = \frac{m_n}{n}$, $0 \leqq m_n \leqq n$ form a special Brouwerian sets with $k_n = n$.

special [18]) if they are restricted to a finite interval (e.g. $\geqq 0$ and $\leqq 1$).

These examples make it, I believe, sufficiently clear that:

1⁰. no contradiction can occur between intuitionistic and classical mathematics [19]) provided the latter is consistently interpreted as a system of *stable* statements;

A *set* ("Spezies") of zero order is a Brouwerian set or an element of such a one. All sets of n^{th} order possessing some "well defined" ("begrifflich fertig definierte"; this definition, of course, is not sufficiently clear) property form a set ("Spezies") of $(n+1)^{th}$ order. Cf. BROUWER [1], p. 135; [2] III p. 15 seq.; [7] p. 1421.

All sequences of rational numbers form a Brouwerian set. All weak fundamental sequences form a set of order 1; a real number, defined as a set of all weak fundamental sequences concurrent with one of them also is a set of order 1; all real numbers form a set of order 2. An (unordered) pair (or triple, etc.) of real numbers also is a set of order 2; an ordered pair, being defined as an unordered triple, two of the elements of which are equal (hence $(x, y) = (x, x, y) = (x, y, x) = (y, x, x)$) is a set of order 3. A ("weakly defined") *function* may be defined as a set S of ordered pairs $P = (x, y)$ of real numbers (x, y) which for every x contains one and only one y (of course in weak interpretation:

$$\forall_x \neg \forall_y \neg_{|} (x, y) \in S_{|}{}^{18}) \text{ and } \forall_x \forall_y \forall_{z\,|||} (x, y) \in S_{|} \wedge_{|} (x, z) \in S_{||} \supset_{|} y = z_{||};$$

(For the meaning of the strokes cf. the next chapter.) A function is a set of order 4, all of them form a set of order 5.

[18]) From a formal point of view it may be of importance to remark that each successive passage to a set of higher order introduces a new type of all-symbol, which can not be defined by means of the previous ones: first we have $\forall_n{}^{\mathbf{N}}$ where n runs through the natural (or integer or rational, etc.) numbers, then $\forall_\xi{}^{\mathrm{S}}$ with ξ running through all (or all weak fundamental) sequences, then $\forall_x{}^{\mathrm{R}}$ where x runs through all real numbers, etc. Of course, the use of one single all-symbol by writing e.g. explicity $\forall_{x\,||} x \in \mathrm{R}_{|} \supset \Omega(x)_{|}$, instead of $\forall_x{}^{\mathrm{R}} \Omega(x)$ does not alter the logical situation.

[19]) Cf. L. E. J. BROUWER [2 I]: "Still, in unjustified use [of the principium tertii] one will never be checked by a contradiction and discover in *this* way the unfoundedness of one's argument".

[3] "Denn auf der Basis der intuitionistischen Einsichten lassen sich ausser den unabhängig from Prinzip des ausgeschlossenen Dritten entwickelbaren *richtigen Theorien* auch unter Heranziehung dieses Prinzips [for finite sets of properties] *nichtkontradiktorische* Theorien herleiten, mit denen sich von der bisherigen Mathematik ein viel grösserer Teil als mit den richtigen Theorien umfassen lässt. Eine geeignete Mechanisierung der Sprache dieser intuitionistisch-nichtkontradiktorischen Mathematik müsste also gerade liefern, was die formalistische Schule sich zum Ziel setzt.

Dagegen kann die gleichzeitige Aussage des Prinzips des ausgeschlossenen Dritten für *beliebige Spezies* von Eigenschaften sehr wohl kontradiktorisch sein. So lässt sich von der der folgenden Aussage die Kontradiktorität beweisen: Alle reelle Zahlen sind entweder rational oder irrational."

The contradiction with classical mathematics, however, is only apparent, because BROUWER uses the words "sind entweder ... oder..." in his strong sense (it can be *decided* whether ... or ...) whereas classical mathematics mean them to be interpreted in the weak sense (it is *not* true that neither ... nor ...). Moreover, it seems to me that BROUWER's higher valuation of the non-stable part of intuitionistic mathematics, which I completely share, should rather not be expressed by the term "richtig".

2⁰. on the contrary, with this interpretation classical mathematics becomes a *part* of intuitionistic mathematics;

3⁰. the main importance of BROUWER's work may be seen in the fact that a *stronger* interpretation of the classical statements than the stable one is possible, and (as we show in Ch. 3) in many respects of considerably greater interest; it may then be expected that the most important interpretation will be the strongest one, which leads to the *affirmative* (Ch. 3) and through it to a consistent *finitistic* interpretation of mathematics.