So we find for the characteristics of skewness $S^{3,0}$, $S^{2,1}$, $S^{1,2}$, $S^{0,3}$ and for the characteristics of excess $E^{4,0}$, $E^{3,1}$, $E^{2,2}$, $E^{1,3}$, $E^{0,4}$:

$$S^{2,1}=\frac{m^{2,1}}{m^{2,0}\,(m^{0,2})^{1/2}}=\frac{\dfrac{m^{1,1}\,m^{3,0}}{m^{2,0}}}{m^{2,0}\,(m^{0,2})^{1/2}}=\frac{m^{1,1}}{\sqrt{m^{2,0}\,m^{0,2}}}\cdot\frac{m^{3,0}}{(m^{2,0})^{3/2}},$$

or

$$S^{2,1}=\gamma\,S^{3,0};\qquad \text{likewise}\qquad S^{1,2}=\gamma\,S^{0,3}; \qquad (18)$$

$$E^{3,1}=\frac{m^{3,1}}{(m^{2,0})^{3/2}\,(m^{0,2})^{1/2}}-3\gamma=\frac{\dfrac{m^{1,1}\,m^{4,0}}{m^{2,0}}}{(m^{2,0})^{3/2}\,(m^{0,2})^{1/2}}-3\gamma=$$

$$=\frac{m^{1,1}}{\sqrt{m^{2,0}\,m^{0,2}}}\cdot\frac{m^{4,0}}{(m^{2,0})^{2}}-3\gamma=\gamma\left(\frac{m^{4,0}}{(m^{2,0})^{2}}-3\right),$$

or

$$E^{3,1}=\gamma\,E^{4,0};\qquad \text{likewise}\qquad E^{1,3}=\gamma\,E^{0,4}. \qquad (19)$$

Mathematics. — *A study of Bessel functions in connection with the problem of two mutually attracting circular discs.* By C. J. BOUWKAMP. (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands.) (Communicated by Prof. J. G. VAN DER CORPUT.)

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Summary.

In this note I discuss the application of Bessel functions to the physical problem of the mutual attraction of two homogeneous circular discs lying in the same plane. It is assumed that the law of force, which describes the interaction of two unit point masses, is derivable from the potential function $V(r)$ depending only on the distance r between the masses. So, the force problem is reducible to a scalar problem: the calculation of the mutual potential energy of the discs.

Special attention is paid to potential functions varying with the distance as r^{-n} where n is any positive number. This includes the gravitational force ($n=1$) as well as the London–Van der Waals force ($n=6$).

The paper is entirely mathematical.

1. *Formulation of the problem in terms of Bessel functions.*

The reader is, of course, familiar with the two-dimensional, logarithmic, potential: $V(r)=\log r$. In this case the mutual energy of two non-overlapping homogeneous discs in the same plane is equal to that obtained when the total masses of the discs are concentrated at the respective centres.

A somewhat more general question arises almost at once. Namely, whether there exists a particular law of interaction such that two non-overlapping discs (radii: a, b; distance between centres: $c>a+b$) in the same plane shall attract each other as if certain reduced masses were located at the centres. More precisely, whether it is possible to choose $V(r)$ such that the mutual energy of the two discs is given by $\varphi(a,b)$ $V(c)$ where $\varphi(a,b)$ is a (symmetric) function of a and b, *not depending on* c.

The answer to the question above is affirmative, even if the trivial case of the logarithmic potential, for which $\varphi(a,b)=1$ (both discs having unit mass), is excluded. As will be shown in due course, the modified Bessel function $K_0(rt)$ serves the purpose, for all values of the parameter t. Once we have succeeded to represent the potential function $V(r)$ as a sum or integral (the weight function depending on t) of the 'invariant' function $K_0(rt)$, it is easy to calculate the interaction energy in question.

We now proceed to the invariance — with respect to the transition from point mass to disc — of the function $K_0(rt)$.

Let R be the distance between a unit point mass and the centre of a disc of radius a. The point mass is assumed in the plane of the disc and lying outside the latter. Further, let polar coordinates (ϱ, θ) be introduced at the centre of the disc, the polar axis being directed to the unit point mass. Then the mutual energy of the point mass and the disc, which will be supposed homogeneous and of unit mass, is given by

$$u_t(a, 0; R) = \frac{1}{\pi a^2} \int_0^a \varrho \, d\varrho \int_0^{2\pi} K_0 (t \sqrt{\varrho^2 - 2\varrho R \cos\theta + R^2})\, d\theta$$

$$= \frac{2}{a^2} K_0(Rt) \int_0^a I_0(\varrho t)\, \varrho d\varrho$$

$$= \frac{2}{at} I_1(at) K_0(Rt), \qquad (a < R)$$

whereby use is made of some well-known formulae of the theory of Bessel functions.

Therefore, the interaction between the point mass and the disc is as if the mass

$$\varphi(a, 0) = \frac{2}{at} I_1(at)$$

were located at the centre of the disc, for all values of the distance $R > a$. This is the invariance property of the Bessel function K_0 as referred to above.

It will further be obvious, by twice applying this process of reduction, that

$$\varphi(a, b) = \varphi(a, 0)\ \varphi(b, 0).$$

Hence, the energy of interaction of the two discs under consideration is given by

$$u_t(a, b; c) = \frac{4}{abt^2} I_1(at) I_1(bt) K_0(ct)$$

when $V(r) = K_0(rt)$.

Let us now assume that the given potential function $V(r)$ can be represented by an integral of the following type:

$$V(r) = \int_0^\infty f(t) K_0(rt)\, dt. \qquad (1)$$

Then, since the energy is additive, the mutual potential energy of the discs becomes

$$U(a, b; c) = \frac{4}{ab} \int_0^\infty I_1(at) I_1(bt) K_0(ct) f(t) t^{-2}\, dt. \qquad (2)$$

2. *Expression for the energy of two non-overlapping discs when* $V(r) = r^{-n}$ $(n > 0)$.

The 'generating' function $f(t)$ occurring in (1) and (2) is known for the particular potential $V(r) = r^{-n}$ $(n > 0)$, as follows from [1])

$$\frac{1}{r^n} = \frac{2^{2-n}}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty K_0(rt)\, t^{n-1}\, dt. \qquad (n > 0)$$

Consequently, the interaction energy of the two discs in response to the law $V(r) = r^{-n}$ is given by

$$U_n(a, b; c) = \frac{2^{4-n}}{ab\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty I_1(at) I_1(bt) K_0(ct)\, t^{n-3}\, dt. \qquad (3)$$

This function will be discussed in sections 4, 5, 6, 7 for $b = 0$, $a = b$, $a \neq b \neq 0$, $b \to \infty$, respectively.

3. *Differential relations.*

Let x, y denote rectangular cartesian coordinates in the plane of a disc of arbitrary shape and mass distribution. Then, the potential outside the disc under action of the law $V(r)$ is given by

$$U(x, y) = \iint D(\xi, \eta)\, V(r)\, d\xi\, d\eta \qquad (r^2 = (x-\xi)^2 + (y-\eta)^2)$$

where the integration has to be carried out over the surface of the disc, and where $D(\xi, \eta)$ stands for the local mass density.

Obviously,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \iint D(\xi, \eta) \left\{ \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} \right\} d\xi\, d\eta.$$

Furthermore,

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = \left\{ \frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} \right\}_{r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2}}.$$

We have thus proved the following

Theorem I:

If U is the potential of an arbitrary disc in response to the potential law $V(r)$, then ΔU is the potential of the same disc in response to the law $V''(r) + V'(r)/r$.

Especially for centre-symmetric discs, $\Delta U = U''(c) + U'(c)/c$ where c denotes the distance from the centre. When in this case the theorem is applied twice in succession, we obtain

[1]) Cf. G. N. WATSON, A treatise on the theory of Bessel functions, Cambridge 1922/1944, p. 388, formula (2).

Theorem II:

If $U(a, b; c)$ denotes the energy of interaction of two non-overlapping circular discs with centre-symmetric mass distributions under influence of the potential law $V(r)$, then

$$U^*(a, b; c) = \frac{\partial^2 U}{\partial c^2} + \frac{1}{c}\frac{\partial U}{\partial c} = 4\frac{\partial}{\partial (c^2)}\left\{c^2 \frac{\partial U}{\partial (c^2)}\right\} = \frac{1}{c}\frac{\partial}{\partial c}\left(c\frac{\partial U}{\partial c}\right)$$

is the energy of interaction of the same discs under influence of the potential law

$$V^*(r) = \frac{d^2 V}{d r^2} + \frac{1}{r}\frac{d V}{d r}.$$

This way of reasoning I owe to Prof. N. G. DE BRUIJN. An alternative proof of Theorem II is based on the integral representation (2). In view of the differential equation satisfied by K_0, the function U^* corresponds to the generating function $t^2 f(t)$ when U corresponds to $f(t)$, and, in its turn, $t^2 f(t)$ corresponds by (1) to the potential law $V^*(r)$. In the latter way I originally found Theorem II.

Theorem II is of particular interest with respect to the potential functions r^{-n} since it reduces the interval of n to be investigated to $0 < n \leq 2$. For greater values of n the function can be found by a process of differentiation, namely,

$$U_{n+2}(a, b; c) = \frac{1}{n^2}\left\{\frac{\partial^2}{\partial c^2} + \frac{1}{c}\frac{\partial}{\partial c}\right\} U_n(a, b; c). \qquad (n > 0) \quad . \quad . \quad (4)$$

Of course, equation (4) is also easily proved directly from (3) when use is made of Bessel's differential equation for $K_0(ct)$.

4. *The potential energy of a point mass outside a circular disc when* $V(r) = r^{-n}$.

The discussion of the function $U_n(a, b; c)$ defined by (3) is comparatively simple when one of the discs reduces to a point mass. In that case we have

$$U_n(a, 0; c) = \frac{2^{3-n}}{a\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty I_1(at)\, K_0(ct)\, t^{n-2}\, dt. \quad . \quad . \quad . \quad (5)$$

This integral is expressible in terms of hypergeometric functions. When use is made of the modified WEBER–SCHAFHEITLIN integral [2]) it is found that

$$U_n(a, 0; c) = \frac{1}{c^n} F\left(\frac{n}{2}, \frac{n}{2}; 2; \frac{a^2}{c^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In passing, it may be noted that (6) holds for *all* values of n, not necessarily positive, as from physical considerations it is obvious that $U_n(a, b; c)$ is an analytic function of the variable n.

[2]) Cf. WATSON, *loc. cit.*, p. 410, formula (1).

For even values of n the function $U_n(a, 0; c)$ is elementary. For instance,

$$U_2(a, 0; c) = -\frac{1}{a^2}\log\left(1 - \frac{a^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$U_4(a, 0; c) = (c^2 - a^2)^{-2} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$U_6(a, 0; c) = (c^2 + \tfrac{1}{2}a^2)(c^2 - a^2)^{-4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Further, the function is expressible in terms of complete elliptic integrals of the first and second kinds when n is an odd integer. For example, let us take $n = 1$; then

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 2; k^2) = \frac{\Gamma(2)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})} \int_0^1 u^{-\frac{1}{2}}(1-u)^{\frac{1}{2}}\,(1-k^2 u)^{-\frac{1}{2}}\, du$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \frac{\cos^2\varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$= \frac{4}{\pi}\left[\left(1 - \frac{1}{k^2}\right)\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \frac{1}{k^2}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\, d\varphi\right]$$

$$= \frac{4}{\pi}\left\{\left(1 - \frac{1}{k^2}\right)K(k) + \frac{1}{k^2}E(k)\right\},$$

in the usual notation of elliptic integrals. Consequently,

$$U_1(a, 0; c) = \frac{4c}{\pi a^2}\left\{E\left(\frac{a}{c}\right) - \left(1 - \frac{a^2}{c^2}\right)K\left(\frac{a}{c}\right)\right\}. \quad . \quad . \quad (10)$$

This result is not new [3]).

In a similar way one can evaluate the function U_n when $n = 3$, *viz.*:

$$F(\tfrac{3}{2}, \tfrac{3}{2}; 2; k^2) = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})} \int_0^1 u^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2 u)^{-\frac{3}{2}}\, du$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\frac{\sin^2\varphi\, d\varphi}{(1-k^2\sin^2\varphi)^{3/2}} = \frac{4}{\pi k}\frac{dK(k)}{dk}$$

$$= \frac{4}{\pi k^2}\left[(1-k^2)^{-1}E(k) - K(k)\right].$$

We therefore obtain that

$$U_3(a, 0; c) = \frac{4}{\pi a^2 c}\left[\left(1 - \frac{a^2}{c^2}\right)^{-1} E\left(\frac{a}{c}\right) - K\left(\frac{a}{c}\right)\right]. \quad . \quad . \quad (11)$$

When comparing (10) with (11), we see that

$$U_1(a, 0; c) = (c^2 - a^2)\, U_3(a, 0; c), \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

[3]) Cf. H. BATEMAN, Partial differential equations of mathematical physics, Cambridge 1932 — New York 1944, p. 417, example 2.

which might also have been obtained by EULER's transformation of hypergeometric functions:

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x). \quad . \quad . \quad (13)$$

The result I found when taking $n = 5$ is

$$U_5(a, 0; c) = \frac{4}{9\pi a^2 c^3}\left[\frac{1+7\frac{a^2}{c^2}}{\left(1-\frac{a^2}{c^2}\right)^3} E\left(\frac{a}{c}\right) - \frac{1+3\frac{a^2}{c^2}}{\left(1-\frac{a^2}{c^2}\right)^2} K\left(\frac{a}{c}\right)\right]. \quad (14)$$

Generally, $U_{2m+1}(a, 0; c)$ (m = non-negative integer) is expressible as a linear combination of $E(a/c)$ and $K(a/c)$ with coefficients rational in a/c.

It is very interesting that also for half-integral values of n the function $U_n(a, 0; c)$ is expressible in terms of elementary functions and complete elliptic integrals. This is easily proved with the aid of KUMMER's relation [4]:

$$F(\tfrac{3}{4}, \tfrac{3}{4}; 2; \sin^2\theta) = F(\tfrac{3}{2}, \tfrac{3}{2}; 2; \sin^2 \tfrac{1}{2}\theta) \quad . \quad . \quad . \quad . \quad (15)$$

The hypergeometric function on the right has already been evaluated; therefore we have at once:

$$\left.\begin{aligned} U_{\frac{3}{2}}(a, 0; c) = \frac{16c^{\frac{1}{2}}}{\pi a^2}\Bigg[E\left\{\sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{a^2}{c^2}}\right)} - \right. \\ \left. -\frac{1}{2}\left(1+\sqrt{1-\frac{a^2}{c^2}}\right) K\left\{\sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{a^2}{c^2}}\right)}\right\}\Bigg] \end{aligned}\right\}. \quad (16)$$

Moreover, the relation (15) immediately leads to the following interesting equation

$$U_3(a, 0; c) = c^{-\frac{3}{2}} U_{\frac{3}{2}}\left(2a\sqrt{1-\frac{a^2}{c^2}}, 0; c\right). \quad . \quad . \quad . \quad . \quad (17)$$

When $n = \frac{5}{2}$ the calculation runs as follows. First, by EULER's transformation (13),

$$F\left(\tfrac{5}{4}, \tfrac{5}{4}; 2; \frac{a^2}{c^2}\right) = \left(1-\frac{a^2}{c^2}\right)^{-\frac{1}{2}} F\left(\tfrac{3}{4}, \tfrac{3}{4}; 2; \frac{a^2}{c^2}\right),$$

and, consequently,

$$U_{\frac{5}{2}}(a, 0; c) = (c^2-a^2)^{-\frac{1}{2}} U_{\frac{3}{2}}(a, 0; c), \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is known by (16).

To determine $U_{\frac{1}{2}}(a, 0; c)$, I once more apply EULER's transformation (13):

$$F\left(\tfrac{1}{4}, \tfrac{1}{4}; 2; \frac{a^2}{c^2}\right) = \left(1-\frac{a^2}{c^2}\right)^{\frac{3}{2}} F\left(\tfrac{7}{4}, \tfrac{7}{4}; 2; \frac{a^2}{c^2}\right)$$

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[4]) Cf. E. T. WHITTAKER–G. N. WATSON, A course of modern analysis, Cambridge 1935, p. 298, example 12.

which leads to the relation

$$U_{\frac{1}{2}}(a, 0; c) = (c^2-a^2)^{\frac{3}{2}} U_{\frac{7}{2}}(a, 0; c). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

On the other hand, we have by (4)

$$U_{\frac{7}{2}}(a, 0; c) = \tfrac{4}{9}\left\{\frac{\partial^2}{\partial c^2} + \frac{1}{c}\frac{\partial}{\partial c}\right\} U_{\frac{3}{2}}(a, 0; c).$$

Consequently, both $U_{\frac{1}{2}}$ and $U_{\frac{7}{2}}$ are expressible in terms of elementary functions and complete elliptic integrals; and so is $U_{m-\frac{1}{2}}(a, 0; c)$ for any positive integer m in virtue of (4).

5. *The case of equal radii.*

In the second place I shall consider the function U_n for discs of equal radii: $a = b$. Thus

$$U_n(a, a; c) = \frac{2^{4-n}}{a^2\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}\int_0^\infty I_1^2(at)\, K_0(ct)\, t^{n-3}\, dt. \quad . \quad . \quad (20)$$

The function (20) can be evaluated in terms of the generalized hypergeometric function ${}_3F_2$. To prove this, let us expand [5]) the square of the Bessel function into ascending powers of t, *viz.*:

$$I_1^2(at) = \sum_{m=0}^{\infty}\left(\frac{at}{2}\right)^{2+2m}\frac{\Gamma(2m+3)}{m!\,\{\Gamma(m+2)\}^2\,\Gamma(m+3)},$$

and integrate term by term. One then finds that

$$U_n(a, a; c) = \frac{1}{c^n}\sum_{m=0}^{\infty}\frac{\Gamma(2m+3)}{\Gamma(m+1)\,\Gamma(m+3)}\left\{\frac{\Gamma\left(m+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(m+2)}\right\}^2\left(\frac{a^2}{c^2}\right)^m$$

$$= \frac{1}{c^n}\,\frac{\Gamma(1)\,\Gamma(2)\,\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}\sum_{m=0}^{\infty}\frac{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\frac{n}{2}\right)\Gamma\left(m+\frac{n}{2}\right)}{\Gamma(m+1)\,\Gamma(m+2)\,\Gamma(m+3)}\left(\frac{4a^2}{c^2}\right)^m.$$

That is to say,

$$U_n(a, a; c) = \frac{1}{c^n}\,{}_3F_2\left(\frac{3}{2}, \frac{n}{2}, \frac{n}{2}; 2, 3; \frac{4a^2}{c^2}\right). \quad . \quad . \quad . \quad . \quad (21)$$

which, as (6), is true for all values of n not necessarily positive.

As in the preceding section, the function is elementary when n is an even integer. Particularly simple are the cases $n = 4$, $n = 6$, since then the

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[5]) Cf. WATSON, *loc. cit.*, p. 147, formula (6).

generalized hypergeometric function (21) reduces to an ordinary hypergeometric function, namely,

$$U_4(a, a; c) = \frac{1}{c^4} F\left(\tfrac{3}{2}, 2; 3; \frac{4a^2}{c^2}\right),$$

$$U_6(a, a; c) = \frac{1}{c^6} F\left(\tfrac{3}{2}, 3; 2; \frac{4a^2}{c^2}\right).$$

Now,

$$F(\tfrac{3}{2}, 2, 3; x) = 2\int_0^1 u\,(1-xu)^{-\frac{3}{2}}\,du,$$

which is easily integrated. The result is found to be

$$U_4(a, a; c) = \frac{1}{c^4}\frac{1}{\sqrt{1-\frac{4a^2}{c^2}}}\left\{\frac{2}{1+\sqrt{1-\frac{4a^2}{c^2}}}\right\}^2 \quad . \quad . \quad . \quad (22)$$

Moreover, by EULER's formula (13),

$$F(\tfrac{3}{2}, 3; 2; x) = (1-x)^{-\frac{5}{2}} F(\tfrac{1}{2}, -1; 2; x).$$

The hypergeometric series on the right consists of two terms; accordingly,

$$U_6(a, a; c) = \frac{1}{c^6}\frac{1-\frac{a^2}{c^2}}{\left(1-\frac{4a^2}{c^2}\right)^{\frac{5}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The evaluation of $U_2(a, a; c)$ can be accomplished by an integration, either by means of (4) or more directly as follows. First, it is easily verified that

$$\frac{d}{dx}\{x\,{}_3F_2(\tfrac{3}{2}, 1, 1; 2, 3; x)\} = F(\tfrac{3}{2}, 1; 3; x)$$

$$= (1-x)^{\frac{1}{2}} F(\tfrac{3}{2}, 2; 3; x) = \left\{\frac{2}{1+\sqrt{1-x}}\right\}^2$$

$$= -8\frac{d}{dx}\left\{\frac{1}{1+\sqrt{1-x}} + \log(1+\sqrt{1-x})\right\}.$$

Therefore, upon integrating while suitably accounting for the constant of integration, we obtain

$${}_3F_2(\tfrac{3}{2}, 1, 1; 2, 3; x) = \frac{8}{x}\left\{\frac{1}{2} - \frac{1}{1+\sqrt{1-x}} - \log\left(\frac{1+\sqrt{1-x}}{2}\right)\right\}$$

from which it follows that

$$U_2(a, a; c) = \frac{1}{a^2}\left[1 - \frac{2}{1+\sqrt{1-\frac{4a^2}{c^2}}} - 2\log\left\{\frac{1+\sqrt{1-\frac{4a^2}{c^2}}}{2}\right\}\right]. \quad . \quad . \quad (24)$$

The analysis is much more complicated when n is an odd integer. As a

matter of fact, only the case $n = 3$ seems tractable; that is to say, I have not been able to sum the series (21) when $n = 1$, this of course being possible for $n = 5, 7, \ldots$, in virtue of (4), once $U_3(a, a; c)$ is known.

To evaluate $U_3(a, a; c)$, I use a relation due to CLAUSEN [6]); it follows from this relation that

$${}_3F_2(\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}; 2, 3; x) = \{F(\tfrac{3}{4}, \tfrac{3}{4}; 2; x)\}^2.$$

Therefore,

$$U_3(a, a; c) = \frac{1}{c^3}\left\{F\left(\tfrac{3}{4}, \tfrac{3}{4}; 2; \frac{4a^2}{c^2}\right)\right\}^2.$$

On comparing this with (6) we see that

$$U_3(a, a; c) = \{U_{\frac{3}{2}}(2a, 0; c)\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Consequently, from (16),

$$U_3(a, a; c) = \frac{16c}{\pi^2 a^4}[E(k) - (1-k^2)K(k)]^2, \quad . \quad . \quad . \quad . \quad (26)$$

where the modulus of the complete elliptic integrals is given by

$$k = \frac{1}{\sqrt{2}}\left\{1 - \sqrt{1-\frac{4a^2}{c^2}}\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

It may be remarked that (26) is equivalent to

$$\int_0^\infty I_1^2(xt)\,K_0(2t)\,dt = \frac{\pi}{16}\left\{P_{-\frac{1}{2}}^{-1}(\sqrt{1-x^2})\right\}^2, \quad . \quad . \quad . \quad . \quad (28)$$

$$(-1 < x < 1)$$

which is closely related to MEIJER's integral representations of the product of two Legendre functions [7]).

6. *The case of unequal discs.*

In this section, the function U_n will be investigated for general values of a and b. However, n will again be restricted to integers since I have not succeeded to evaluate the integral (3) for other values of n, apart from the fact that (3) is expressible in terms of APPELL's hypergeometric function F_4, *viz.*:

$$U_n(a, b; c) = \frac{1}{c^n} F_4\left(\frac{n}{2}, \frac{n}{2}; 2, 2; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad (29)$$

which is readily proved by expanding $I_1(at)\ I_1(bt)$ into ascending powers of t and integrating term by term [8]).

To begin with, let us take $n = 4$. Then from (3),

$$U_4(a, b; c) = \frac{1}{ab}\int_0^\infty I_1(at)\,I_1(bt)\,K_0(ct)\,t\,dt.$$

[6]) Cf. WHITTAKER–WATSON, *loc. cit.*, p. 298, example 11.
[7]) Cf. C. S. MEIJER, Nieuw Arch. Wiskunde 19, 207—234, 1938.
[8]) For the definition of APPELL's function, cf. P. APPELL–J. KAMPÉ DE FÉRIET, Fonctions hypergéométriques et hypersphériques — Polynomes d'Hermite, Paris 1926.

This integral can be evaluated with the aid of a result due to MACDONALD[9]:

$$\int_0^\infty K_\mu(at)\, J_\nu(bt)\, J_\nu(ct)\, t^{\mu+1}\, dt = -\frac{a^\mu \cos\nu\pi\,(X^2-1)^{-\frac{1}{2}\mu-\frac{1}{4}}}{\sqrt{2\pi}\,(bc)^{\mu+1}\sin(\mu+\nu)\pi}\, Q^{\mu+\frac{1}{2}}_{\nu-\frac{1}{2}}(X), \quad (30)$$

in which $2bcX = a^2 + b^2 + c^2$ and Re $(a \pm ib \pm ic) > 0$; Q is the Legendre function of the second kind in the sense of BARNES.

As is well known, the right-hand side of (30) can be written in terms of hypergeometric functions. The result is found to be

$$\frac{a^\mu\, \Gamma(\mu+\nu+1)}{2^{\nu+1}(bc)^{\mu+1}\,\Gamma(\nu)\, X^{\mu+\nu+1}}\, F\left(\frac{\mu+\nu+2}{2}, \frac{\mu+\nu+1}{2}; \nu+1; X^{-2}\right).$$

Upon taking $\mu = 0$, $\nu = 1$ we obtain

$$\int_0^\infty K_0(at)\, J_1(bt)\, J_1(ct)\, t\,dt = \frac{1}{4bcX^2}\, F(\tfrac{3}{2}, 1; 2; X^{-2})$$

$$= \frac{1}{4bcX^2}\int_0^1 \left(1-\frac{u}{X^2}\right)^{-\frac{3}{2}} du = \frac{1}{2bc}\left\{\left(1-\frac{1}{X^2}\right)^{-\frac{1}{2}}-1\right\}.$$

Now replace a, b, c by c, ia, ib respectively. It has thus been proved that

$$\int_0^\infty I_1(at)\, I_1(bt)\, K_0(ct)\, t\,dt = \frac{1}{2ab}\left[\left\{1-\frac{4a^2b^2}{(c^2-a^2-b^2)^2}\right\}^{-\frac{1}{2}}-1\right],$$

and, consequently,

$$U_4(a, b; c) = \frac{1}{2a^2b^2}\left[\frac{c^2-a^2-b^2}{\{c^2-(a-b)^2\}^{\frac{1}{2}}\{c^2-(a+b)^2\}^{\frac{1}{2}}}-1\right]. \quad (31)$$

As is easily verified, formula (31) is in accordance with the special case $a = b$ given before at (22).

For greater, even, values of n the function $U_n(a, b; c)$ can be calculated by differentiation of (31) in virtue of (4). Omitting the rather tedious, though elementary, intermediate computations, I merely state the final result when $n = 6$:

$$U_6(a, b; c) = \frac{c^2(2c^2-a^2-b^2)-(a^2-b^2)^2}{2\{c^2-(a-b)^2\}^{\frac{3}{2}}\{c^2-(a+b)^2\}^{\frac{3}{2}}}, \quad . \quad . \quad . \quad (32)$$

which, by the way, is in complete agreement with the result obtained by DUBE and DASGUPTA[10]).

Whereas U_6 has been obtained by differentiation of U_4, the function U_2 can be calculated by the inverse process, that is, by integration of

$$U_4(a, b; c) = \frac{\partial}{\partial(c^2)}\left\{c^2\frac{\partial U_2(a, b; c)}{\partial(c^2)}\right\}.$$

[9]) Cf. WATSON, *loc. cit.*, p. 412, formula (6).

[10]) G. P. DUBE and H. K. DASGUPTA, On the London–Van der Waals forces between two disc-like particles, Indian J. Phys. **13**, 411—416, 1939.

For this purpose, let

$$u = c^2; \quad W(u) = \sqrt{u^2-2(a^2+b^2)u+(a^2-b^2)^2}.$$

Then, by integrating once,

$$u\frac{\partial U_2}{\partial u} = \frac{1}{2a^2b^2}\int^u \left\{\frac{t-a^2-b^2}{W(t)}-1\right\} dt$$

$$= \frac{1}{2a^2b^2}\,[a^2+b^2-u+W(u)],$$

where the constant of integration has been chosen so as to make $u\dfrac{\partial U_2}{\partial u}$ vanish at infinity (as a matter of fact, this condition is sufficient; the more stringent, and physically necessary, condition $u^2\dfrac{\partial U_2}{\partial u} \to -1$ $(u \to \infty)$ is then automatically fulfilled).

Further, a second integration yields:

$$2a^2b^2U_2 = W(u)-u+a^2+b^2-(a^2+b^2)\log\left\{\frac{W(u)+u-a^2-b^2}{2u}\right\}$$

$$+(a^2-b^2)^2\int_\infty^u \frac{dt}{tW(t)},$$

where the constant of integration is taken such that $U_2(\infty) = 0$.

Moreover we have

$$(a^2-b^2)^2\int_\infty^u \frac{dt}{tW(t)} = -|a^2-b^2|\log\left\{\frac{u(a^2+b^2)-(a^2-b^2)^2-|a^2-b^2|\,W(u)}{u(a^2+b^2-|a^2-b^2|)}\right\}.$$

Accordingly, the final result is found to be:

$$U_2(a, b; c) = \frac{1}{2a^2b^2}\left[W-c^2+a^2+b^2-(a^2+b^2)\log\left\{\frac{W+c^2-a^2-b^2}{2c^2}\right\}-\right.$$
$$\left.-|a^2-b^2|\log\left\{\frac{c^2(a^2+b^2)-(a^2-b^2)^2-|a^2-b^2|\,W}{c^2(a^2+b^2-|a^2-b^2|)}\right\}\right], \quad (33)$$

in which W is an abbreviation for

$$W = W(c^2) = \sqrt{\{c^2-(a-b)^2\}\{c^2-(a+b)^2\}}. \quad . \quad . \quad . \quad (34)$$

As may have been anticipated from section 5, the function $U_n(a, b; c)$ is expressible in terms of complete elliptic integrals when n is an odd integer greater than 1.

To begin with, let us take $n = 3$. Then, the fourth type of APPELL's hypergeometric function of two variables reduces to a product of two ordinary hypergeometric functions, as follows from BAILEY's formula[11])

$$F_4(a, b; c, a+b-c+1;\ x(1-y),\ y(1-x)) =$$
$$= F(a, b; c; x)\ F(a, b; a+b-c+1; y);$$

[11]) W. N. BAILEY, Generalized hypergeometric series, Cambridge 1935, p. 81.

namely,

$$F_4\left(\tfrac{3}{2}, \tfrac{3}{2}; 2, 2; x(1-y), y(1-x)\right) = F\left(\tfrac{3}{2}, \tfrac{3}{2}; 2; x\right) F\left(\tfrac{3}{2}, \tfrac{3}{2}; 2; y\right).$$

Since the remaining ordinary hypergeometric function is known from the preceding section, we at once have

$$U_3(a, b; c) = \frac{16c}{\pi^2 a^2 b^2} \{E(k_1) - (1-k_1^2) K(k_1)\} \{E(k_2) - (1-k_2^2) K(k_2)\}, \tag{35}$$

in which the moduli of the complete elliptic integrals are defined by

$$k_1^2 (1-k_2^2) = a^2/c^2,$$
$$k_2^2 (1-k_1^2) = b^2/c^2;$$

thus

$$\left.\begin{aligned} k_1^2 &= \frac{1}{2c^2} \{c^2 + a^2 - b^2 - W\} \\ k_2^2 &= \frac{1}{2c^2} \{c^2 - a^2 + b^2 - W\} \end{aligned}\right\}, \quad \ldots \ldots \tag{36}$$

where W is as defined in (34).

It is to be remarked that (35) is completely symmetric in a and b; further, it reduces to (26) when $a = b$.

By means of (4), the functions $U_5, U_7, \ldots$, can be expressed in terms of elliptic integrals also. Unfortunately, I have not succeeded to sum the series (29) in the newtonian case $n = 1$.

7. *Energy of interaction between a circular disc and a half-plane.*

In conclusion, it may be worth while to consider briefly the degenerated problem of one of the discs becoming infinitely large. Thus, let us calculate the mutual energy of a disc of radius a, of homogeneous mass distribution and *total mass* equal to 1, and a homogeneous half-plane of *mass density* equal to 1.

Let $\triangle$ denote the distance between the centre of the disc and the boundary of the half-plane. Confining ourselves to the work function $V(r) = r^{-n}$ $(n > 2)$, we have for the energy in question

$$u_n(a; \triangle) = \lim_{b \to \infty} \{\pi b^2 U_n(a, b; \triangle + b)\} \quad \ldots \tag{37}$$

This limit is most easily evaluated by replacing the functions $I_1(bt)$ and $K_0(ct)$ in (3) by their asymptotic expressions for $bt \to \infty$. The result is found to be

$$u_n(a; \triangle) = \frac{2^{3-n}\pi}{a\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty e^{-\triangle t} I_1(at)\, t^{n-4}\, dt \quad \ldots \tag{38}$$

The analogue of the differential relation (4) is very simple; it now connects the functions of orders n and $n+1$, namely,

$$u_{n+1}(a; \triangle) = -\tfrac{1}{2} \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \right\}^2 \frac{\partial u_n(a; \triangle)}{\partial \triangle} \quad \ldots \tag{39}$$

as is readily verified by differentiation of (38) with respect to $\triangle$.

The integral (38) is generally expressible in terms of Legendre functions:

$$u_n(a; \triangle) = -\frac{\sqrt{2\pi a}}{2^{n-3} a^2} \frac{(\triangle^2 - a^2)^{\frac{7}{4} - \frac{n}{2}}}{\sin n\pi \left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} Q_{\frac{1}{2}}^{n-\frac{7}{2}}(\triangle/a) \quad \ldots \tag{40}$$

In terms of hypergeometric functions, one has

$$u_n(a; \triangle) = \frac{\pi}{(2\triangle)^{n-2}} \frac{\Gamma(n-2)}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} F\left(\frac{n-2}{2}, \frac{n-1}{2}; 2; \frac{a^2}{\triangle^2}\right), \quad \ldots \tag{41}$$

or, alternatively,

$$u_n(a; \triangle) = \frac{\pi}{(2\sqrt{\triangle^2 - a^2})^{n-2}} \frac{\Gamma(n-2)}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} F\left(\frac{n-2}{2}, \frac{5-n}{2}; 2; -\frac{a^2}{\triangle^2 - a^2}\right), \tag{42}$$

by means of which the function $u_n(a; \triangle)$ is easily evaluated for integral values of n. The latter functions are elementary; for instance,

$$u_3(a; \triangle) = \frac{4}{\triangle + \sqrt{\triangle^2 - a^2}}, \quad \ldots \tag{43}$$

$$u_4(a; \triangle) = \frac{\pi}{2a^2} \left\{ \frac{\triangle}{\sqrt{\triangle^2 - a^2}} - 1 \right\}, \quad \ldots \tag{44}$$

$$u_5(a; \triangle) = \frac{4}{9} \frac{1}{(\triangle^2 - a^2)^{3/2}}, \quad \ldots \tag{45}$$

$$u_6(a; \triangle) = \frac{3\pi}{32} \frac{\triangle}{(\triangle^2 - a^2)^{5/2}}, \quad \ldots \tag{46}$$

$$u_7(a; \triangle) = \frac{4}{75} \frac{4\triangle^2 + a^2}{(\triangle^2 - a^2)^{7/2}}. \quad \ldots \tag{47}$$

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