Physics. — *Reflection of light by rippled water surfaces.* By J. S. VAN WIERINGEN. (Communication from the Laboratories of the N.V. KEMA, Arnhem.) (Communicated by Prof. M. MINNAERT.)

(Communicated at the meeting of September 27, 1947.)

In this article some results are given of an investigation concerning the reflection of light from the sun or artificial lightsources by the rippled surface of the sea, canals and rivers. The following calculations complete the observations described by Prof. MINNAERT [1] [4]).

The reflected image of a light source in a rippled water surface is not a single image of the source, as in the case of a flat surface, but a juxtaposition of a number of these images. They are not normal images as seen in a plane mirror, but they are deformed because of the curvature of the surface.

We consider two simple cases.

1. If the ripples are totally irregular, a light pillar is seen, extending over an area which generally is elongated in a vertical direction. In each

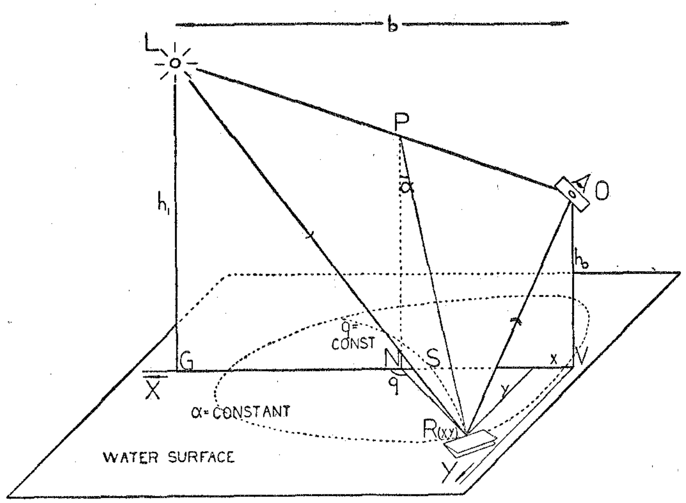

Fig. 1. The light of the point source L reaches the observer O via the points R of the water surface. The coordinates R are functions of the inclination α and the azimuth q of the surface element at R.

point R of the water surface there is a definite inclination α and an azimuth q for which light from the source L can be reflected towards the observer O (fig. 1). The observed reflection points are all lying within a closed contour, a limiting curve which will be shown to be a curve of

1) MINNAERT, Natuurkunde van het Vrije Veld, I, p. 20.

the 6th degree. If the positions of the light source and of the observer are fixed, this curve is determined by the maximum angle of inclination α_m of the rippled surface. For the points *within* this contour, $|\alpha| < \alpha_m$.

2. If the waves are strictly parallel, al surface elements satisfy the condition $q =$ const.; this penomenon occurs in canals and rivers where the waves by preference move, in the direction of the canal and have their fronts perpendicular to the direction of propagation. The reflected images are now lying on a curve which generally is slanting and which will be shown to be of the 3d degree. This curve is entirely determined by the positions of the light source and the observer and by the direction of propagation of the waves. The maximum inclination α_m of the waves only determines whether the curve is illuminated over a greater or smaller extent. Of course the point of normal reflection S, where $\alpha = 0$, belongs to it.

Practically, in most cases an intermediate phenomenon is seen. The waves have all directions but with a preference for one of these. The reflection points are then lying within a closed curve which is neither the contour first mentioned nor the curve mentioned in the second case. Within this curve a number of images of the light source are seen. Just as in the other cases they are deformed because of the curvature of the surface, and they are constantly moving as a result of the propagation of the waves.

The light pillar observed on totally irregular waves.

As was shown above, the limiting curves are the curves $\alpha =$ const. $= \alpha_m$. They can be calculated as follows (fig. 1).

If R (coordinates x and y, as in fig. 1) is a point which reflects light from the source L to the observer O, the line normal to the reflecting surface element in R should intersect the line LO at a point which we will call P. Besides, PR should divide $\angle LRO$ into two equal parts, or in other words:

$$PR = \sqrt{LR \cdot OR - LP \cdot OP} = \frac{PN}{\cos \alpha_m},$$

or

$$\sqrt{LR \cdot OR - \frac{LR \cdot OR}{(LR+OR)^2} \cdot LO^2} = h_o + (h_l - h_o)\frac{OR}{LR+OR} =$$

$$= \frac{h_o \cdot LR + h_l \cdot OR}{LR + OR}$$

Expressing LR and OR in terms of x, y, b, h_l and h_0, h_l and h_0 being the heights of L and O above the water surface and b their horizontal distance, the equation for (x, y), when $\alpha = \alpha_m$ results in:

$$\left.\begin{aligned}\cos\alpha_m \sqrt{\sqrt{(x^2+y^2+h_0^2)\{(b-x)^2+y^2+h_l^2\}}\,[\{\sqrt{x^2+y^2+h_0^2}+\sqrt{(b-x)^2+y^2+h_l^2}\}^2-b^2-h_l-h_0)^2]} &= \\ = h_0\sqrt{(b-x)^2+y^2+h_l^2}+h_l\sqrt{x^2+y^2+h_0^2} &\end{aligned}\right\} \quad (1)$$

It is somewhat simplified if the source is at an infinite distance, as in the case of the sun. Then the direction of all incident rays is the same. The curve in this case is found by taking the limit of (1 for $b = \infty$. Then

$$\frac{\sqrt{(b-x)^2+y^2+h_l^2}}{b} = \frac{1}{\sin i}$$ (i = angle of incidence) and (1 is simplified to:

$$\left.\begin{aligned}\cos a_m \sqrt{2\,(x^2+y^2+h_0^2)-2\,\sqrt{x^2+y^2+h_0^2}\,(x\cdot\sin i - h_0\cdot\cos i)} = \\ = h_0 + \cos i\sqrt{x^2+y^2+h_0^2}\end{aligned}\right\} \quad . \quad . \quad (2)$$

This is the limiting curve, *drawn in the plane of the reflecting water surface*.

The limiting curves are of the 6th degree in the general case of a source at a finite distance. The same result was found by PICCARD [2]) for the more special case that the heights of observer O and light source L to the water surface are equal ($h_0 = h_l$). In (1 and (2 the complete 6th and 4th degree curves include negative as well as positive roots. The negative roots describe the case of PR being external bissectrix of $\angle LRO$ and consequently that part of the complete curve has no physical meaning; it will not be considered further.

The equations (1 and (2 are too complicated for computation. It is much easier to find the curves experimentally. A lamp is placed above a table at L. A little mirror makes an angle a_m with the table and can be moved over it. By looking through a fixed hole at O we can seen the points where reflection occurs (fig. 1).

The curves found by this method prove to be well approximated by an ellipse, as was already found by PICCARD [2]). The observer looks at this ellipse, as was already found by PICCARD [2]). The observer looks at this pillar under an angle i, and sees the projection on a plane perpendicular to his direction of sight OS. The ellipse is therefore foreshortened.

We now consider the shape of the limiting curve projected on the plane perpendicular to OS. For general orientation it is sufficient to consider the case of a source at an infinite distance, the shape of the curve being mainly determined by i, the angle of incidence for normal reflection. The length of the short (horizontal) axis of the ellipse is repersented as an angle γ in fig. 2: $tg\,\frac{1}{2}\,\gamma = \cos i.\; tg\, 2a$. The ratio $\frac{\gamma}{a}$ of the short over the long axis is $\cos i$, if $90 - i > 2a_m$. If $90 - i \leq 2a_m$ the pillar reaches the horizon and so the long axis is partly cut off.

Then the ratio is

$$\frac{\gamma}{a'} = \frac{2\ \mathrm{tg}\ 2a}{\dfrac{\mathrm{tg}\ 2a}{\cos i} + \dfrac{1}{\sin i}}. \qquad \text{(fig. 3).}$$

2) PICCARD, Arch. sc. phys. et nat. 21, 481 (1889).

It is remarkable that in general the ratio $\frac{\gamma}{a}$, that is to say the shape of the ellipse is independent of the maximum inclination a_m of the waves. However, if the ellipse is cut off by the horizon, a_m comes into account. By measuring the arc of the pillar at the horizon the maximum inclination a_m of the waves can be easily determined [3]).

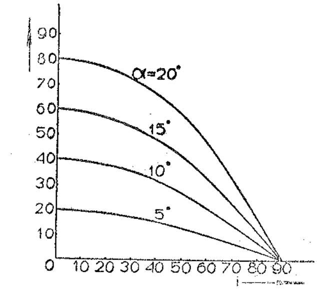

Fig. 2. The angle γ under which the horizontal axis of the pillar is seen for different angles of incidence i. Parameter: the maximum inclination a of the water surface. The light source is at an infinite distance.

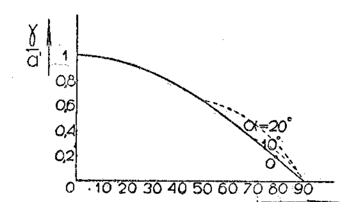

Fig. 3. The ratio of the angles under which the horizontal and the vertical axis of the pillar are seen as a function of the angle of incidence i (infinitely distant source). The dashed lines are corrections for the cutting-off effect of the horizon.

Properly speaking the cutting off does already occur before the reflected ray reaches the horizon. When grazing over the water surface, it could be withdrawn from the observer's eye by a wave top (fig. 4). This is the case if $90 - i \leq 2a + \beta$. Owing to the smallness of β ($tg\,\beta \approx \frac{2\,tg\,a_m}{3\,\pi} \approx 1/5\; tg\, a_m$ for sine waves, so $\beta \approx 4°$), the correction due to this effect is not large.

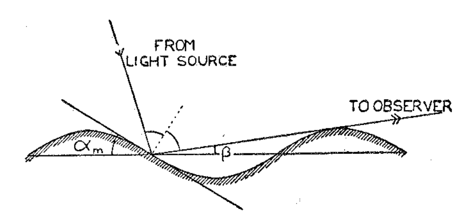

Fig. 4. Part of the pillar may be cut off by wave tops.

3) SPOONER, Corresp. du Baron de Zach, p. 331 (1822).

From figs. 2 and 3 it follows that in a reflected landscape the pillars are al about equally high, but that these of low objects are narrower than those of high objects.

The luminous line seen on plane waves.

The symbols h_o, h_l and b are the same as in the preceding paragraph and in fig. 1; the azimuth q is the angle between the direction of propagation of the waves and the vertical plane through L and O (fig. 1). The points R (x, y) where reflection occurs are lying on a curve with $q =$ constant. Of course the point S of normal reflection is one of them ($\alpha = 0$). The terminal points of the curve are lying on the ellipse of the previous case. In the other points of the curve the azimuth q is the same, but $|\alpha| < \alpha_m$.

The calculation runs as follows (fig. 1):
As RP is the bissectrix of $\angle LRO$,

$$\frac{LR}{RO} = \frac{LP}{PO} = \frac{GN}{NV} \text{ or } LR \,.\, NV = RO \,.\, GN.$$

By expressing these in terms of x, y and q, the result obtained is:

$$(x \sin q - y \cos q) \sqrt{(b-x)^2+y^2+h_l^2} = \{(b-x) \sin q + y \cos q\} \sqrt{x^2+y^2+h_0^2} \quad . \quad (3)$$

This is the curve *drawn in the plane of the water surface.*

It is a curve of the third degree, for, taking the squares, the terms of the fourth degree disappear.

Again the equation becomes much simpler if we consider a light source at an infinite distance such as the sun. By taking the limit of (3 for $b = \infty$ and substituting $\dfrac{h_l - h_o}{b} = \text{cotg } i$

it is found that:

$$\sin i \; \sin q \sqrt{x^2 + y^2 + h_0^2} = x \sin q - y \cos q \quad . \; . \; . \; . \quad (4)$$

If the coordinate system is rotated in its plane over the angle q, the equation expressed in the new coordinates ξ and η is:

$$\eta = -\sin i \sin q \sqrt{\xi^2 + \eta^2 + h_0^2} \quad . \; . \; . \; . \; . \; . \quad (5)$$

So the images of a source at an infinite distance over a plane-wave surface are lying on one branch of a hyperbola (the root in (5 being positive only). They are lying on a straight line (the x-axis) if $q = 0$, that is to say if the waves are running in the direction of the observer; and also on a straight line (the ξ-axis) if $i = 0°$, when the sun is over the head of the observer. If at the same time $i = 90°$ and $q = 90°$ the curve consists of two imaginary lines of which only the intersection is observed, being the infinitely distant point of the x-axis.

The curves were computed for a light source at an infinite distance and for $i = 30°$ (fig. 5). From this figure it is seen that a gradual increase in

the azimuth q of the waves has only a small influence on the direction of the pillar as long as the observer is looking more or less perpendicularly to the waves. If the direction of observation exceeds 45°, the influence rapidly increases and becomes very strong near $q = 90°$: This effect is the more striking the greater the angle of incidence i.

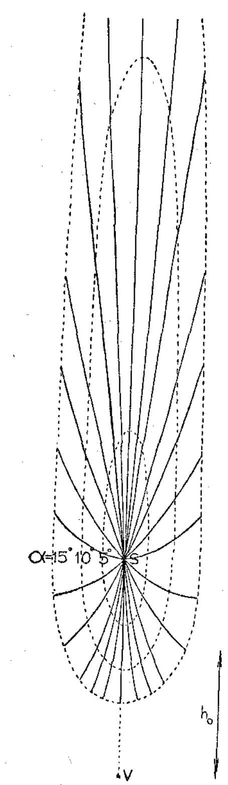

Fig. 5. Dashed: computed limiting curves of the pillar seen on an irregularly rippled surface with maximum inclination 5°, 10°, and 15°. Solid: the luminous lines seen on plane waves for different directions of propagation (q = 0°, 15°, 30°, 45°, 60°, 75° and 90°) for the case i = 60°. These lines are drawn in the plane of the reflecting water surface. The observer is at a point O directly over V at a height h_0. The infinitely distant soure L is at a height of 30° above the horizon (i = 60°).

However the computation is rather tedious and an experimental measurement is much simpler. It is easy to investigate the influence of the azimuth q by placing on the table a sheet of parallel-rippled glass, which is rotated in its plane. Here also the difference between a light source at a finite or an infinite distance proves to be small. In some cases with a source at at finite distance it is possible to see the inflexion point of the third degree curve.

In the reflections by rippled water surfaces it is possible to discover a curved pillar under very favorable conditions. However in most cases the pillar is practically straight, because the waves are not running exactly in one direction and the illuminated curve is more or less blurred. Therefore it is ordinarily sufficient, to know the mean inclination of the pillar. The angle p' in the surface between the curve $q =$ constant and the x-axis follows from $tg\, p' = tg\, q \,.\, \cos^2 i$, as can be easily calculated from (3 or (4. The observer sees this angle projected on a plane perpendicular to his direction of sight. This projection p of p' is found from $tg\,p = tg\,q.\cos i$. [4]) This formula describes completely the above mentioned effect: if q, the direction of the waves, is changed at an uniform rate, the inclination p of the pillar changes first at a slow, then at an increasingly quick rate; the effect being the more pronounced the larger i.

Summary.

It is shown that the reflected image of a light source in a rippled water surface is a light pillar which generally is limited by a curve of the 6th degree. If the source is at an infinite distance, the curve is of the 4th degree. The shape of the limiting curve is discussed. In parallel ripples the light pillar is a single curve which is generally slanting. It is a curve of the 3d degree, in the special case of an infinitely distant source of the 2d degree (cf. the solid lines in fig. 5). The influence of the direction of the ripples is discussed.

4) MINNAERT, Physica 9, 925 (1942).

Physical Geography. — *Theory on central rectilinear recession of slopes.* I. By J. P. BAKKER and J. W. N. LE HEUX. (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of May 31, 1947.)

Introduction.

After the early publications of OSMOND FISHER (1) [1]) and LAWSON (2), in which a brief mathematical derivation was used, the theories of OTTO LEHMANN (3) and ED. GERBER (4) were the first, in which a problem of physionomic geomorphology was more thoroughly treated from a quantatively exact point of view.

LEHMANN started from parallel recession of steep mountain slopes and constant height. GERBER accepted the former condition, but added a new one, assuming that the fault scarp due to crustal movements, or the valley-slope owing to vertical erosion increased in height during parallel recession. For various parts of the earth, especially in regions where permeable resistant sand- and limestones occur (the Dolomites in the Alps, many cuesta- and mesa-landscapes) there are several reasons for assuming parallel retreat of slopes as a real condition. (See fig. 4 and 5.)

Nevertheless we may note that as early as sixty years ago geomorphologists based deductions on the assumption of the recession of steep slopes with decreasing slope angle, the simplest case of which we find reproduced in PHILIPPSON's figure (5, II, 2, p. 63) for the development of denudation landscapes (our fig. 1). This figure, therefore, indicates that the intensity

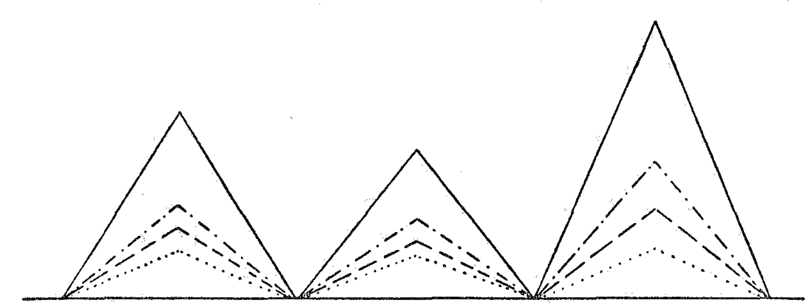

Fig. 1. (After PHILIPPSON). See text.

of weathering-removal increases rectiliniarly with the height of the wall, equalling zero at the basic point of the slope. We shall call this type of wall recession "*central rectilinear recession*". The basic point of the steep slope functions in this case as recession centre.

1) The numbers in parenthesis refer to the list of literature at the end of this article.