Mathematics. — *A matric representation of binary modular congruence groups of degree m.* (First communication.) By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

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In this publication we continue the study of the behaviour of general theta functions of degree m under substitutions of the modular group of degree m [1]).

In chapter 1 we mention without proofs some theorems concerning the behaviour of the theta functions. For the proofs we may refer to our thesis. The theta functions are functions of the complex elements of a symmetric matrix T. We replace this matrix by $(AT+B)(CT+D)^{-1}$, where $\mathbf{U}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a modular matrix, that is to say the matrix $\mathbf{U}$ satisfies the equation $\mathbf{U}'\mathbf{I}\mathbf{U}=\mathbf{I}$ where $\mathbf{I}=\begin{pmatrix} N & E \\ -E & N \end{pmatrix}$.

In chapter 2 we normalize the general theta functions. Under certain assumptions about the parameters of the theta functions, we obtain a set of functions with the property that the function obtained from a function of the set by applying a modular substitution on T, can be represented as a linear aggregate of functions of this set.

In chapter 3 we deduce from the results of chapter 2 a matric representation of the group $G(\varepsilon_n)$ of modular substitutions $\mathbf{U}\equiv\mathbf{E} \pmod{\varepsilon_n}$. In this representation the unity element corresponds with all substitutions $\mathbf{U}\equiv\mathbf{E} \pmod{\nu\varepsilon_n}$. The representation of the quotient group $G(\varepsilon_n)/G(\nu\varepsilon_n)$ gives a matric representation of the binary modular congruence groups of degree m modulo ν, since this group is simply isomorphic with this quotient group. We restrict ourselves to odd moduli ν.

In chapter 4 we prove in a direct way, that the formulas of chapter 3 define a matric representation of the binary modular congruence group of degree m and to the modulus ν.

1. 1. *Definitions and notation.*

A matrix will be called integral, if its elements are rational integers. An integral matrix will be said to be divisible by a given integer, if all its elements are divisible by this integer. Two matrices A and B will be called congruent modulo an integer ν, if their difference is divisible by ν. The well-known notation $A\equiv B \pmod{\nu}$ will often be abbreviated to $A\equiv B(\nu)$.

The transposed matrix of a matrix A will be denoted by A'. The usual definition of the product of two matrices will be used consistently. If a and b are n-vectors (that is to say matrices of one column and n rows), the product $a'b\equiv b'a$ is a scalar and the product ab' (or ba') is a matrix of n rows and n columns.

The matrix which we obtain by writing the columns of a matrix B to the right of those of a matrix A is denoted by (AB). Analogous meaning have the symbols $\begin{pmatrix} A \\ B \end{pmatrix}$ and $\begin{pmatrix} A B \\ C D \end{pmatrix}$.

A matrix A with n rows and m columns will also be denoted by $A=A^{(n,\,m)}$. The matrix $A=A^{(n,\,n)}=A^{(n)}$, in the special case $n=m$, will be called a square matrix of degree n. The value of the determinant of the square matrix A will be denoted by $|A|$.

The following letters will retain the same meaning hereafter:

n and m	positive integers.
N	a matrix the elements of which are all zero.
E	the unity matrix.
Q	an integral, symmetric, positive (definite) matrix of degree n.
Δ	the determinant $\lvert Q\rvert$ of Q.
T	a symmetric matrix of degree m with variable (complex) elements. The imaginary part of the matrix T will supposed to be positive.
Z	a matrix $Z^{(n,\,m)}$ of complex variable elements.
ν	a positive integer.
ν_0	$\nu_0=\Delta\,\nu$.

An integral matrix A with n rows is called *special* (with respect to Q) if the matrix QA is divisible by Δ.

The letters

P, R, S, L, G, H, F	denote henceforth special matrices of n rows and m columns.
U and V	denote integral matrices with determinant unity.

The trace $\sigma(A)$ of a matrix A is the sum of its diagonal elements. It can be seen readily that $\sigma(A)=\sigma(A')$ and $\sigma(AB)=\sigma(BA)$.

For abbreviation we write

$$e\,[x]=e^{\pi i x} \text{ and } e\,\{X\}=e^{\frac{\pi i}{\Delta}\sigma(QX)}.$$

In this notation we have the following identities

$$e\{X+Y\}=e\{X\}.e\{Y\} \text{ and } e\{X'\}=e\{X\}.$$

If a summation must be extended over special matrices only, we put an asterisk after the sign of summation (Σ^*).

The column vector, the elements of which are the diagonal elements of a square matrix K is denoted by $\langle K\rangle'$. The column vector v is defined by $Q\,v=\Delta\,\langle Q\rangle'$.

[1]) See F. VAN DER BLIJ, Theta functions of degree m. Thesis, Leiden 1947.

1. 2. *Theta functions and the sums $\varphi(P,R)$.*

General theta functions of degree m were introduced by the following definition:

$$\theta_{GH}(Z|T;P,\nu)=$$

(1.01) $$\sum_{M\equiv P(\nu_0)} e\left\{\frac{(M-P)H'}{\nu_0}\right\} e\left\{\frac{(M+\tfrac{1}{2}G)T(M+\tfrac{1}{2}G)'}{\nu_0}\right\} e\{2Z(M+\tfrac{1}{2}G)'\}.$$

Here the summation must be extended over all integral matrices $M=M^{(n,m)}$, which are a multiple of ν_0 plus P. Under the assumptions made about T and Q the series converges absolutely and uniformly in Z in each finite domain, uniformly in T in those finite domains, the closure of which contains only points with matrices T with positive imaginary part.

The following relations are easily established:

(1.02) $$\theta_{G+2L,H}(Z|T;P,\nu)=\theta_{GH}(Z|T;P+L,\nu),$$

(1.03) $$\theta_{G,H+2L}(Z|T;P,\nu)=\theta_{GH}(Z|T;P,\nu).$$

Let σ denote a complete system of non-congruent special matrices $P^{(n,m)}$ mod ν_0. If P runs through a system σ, the corresponding theta functions are linearly independent.

Modular matrices of degree m were introduced by C. L. SIEGEL [2]). We consider the matrix

(1.04) $$\mathbf{U}=\begin{pmatrix} A & B \\ C & D\end{pmatrix},$$

which is composed of four integral square matrices A, B, C and D of degree m. This matrix is termed modular if it satisfies the equation

(1.05) $$\mathbf{U}'\,\mathbf{I}\,\mathbf{U}=\mathbf{I} \text{ where } \mathbf{I}=\begin{pmatrix} N & E \\ -E & N\end{pmatrix}.$$

We can prove by a straightforward calculation that

(1.06) $$\begin{aligned} A'D-C'B&=D'A-B'C=AD'-BC'=DA'-CB'=E,\\ A'C-C'A&=AB'-BA'=DC'-CD'=D'B-B'D=N.\end{aligned}$$

Hereafter we shall reserve the letters A, B, C and D for the elements of a modular matrix $\mathbf{U}$.

The sums φ_{FGH} are multiple sums which are closely connected with a generalization of the ordinary Gaussian sums.

If C and $\bar{C}$ are non-singular, integral, square matrices of degree m and if γ is a positive integer, such that $C\bar{C}=\gamma E$, the sums φ are defined by

(1.07) $$\varphi_{FGH}(P,R;\nu,\mathbf{U})=\sum_{\substack{X\bar{C}' \bmod \gamma\nu_0\\ X\equiv P(\nu_0)}} e\left\{\frac{(X-P)H'}{\nu_0}\right\} e\left\{\frac{(X+\tfrac{1}{2}G)A\bar{C}(X+\tfrac{1}{2}G)'-2(R+\tfrac{1}{2}F)\bar{C}(X+\tfrac{1}{2}G)'+(R+\tfrac{1}{2}F)\bar{C}D(R+\tfrac{1}{2}F)'}{\gamma\nu_0}\right\}.$$

[2]) C. L. SIEGEL, Ueber die analytische Theorie der quadratischen Formen. Ann. of Math. (2) 36, 527—605 (1935).

Here the parameters are supposed to satisfy the relation

(1.08) $$Q(F+GA+HC+\nu v\langle A'C\rangle)\equiv 0 \qquad (\text{mod } 2\triangle).$$

The matrices X run through a system of integral matrices, all congruent P mod ν_0, such that the matrices $X\bar{C}'$ run through a system of non-congruent matrices mod $\gamma\nu_0$.

1. 3. *The behaviour of theta functions under the modular group.*

If F is a function of Z and T, we define the left-hand operator $\mathbf{U}$ by

(1.09) $$\mathbf{U}\,F(Z|T)=F(Z(CT+D)^{-1}\,|\,(AT+B)(CT+D)^{-1}).$$

If $\mathbf{U}$ is a modular matrix with non-singular matrix C, it can be proved that

(1.10) $$\mathbf{U}\,\theta_{GH}(Z|T;P,\nu)=W\sum^{*}_{S \bmod \nu_0}\varphi_{G_1GH}(P,S;\nu,\mathbf{U})\,\theta_{G_1H_1}(Z|T;S,\nu).$$

Here the symbols G_1 and H_1 have the following meaning

(1.11) $$\begin{aligned} G_1&=GA+HC+\nu v\langle A'C\rangle,\\ H_1&=GB+HD+\nu v\langle B'D\rangle.\end{aligned}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of the equation $|\mathfrak{R}(T)-\lambda\mathfrak{J}(T)|=0$ we define

$$\sqrt{|-iT|}=\sqrt{|\mathfrak{J}(T)|}\prod_{k=1}^{m}\sqrt{(1-i\lambda_k)} \text{ where } \mathfrak{R}(\sqrt{\ })>0.$$

With this definition of the square root we have

(1.12) $$W=\frac{\{\sqrt{|-i(T+C^{-1}D)|}\}^{n}}{\nu^{\frac{1}{2}nm}\triangle^{\frac{1}{2}m}}\,e\{\nu_0 Z(CT+D)^{-1}CZ'\}.$$

The formula (1. 10) is a special case of a more general transformation formula of the theta functions under modular substitutions.

If $\mathbf{U}=\begin{pmatrix} A\,B\\ C\,D\end{pmatrix}$ is a modular matrix with a matrix C of rank r, the effect of the operator $\mathbf{U}$ is given by (we suppose either $\nu\equiv 1$ or $v\equiv 0$ (mod 2).):

(1.13) $$\mathbf{U}\,\theta_{GH}(Z|T;P,\nu)=W\sigma_\nu(\mathbf{U})\sum^{*}_{XC \bmod \nu_0}\eta_{GH}(P,X;\nu,\mathbf{U})\,\theta_{G_1H_1}(Z|T;R,\nu).$$

Here we write as an abbreviation $R=PA+XC-\tfrac{1}{2}\nu v\langle A'C\rangle$; and the sum over X runs over special matrices X if the matrix R is integral, otherwise the sum runs over special matrices $2X$ with $2XC\equiv\nu v\langle A'C\rangle$ (mod 2).

G_1 and H_1 are defined by (1. 11);

(1.14) $$\eta_{GH}=e\left\{\frac{(P+\tfrac{1}{2}G)AB'(P+\tfrac{1}{2}G)'+2(P+\tfrac{1}{2}G)BC'(X+\tfrac{1}{2}H)'+(X+\tfrac{1}{2}H)DC'(X+\tfrac{1}{2}H)'}{\nu_0}\right\},$$

(1.15) $$\sigma_\nu(\mathbf{U})=\gamma^{-n(m-r)}\sum_{\substack{Y \bmod \gamma\\ YC'\equiv 0\,(\gamma)}} e\left\{\frac{\nu_0 YA'CY'}{\gamma^2}\right\}.$$

(the positive integer γ is a multiple of the discriminant of C, that is to say of the greatest common divisor of all minors of degree r from C.)

$$W = \frac{\omega^n (CT+D)}{\nu^{\frac{1}{2}nr} \triangle^{\frac{1}{2}r}}\, e\,\{\nu_0\, Z (CT+D)^{-1}\, CZ'\}. \tag{1.16}$$

In order to define $\omega(CT+D)$ we need the following lemma [3]:

There exist unimodular matrices U and V such that

$$UCV^{-1} = \begin{pmatrix} C_1 & N \\ N & N \end{pmatrix}; \quad U(CT+D)\,V' = \begin{pmatrix} C_1 T_1 + D_1 & * \\ N & E \end{pmatrix},$$

where $|C_1{}^{(r)}| \neq 0$ and where the asterisk must be replaced by a noninteresting matrix. Then we define

$$\omega\,(CT+D) = \sqrt{|-i\,(T_1 + C_1^{-1} D_1)|}\,.$$

2\. 1. *Construction of a special set of theta functions.*

Now we determine a necessary and sufficient condition which must be satisfied by a modular matrix $\mathbf{U}$ in order that we obtain in (1.13) a sum of one term only. Then every special matrix X must satisfy $XC \equiv N$ (mod ν_0). Denoting the largest elementary divisor of Q by ε_n, we find the necessary and sufficient condition: $C \equiv N$ (mod $\nu\varepsilon_n$).

In order to deduce a representation of the binary modular congruence group we must determine a set of theta functions with variable P, such that the function obtained from a function of the set by the application of a modular substitution can be represented as a linear aggregate of functions of the set.

Thus we have to determine the matrices G and H such that for all modular matrices $\mathbf{U}$ the congruences

$$\begin{aligned} G &\equiv GA + HC + \nu v \langle A'C \rangle \quad &&(\text{mod } 2), \\ H &\equiv GB + HD + \nu v \langle B'D \rangle &&(\text{mod } 2) \end{aligned} \tag{2.01}$$

are satisfied. Hence, if we denote the elements of the vector v by v_i $(1 \leq i \leq n)$, we have to determine vectors g_i and h_i such that for $i = 1, \ldots n$:

$$\begin{aligned} g_i' &\equiv g_i' A + h_i' C + \nu v_i \langle A'C \rangle \quad (\text{mod } 2), \\ h_i' &\equiv g_i' B + h_i' D + \nu v_i \langle B'D \rangle \quad (\text{mod } 2). \end{aligned} \tag{2.02}$$

If $m > 1$, this system of congruences, where $\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs through all modular matrices of degree m, modulo 2, is incompatible unless $\nu v \equiv 0$ (mod 2).

In order to prove this we use the following lemma:

If m is an integer > 1, an integral matrix A of degree m can be constructed, such that $|A| = \pm 1$ and $|A - E| = \pm 1$.

[3]) l.c. [1]) lemma 5, pag. 32.

Construction:

First let m be even, $m = 2k$. The matrix

$$A = \begin{pmatrix} E^{(k)} & E^{(k)} \\ E^{(k)} & N^{(k)} \end{pmatrix}; \quad A - E = \begin{pmatrix} N^{(k)} & E^{(k)} \\ E^{(k)} & -E^{(k)} \end{pmatrix}$$

satisfies the conditions.

If $m = 2k + 1$ the following matrix satisfies the conditions

$$A = \begin{pmatrix} E^{(k-1)} & N^{(k-1,3)} & E^{(k-1)} \\ N^{(3,k-1)} & X^{(3,3)} & N^{(3,k-1)} \\ E^{(k-1)} & N^{(k-1,3)} & N^{(k-1)} \end{pmatrix}; \quad A - E = \begin{pmatrix} N_{(k-1)} & N^{(k-1,3)} & E^{(k-1)} \\ N^{(3,k-1)} & Y^{(3,3)} & N^{(3,k-1)} \\ E^{(k-1)} & N^{(k-1,3)} & -E^{(k-1)} \end{pmatrix}$$

Here

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Now we shall prove the system (2.02) to be incompatible, unless $\nu v \equiv 0$ (mod 2). The proof consists of two parts.

I. We choose the integral matrix A such that $|A| = \pm 1$ and $|A - E| = \pm 1$. Further we take $B = C = N$ and $D' = A^{-1}$. These four matrices are the "elements" of a modular matrix, since $AD' - BC' = E$ and $AB' = BA'$, $DC' = CD'$. The first congruence of (2.02) can now be written

$$g_i'(A-E) + h_i' C + \nu v_i \langle A'C \rangle = g_i'(A-E) \equiv 0 \ (\text{mod } 2). \tag{2.03}$$

Hence $g_i \equiv 0$ (mod 2).

II. We choose the "elements" of the modular matrix by

$$A = B = D = E, \; C = N.$$

The second congruence of (2.02) can now be written

$$g_i' B + h_i'(D-E) + \nu v_i \langle B'D \rangle = g_i' + \nu v_i\,(1, 1, \ldots, 1) \equiv 0 \ (\text{mod } 2). \tag{2.04}$$

Thus we have $g_i \equiv \nu v_i\,(1, 1, \ldots 1)'$ (mod 2).

From this it is quite obvious that the system (2.02) is incompatible unless $\nu v \equiv 0$ (mod 2).

Henceforth we suppose $v \equiv 0$ (mod 2) and $\nu \equiv 1$ (mod 2). Since $\triangle$ must be an odd integer, the diagonal elements of Q must be even. *Thus the matrix Q must be of even degree.* (Q is a skew symmetric matrix modulo 2.)

2\. 2. *The normalized functions $X(P)$.*

We consider the behaviour of certain functions $X(Z \mid T; P)$ under substitutions of the modular group of degree m, which belong to the principal congruence group of "Stufe" ε_n, — ε_n is the largest elementary divisor of Q—,

$$\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} E & N \\ N & E \end{pmatrix} \pmod{\varepsilon_n}. \tag{2.05}$$

We define the symbol $\mathbf{U}$, apart from its meaning as a left-hand operator, as a right-hand operator on functions F of Z and T by

$$F(Z|T)\,\mathbf{U} = |CT+D|^{\frac{1}{2}n(\nu-1)} F(Z(CT+D)^{-1}|(AT+B)(CT+D)^{-1}) \tag{2.06}$$

Now we introduce the function

$$X(P) = X(Z|T;P) = \frac{\theta_{NN}(Z|T;P,\nu)}{\theta_{NN}^{\nu}(Z|T;N,1)}. \tag{2.07}$$

If we use the formula (1.13) we get

$$\begin{aligned} &\mathbf{U}\,\theta_{NN}(Z|T;P,\nu) = \\ &\quad = W\sigma_\nu(\mathbf{U}) \sum_{XC \bmod \nu_0}^{*} \eta_{NN}(P,X;\nu,\mathbf{U})\,\theta_{NN}(Z|T;PA+XC,\nu). \end{aligned} \tag{2.08}$$

And in the special case $\nu = 1$, $P = N$ we have

$$\begin{aligned} &\mathbf{U}\,\theta_{NN}(Z|T;N,1) = \\ &\quad = \frac{\omega^n(CT+D)}{\Delta^{\frac{1}{2}r}}\, e\{\Delta Z(CT+D)^{-1}CZ'\}\,\sigma_1(\mathbf{U})\,\theta_{NN}(Z|T;N,1). \end{aligned} \tag{2.09}$$

We obtain from (2.07), (2.08) and (2.09)

$$X(P)\,\mathbf{U} = \sum_{R \bmod \nu_0}^{*} A(P,R)\,X(R), \tag{2.10}$$

where

$$\begin{aligned} A(P,R) = {} & \nu^{-\frac{1}{2}nr}\,\Delta^{\frac{1}{2}r(\nu-1)}\,i^{\frac{1}{2}nr(\nu-1)}\,(disr\,C)^{\frac{1}{2}n(\nu-1)}\,\sigma_\nu(\mathbf{U})\,\sigma_1^{-\nu}(\mathbf{U}) \\ & e\left\{\frac{PAB'P' + 2PBC'X' + XDC'X'}{\nu_0}\right\}. \end{aligned} \tag{2,11}$$

if there exists a special matrix X with $R \equiv PA + XC \pmod{\nu_0}$.
$A(P,R) = 0$ otherwise.

Here *disr* C denote the product of the non-vanishing elementary divisors of C.

Now we calculate the value of $\sigma_\nu(\mathbf{U})\,\sigma_1^{-\nu}(\mathbf{U})$ under the assumption that $\mathbf{U}$ belongs to the principal congruence group of "Stufe" ε_n.

First we write the sum $\sigma_\nu(\mathbf{U})$ in a somewhat more convenient form. It is possible to determine matrices V and W of determinant unity such that

$$W^{-1}CV' = \begin{pmatrix} C_1 & N \\ N & N \end{pmatrix} \quad ; \quad W'AV' = \begin{pmatrix} A_1 & N \\ N & E \end{pmatrix},$$

where $|C_1| = |C_1^{(r)}| \neq 0$.

After an obvious modefication of the matrix of summation in the definition of $\sigma_\nu(\mathbf{U})$ we get

$$\sigma_\nu(\mathbf{U}) = \sum_{\substack{Y \bmod \gamma \\ YC_1' \equiv 0(\gamma)}} e\left\{\frac{\nu_0 YC_1'A_1Y'}{\gamma^2}\right\}. \tag{2.12}$$

It is possible to determine a matrix A_1^* with $(|A_1^*|, \gamma) = 1$ such that for every integral matrix Y satisfying $YC_1' \equiv 0(\gamma)$ we have

$$YC_1'A_1Y' \equiv YC_1'A_1^*Y' \pmod{\gamma^2}.$$

Let p be a prime, $p|\gamma$. If $C_1 = N \pmod p$ it follows $|A_1| \not\equiv 0 \pmod p$ and we can determine an integral matrix K with $(|A_1 + KC_1|, p) = 1$. If C_1 has rank r_p ($r_p > 0$) modulo p there exist p-adic unimodular matrices V and W such that

$$W^{-1}C_1V' = \begin{pmatrix} C_2 & N \\ N & N \end{pmatrix} \quad ; \quad W'A_1V' = \begin{pmatrix} A_2 & N \\ N & E \end{pmatrix}$$

and $|C_2^{(r_p)}| \not\equiv 0 \pmod p$. Now it may be seen readily that there can be found an integral matrix K with $(|A_1 + KC_1|, p) = 1$.

The sum $\sigma_\nu(\mathbf{U})$ may then be written as

$$\sigma_\nu(\mathbf{U}) = \sum_{\substack{Y \bmod \gamma \\ YC_1'A_1^* \equiv 0(\gamma)}} e\left\{\frac{\nu_0 YC_1'A_1^*Y'}{\gamma^2}\right\}. \tag{2.13}$$

We suppose now γ to be equal to disr C.

In order to calculate $\sigma_\nu\sigma_1^{-\nu}$ we suppose $\gamma = 2^\varkappa\gamma_0$, $(\gamma_0, 2) = 1$. Then we have

$$\sigma_\nu = \sigma_\nu^{\bullet}\,,\ \sigma_\nu^{\circ}$$

where

$$\sigma_\nu^{\bullet} = \sum_{\substack{X \bmod 2^\varkappa \\ XC_1'A_1^* \equiv 0(2^\varkappa)}} e\left\{\frac{\nu_0 XC_1'A_1^*X'}{2^{2\varkappa}}\right\},$$

$$\sigma_\nu^{\circ} = \sum_{\substack{X \bmod \gamma_0 \\ XC_1'A_1^* \equiv 0(\gamma_0)}} e\left\{\frac{\nu_0 XC_1'A_1^*X'}{\gamma_0^2}\right\},$$

Now there exist γ_0-adic unimodular matrices U and V such that $U'QU$ is a diagonal matrix, the elements of which are $\alpha_i\varepsilon_i$, $(\alpha_i, \gamma_0) = 1$, $\varepsilon_i|\gamma_0$; and $VC_1'A_1^*V'$ is a diagonal matrix, the elements of which are

$$\beta_j\gamma_j \quad , \quad (\beta_j, \gamma_0) = 1 \quad , \quad \gamma_j|\gamma_0 \quad , \quad (j = 1, 2, \ldots r).$$

This follows at once from a theory of MINKOWSKI [4]).

If we replace the matrix of summation X by UXV we get

$$\sigma_\nu^{\circ} = \prod_{j=1}^{r}\prod_{i=1}^{n}\sum_{z \bmod \gamma_j} e\left[\frac{\nu\alpha_i\beta_j\varepsilon_i z^2}{\gamma_j}\right].$$

We supposed $C \equiv N \pmod{\varepsilon_n}$ thus it may be seen readily that $\gamma_j \equiv 0$

4) H. MINKOWSKI. Grundlagen für eine Theorie der quadratischen Formen mit ganzzahligen Koeffizienten. Gesam. Abh. I. Nr. 1, S. 18—25.

(mod ε_i). *Now we suppose* $(\triangle, \nu) = 1$ and introduce $(\nu, \gamma_j) = \delta_j$ and

$$\nu = \delta_j \nu_j \quad ; \quad \prod_{j=1}^{r} \delta_j = \delta \quad ; \quad \nu = \nu_+ \delta.$$

Then we have

$$\sigma_\nu^{\circ} = \prod_{j=1}^{r} \prod_{l=1}^{n} \varepsilon_l \delta_j \sum_{z \bmod \gamma_j/\varepsilon_l \delta_j} e\left[\frac{\nu_j a_l \beta_j z^2}{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}\right].$$

Using the well-known formula for the ordinary Gaussian sums, we obtain

$$\sigma_\nu^{\circ} = \triangle^{\frac{1}{2} r} \delta^{\frac{1}{2} n} \gamma_0^{\frac{1}{2} n} \prod_{j=1}^{r} \prod_{l=1}^{n} \left(\frac{a_l \beta_j \nu_j}{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}\right) e\left[-\tfrac{1}{4}(\gamma_j \varepsilon_l \delta_j - 1)\right],$$

where (—) denotes the symbol of JACOBI from the theory of quadratic residues. We have

$$\left(\frac{a_l \beta_j \nu_j}{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}\right) e\left[-\tfrac{1}{4}(\gamma_j \varepsilon_l \delta_j - 1)\right] =$$

$$= \left(\frac{a_l \beta_j}{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}\right) \left(\frac{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}{\nu_j}\right) e\left[-\tfrac{1}{4}(\nu_j - 1)(\gamma_j \varepsilon_l \delta_j - 1) - \tfrac{1}{4}(\gamma_j \varepsilon_l \delta_j - 1)\right],$$

$$= \left(\frac{a_l \beta_j}{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}\right) \left(\frac{\gamma_j \varepsilon_l^{-1} \delta_j^{-1}}{\nu_j}\right) e\left[-\tfrac{1}{4} \nu_j (\gamma_j \varepsilon_l \delta_j - 1)\right].$$

If $\nu = 1$ we deduce

$$\sigma_1^{\circ} = \triangle^{\frac{1}{2} r} \gamma_0^{\frac{1}{2} n} \prod_{j=1}^{r} \prod_{i=1}^{n} \left(\frac{a_l \beta_j}{\gamma_j \varepsilon_i^{-1}}\right) e\left[-\tfrac{1}{4}(\gamma_j \varepsilon_i - 1)\right].$$

We thus find, since n is even

(2. 14) $$\sigma_\nu^{\circ} (\sigma_1^{\circ})^{-\nu} = \triangle^{\frac{1}{2} r (1-\nu)} \gamma_0^{\frac{1}{2} n (1-\nu)} \delta^{\frac{1}{2} n} \left(\frac{\triangle}{\nu_+}\right) e\left[-\tfrac{1}{4} n \sum_j (\nu - \nu_j)\right].$$

Hereafter we calculate $\sigma_\nu^{\bullet}$.

It can be deduced from the theory of MINKOWSKI [4]) that there exists a 2-adic unimodular matrix V such that

$$V' C_1' A_1^* V = \begin{bmatrix} 2^{\varkappa_1} F_1 & N \cdots\cdots & & & \\ N & 2^{\varkappa_2} F_2 \cdots & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & 2^{\lambda_1} f_1 & 0 \cdots \\ \vdots & \vdots & & 0 & 2^{\lambda_2} f_2 \cdots \\ \vdots & \vdots & & \vdots & \vdots \quad \ddots \end{bmatrix}; \quad F_i = \begin{pmatrix} 2a_i & b \\ b_i & 2c_i \end{pmatrix},$$

where a_i, b_i and f_i are odd integers.

Replacing the matrix X in the definition formula of $\sigma_\nu^{\bullet}$ by XV' we get

$$\sigma_\nu^{\bullet} = \prod_i \sum_{x, y \bmod 2^{\varkappa_i}} e\left[\frac{2\nu(a_i x' Qx + b_i x' Qy + c_i y' Qy)}{2^{\varkappa_i}}\right]$$

$$\prod_j \sum_{z \bmod 2^{\lambda_j}} e\left[\frac{\nu f_j z' Qz}{2^{\lambda_j}}\right].$$

We consider the sum over x if y is fixed. If one of the elements of y is odd, it may be seen readily — by replacing x by $x + 2^{\varkappa_i - 1} k$, with an arbitrary integral vector k — that the sum over x vanishes. Thus we may replace y by $2 a_i y$ and if we suppose $\varkappa_i > 1$ we have, putting $d_i = a_i c_i - b_i^2$,

$$\sigma_\nu^{\bullet} = \prod_i \sum_{y \bmod 2^{\varkappa_i - 1}} e\left[\frac{\nu a_i d_i y' Qy}{2^{\varkappa_i - 1}}\right]$$

$$\sum_{x \bmod 2^{\varkappa_i}} e\left[\frac{2\nu a_i x' Qx}{2^{\varkappa_i}}\right] \prod_j \sum_{z \bmod 2^{\lambda_j}} e\left[\frac{\nu f_j z' Qz}{2^{\lambda_j}}\right].$$

Now we use the following formula [5])

$$\sum_{y \bmod 2^\lambda} e\left[\frac{a y' Q y}{2^\lambda}\right] = \left(\frac{2^\lambda}{\triangle}\right) 2^{\frac{1}{2}\lambda n}.$$

We use $\varkappa = 2 \sum_i \varkappa_i + \sum_j \lambda_j$ and thus we have

$$\sigma_\nu^{\bullet} = \prod_i \left(\frac{2^{\varkappa_i - 1}}{\triangle}\right) 2^{\frac{1}{2}(\varkappa_i - 1) n} \left(\frac{2^{\varkappa_i - 1}}{\triangle}\right) 2^{\frac{1}{2}(\varkappa_i + 1) n} \prod_j \left(\frac{2^{\lambda_j}}{\triangle}\right) 2^{\frac{1}{2} n \lambda_j},$$

$$\sigma_\nu^{\bullet} = 2^{\frac{1}{2} n \varkappa} \left(\frac{2^\varkappa}{\triangle}\right).$$

It may be seen readily that this formula remains true if there exists an index i with $\varkappa_i \leq 1$. We get also

$$\sigma_1^{\bullet} = 2^{\frac{1}{2} n \varkappa} \left(\frac{2^\varkappa}{\triangle}\right).$$

At last we write

(2.15) $$\sigma_\nu^{\bullet} (\sigma_1^{\bullet})^{-\nu} = 2^{\frac{1}{2} n \varkappa (1-\nu)}.$$

Thus we have proved

(2.16) $$\sigma_\nu \sigma_1^{-\nu} = \triangle^{\frac{1}{2} r (1-\nu)} \gamma^{\frac{1}{2} n (1-\nu)} \delta^{\frac{1}{2} n} \left(\frac{\triangle}{\nu_+}\right) e\left[-\tfrac{1}{4} n \sum_{j=1}^{r} (\nu - \nu_j)\right].$$

We introduce these results in (2. 11). If there exists a special matrix X with $R \equiv PA + XC \pmod{\nu_0}$ we have

$$A(P, R) = \nu^{-\frac{1}{2} n r} \delta^{\frac{1}{2} n} \left(\frac{\triangle}{\nu_+}\right) e\left[\tfrac{1}{4} n \sum_{j=1}^{r} (\nu_j - 1)\right]$$

$$e\left\{\frac{P A B' P' + 2 P B C' X' + X D C' X'}{\nu_0}\right\}.$$

And

$$A(P, R) = 0$$

otherwise.

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[5]) See H. D. KLOOSTERMAN, The behaviour of general theta functions under the modular group...... (I). Ann. of Math. (2) **47**, 339 (1946).