Mathematics. — *On the principles of intuitionistic and affirmative mathematics.* II. By D. VAN DANTZIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Ch. 3. *The strong interpretation. Affirmative mathematics. (Intuitionistic mathematics from a formal standpoint.)*

Some classical mathematicians might perhaps say that this weak interpretation, scetched roughly in Ch. 2, were exactly what they needed, and that they therefore had no need of BROUWER's strong interpretation. Against this view however several remarks can be made.

First: scientific interest, and even mere curiosity makes us desire to obtain other than only negative results. If it is true that "totally normal" numbers "exist", we want to know such a number, and if every real number except those of a set of measure 0 possesses this property, we might know whether particular numbers like $\sqrt{2}$, or e possess it. Then: for the *applications* of mathematics these negative statements are entirely worthless. There we have no use for "existence" of a real number in the weak sense; we need effective approximations, the possible errors of which we can estimate explicitly. Third: the use of the weak interpretation may lead to difficulties with regard to the *foundations* of mathematics and physics. E.g. this is the case with the theory of probability, as I have tried to show in a recent conference [20]). Finally it is a good and current habit of mathematicians, always to prove their theorems in as strong a form and with as few undefined notions as can be done without unduly augmenting the difficulties. There is no reason, not to follow this habit with respect to the *logical* part also.

In this chapter we shall follow the inverse way as in Ch. 2, and take the standpoint of a "classical" mathematician who wants to deduce the fundamental theorems of arithmetics and analysis from as few hypotheses and undefined notions and in as strong a form and as rigorously as he can. We shall not assume here anything about intuitionism but proceed in a purely formal way, though here also only a rough scetch can be given.

Of course we begin our analysis with a theory of natural numbers as a base for a theory of real numbers.

Before doing so, we have to say a few words about the logical notations used here. All constant symbols are printed in vertical type; all variables are denoted by letters in italics. The implication, conjunction and

[20]) D. VAN DANTZIG [2].

disjunction of two statements, denoted by A and B are denoted by

$$_{|}A \supset B_{|}$$
$$_{|}A \wedge B_{|}$$
$$_{|}A \vee B_{|}$$

respectively *inclusive* the vertical strokes, which replace brackets or dots. This notation has the advantage, that complex formulae are obtained by direct *substitution* of these formulae for the cursive letters, so that no *counting* of dots or strokes is necessary, e.g.

$$_{|||}A \supset C_{|} \wedge_{|} B \supset C_{||} \supset_{||} A \vee B_{|} \supset C_{||}$$

and that dots and brackets may retain their original algebraical meaning. In the same way the sum and product of two numbers x and y are denoted by $(x+y)$ and $(x \cdot y)$ respectively, *inclusive* the brackets. This makes every further convention with respect to "strength" of brackets superfluous [21]). As, however, we don't intend to give here a complete formalisation, in particular of the logical relations, we sometimes omit brackets or strokes which are not necessary for reading the formulae.

In order to avoid a calculus of classes, we write $\mathrm{N}[x]$ and $\mathrm{R}[x]$ instead of $x \in \mathrm{N}$ and $x \in \mathrm{R}$ respectively. We read these symbols as "x *denotes* (not: "is") a natural (real) number" so that we may avoid defining what a natural or real number "really is". Constant symbols (e.g. N, C, T, R, d) are printed in vertical type.

The axioms which we introduce are: the axioms $A\,1$, $A\,2$, $A\,3$ the latter of which is introduced below, and the axioms $E_1, \ldots, E_4$ of equality. They are:

A 1. $\vdash \mathrm{N}\,[0]$

A 2. $\vdash_{|} \mathrm{N}\,[x] \supset \mathrm{N}\,[x']_{|}$

E 1. $\vdash_{|} \mathrm{N}\,[x] \supset_{|} x = x_{||}$

E 2. $\vdash_{|} \mathrm{N}\,[x, y] \supset_{||} x = y_{|} \supset_{|} x' = y'_{||}$

E 3. $\vdash_{|} \mathrm{N}\,[x, y, z] \supset_{|||} x = y_{|} \wedge_{|} y' = z'_{||} \supset_{|} z = x_{|||}$

E 4. $\vdash_{|} \mathrm{N}\,[x] \supset_{||} x = y_{|} \supset \mathrm{N}\,[y]_{||}$

Here we wrote for abbrevation $\mathrm{N}\,[x, y]$ and $\mathrm{N}\,[x, y, z]$ in stead of $_{|}\mathrm{N}\,[x] \wedge \mathrm{N}\,[y]_{|}$ and $_{||}\mathrm{N}\,[x] \wedge \mathrm{N}\,[y]_{|} \wedge \mathrm{N}\,[z]_{|}$ respectively.

We did not — as it is sometimes done — *define* equality of x and y by requiring $A\,[x]$ and $A\,[y]$ to be equivalent for any statement A, depending on one variable. In fact, this condition is too strong, as we may consider statements, concerning the logical calculus itself. If e.g. $A\,[x]$ is replaced by the statement that the statement $x = 4$ may be

[21]) Strictly spoken the use of two kinds of brackets (opening and closing ones) is superfluous, though of course it makes reading easier. For the same reason we might replace the strokes by inverted or ordinary comma's, writing e.g. $,A \supset B,$ in stead of $_{|}A \supset B_{|}$. The advantage however is not important.

obtained from the axioms and definitions by at most a given number of applications of the deduction schemes, we may very well have $A\,[4]$ without having $A\,[(2+2)]$. It would therefore be necessary to limit in advance the set of statements for which the equivalence of $A\,[x]$ and $A\,[y]$ must hold. We therefore preferred to define equality implicitly by four axioms, from which the other properties are easily deduced.

Complete induction is introduced as a deduction-scheme (not as an axiom), viz

$$\frac{\vdash A\,[0] \qquad \vdash_{|} \mathrm{N}\,[x] \supset_{|} A\,[x] \supset A\,[x']_{||}}{\vdash_{|} \mathrm{N}\,[x] \supset A\,[x]_{|}}$$

for any statement $A\,[x]$ such that $A\,[y]$ is independent of x. We can then define in the ordinary way sums and products of natural numbers and deduce their formal algebraic properties.

We remark that we did not introduce PEANO's fourth axiom:

$$\vdash_{|} N\,[x] \supset_{|} \neg_{|}\, x' = 0_{|||},$$

nor did we until now use any negation or disjunction. The question then naturally arises whether we can do entirely without them.

From the formal point of view mathematics is a system of formulae, the mathematician is willing to "accept" or "admit", whatever these words may mean. The symbol ${}_{|}A \supset B_{|}$ means that he is prepared to take B in his list of accepted formulae as soon as A is admitted. The symbol ${}_{|}A \wedge B_{|}$ means that A as well as B is admitted.

The symbol $\neg A$, however, has quite another nature. It does not describe the admittance of any formula, but the rejection of A, i.e. the mathematician's refusal to accept A. Of course he *may* refuse A, but why should he mention the fact at all? We may make our list without telling anything about formulae we reject, or we eventually or conditionnally would refuse to admit [22]).

And if we *can* do so — and we shall see we can — the mathematician's principle not to introduce any superfluous primitive notion urges us to avoid the negation at all. Such a mathematical (or logical) system in which negation occurs *not* — of course it may occur in discussions *about* (the "metamathematics" of) the system — will be called *affirmative* [23]).

With respect to the disjunction the situation is still somewhat different.

[22]) From the intuitionist's standpoint the situation is not very different. For him $\neg A$ denotes the impossibility ("absurdity") of a construction as stated by A. This is a warning, useful for future investigators, that attempts to construct A necessarily must fail, but strictly indispensible it is not.

[23]) The principles of affirmative mathematics were first exposed in my lectures at the university of Amsterdam and in a subsequent conference before the Amsterdam mathematical society in the spring of 1939. I found since that I. JOHANSSEN [1] had put forward more or less analogous ideas before. Recently the idea of eliminating negations occurred independently to Dr. G. F. C. GRISS also, who, however, does not avoid unrestricted existence statements.

The disjunction ${}_{|}A \vee B_{|}$ describes our will to admit either A or B (or both). But if we are prepared to do so, we still have not admitted anything at all, nor have we — if we avoid negations — engaged ourselves to any definite acceptation. It expresses therefore a certain hesitation, so that one would be inclined to avoid it also. On the other hand we may remark that, if we do so, we necessarily loose a large number of statements of ordinary mathematics. The simplest example is the theorem

$${}_{||}(a \cdot b) = 0_{|} \supset_{||} a = 0_{|} \vee_{|} b = 0_{|||}$$

for natural numbers [24]), and all its consequences.

On another occasion I hope to show that avoidance of the disjunction also is possible without losing anything essential of mathematics. But as the procedure becomes somewhat inelegant, we shall *here* retain the disjunction. For the same reason we introduce here the *restricted* existential symbol, though SKOLEM [1] showed that it can be avoided. It is defined as follows

$${}_{|}\exists^{0}_{\xi}\, A\,[\xi]_{|} \overset{\mathrm{df}}{\equiv} A\,[0]$$

$${}_{|}\exists^{n'}_{\xi}\, A\,[\xi]_{|} \overset{\mathrm{df}}{\equiv} {}_{||}\exists^{n}_{\xi}\, A\,[\xi]_{|} \vee A\,[n']_{|}$$

The unrestricted existence-symbol can not be defined by complete induction and will not be introduced at all.

Instead of PEANO's fourth axion we introduce the axiom

A 3. $$\vdash {}_{|}\mathrm{N}\,[x, y] \supset_{||} (x+y) = 0_{|} \supset_{|} y = 0_{|||}$$

It evidently is independent of the preceding axioms, which are satisfied e.g. in a prime field of characteristic $p > 1$.

The system of axioms introduced here is certainly exempt of formal contradictions, as it would become trivial if the statement

T $${}_{|}\mathrm{N}\,[x] \supset_{|} x = 0_{||}$$

were introduced as another axiom. Evidently T would follow already from

$${}_{|}0' = 0_{|}$$

That this statement is not a consequence of the other axioms follows from the fact that this would imply a contradiction in intuitive mathematics, a possibility, the admittance of which does not seem to have any meaning. A formal proof of non-triviality of the set of axioms still stands out. Perhaps it might be less hopeless than the proof of non-contradictority in the case of HILBERT's axioms.

One might perhaps think of *defining* the negation by

$$\neg A \overset{\mathrm{df}}{\equiv} {}_{|}A \supset \mathrm{T}_{|}$$

analogous to the proceduce of GENTZEN [1] and I. JOHANSSON [1]. But

[24]) For real numbers it is not valid under the strong interpretation.

apart from the fact that the negation then were not a primitive but a defined notion, one should remind that T has *not* the nature of a contradiction, whereas GENTZEN's $\curlywedge$ *does* denote a contradiction. For one may build several arithmetic systems, e.g. by replacing the first axiom N [0] by N [$\square$] and varying the other axioms in some way or another. The relations $0' = 0$ and $\square' = \square$ may then be entirely independent. The fact that some formal arithmetical system reduces to trivialty is of no more importance than the fact that some group consists of its unit-element only. If we would insist upon calling ${}_{|}A \supset \mathrm{T}_{|}$ the "negation" of A, this negation would loose the "universal" character usually ascribed to it and become a relative notion [25]. Of course this does not prevent that in a *single* formal arithmetic system ${}_{|}A \supset \mathrm{T}_{|}$ is *practically* equivalent with $\neg A$ [26].

The further development of the theory of natural numbers, as well as the introduction of integers and rational numbers does not lead to any essential difficulties. We omit it here, and mention only the fact that the relations $\leqq$, $<$ and $\neq$ are defined by

$$ {}_{|}a \leqq b_{|} \stackrel{\mathrm{df}}{\equiv} {}_{|}\exists^{b}_{\xi}\, {}_{|}a = \xi_{||} $$
$$ {}_{|}a < b_{|} \stackrel{\mathrm{df}}{\equiv} {}_{|}a' \leqq b_{|} $$
$$ {}_{|}a \neq b_{|} \stackrel{\mathrm{df}}{\equiv} {}_{||}a < b_{|} \vee_{|} b < a_{|}, $$

the first of which is equivalent with

$$ {}_{|}a \leqq 0_{|} \stackrel{\mathrm{df}}{\equiv} {}_{|}a = 0_{|} $$
$$ {}_{|}a \leqq b'_{|} \stackrel{\mathrm{df}}{\equiv} {}_{||}a \leqq b_{|} \vee_{|} a = b'_{||} $$

Real numbers may be introduced by means of a constant function symbol d, such that $r_n(x) = 2^{-n}\, \mathrm{d}_n(x)$ is the n^{th} approximating rational dual number of x. If we denote the statements "x represents a real number" and "x represents an integer" by R [x] and I [x] respectively, the definition becomes:

$$ \mathrm{R}\,[x] \stackrel{\mathrm{df}}{\equiv} \forall_n\, {}_{|}\mathrm{N}\,[n] \supset_{|} \mathrm{I}\,[\mathrm{d}_n(x)] \wedge_{|} |\mathrm{d}_{n'}(x) - 2\,\mathrm{d}_n(x)| \leqq 1_{|||} $$

Expressed by the rational numbers $r_n(x)$ the inequality means

$$ |r_{n+1}(x) - r_n(x)| \leqq 2^{-n-1}. $$

Though less general in form, the definition is substantially equivalent

[25]) This does *not* imply that the "exclusion-negation" (according to MANNOURY's terminology [5], p. 333, [10]) would become a "choice-negation".

[26]) Anyhow we can not define ${}_{|}A \vee B_{|}$ here by $\neg_{|} \neg A \wedge \neg B_{|}$ i.e. by ${}_{||}A \supset \mathrm{T}_{|} \wedge {}_{|}B \supset \mathrm{T}_{||} \supset \mathrm{T}_{|}$, as this would lead us back to the weak interpretation.

with BROUWER's definition, the main difference being that BROUWER's λ-intervals [27]) have been replaced by their centers, and that the set of sequences, defining equal real numbers ("coinciding points") is somewhat narrower than with BROUWER. The relevant point is that the real numbers are obtained by a limit-procedure with a prescribed "velocity of convergence [28])."

Equality of real numbers is defined by

$$ {}_{||}\mathrm{R}\,[x] \wedge \mathrm{R}\,[y]_{|} \supset_{||} x = y_{|} \stackrel{\mathrm{df}}{\equiv} \forall_n\, {}_{|}\mathrm{N}\,[n] \supset_{|} |\mathrm{d}_n(x) - \mathrm{d}_n(y)| \leqq 1_{||||} $$

The reflexivity and symmetry of the relation are trivial, the transitivity is easily proved by substituting n' for n and considering that $\mathrm{d}_n(x)$ etc. are integers. Also the definitions and the demonstration of the algebraic properties of sums and products of real numbers do not lead to any essential difficulties. Nomore is this the case with the identification of particular real numbers with integers p or dual rationals $p \,.\, 2^{-k}$ by means of the definitions

$$ {}_{|}\mathrm{d}_n(p) \stackrel{\mathrm{df}}{=} p \,.\, 2^{n}{}_{|} $$

and

$$ {}_{|}\mathrm{d}_n(p \,.\, 2^{-k}) \stackrel{\mathrm{df}}{=} \mathcal{E}\,(p \,.\, 2^{n-k})_{|} $$

respectively, where $\mathcal{E}\,(r)$ denotes the entier of the *rational* number r.

The relation $\leqq$ for real numbers is defined as follows:

$$ {}_{||}\mathrm{R}\,[x] \wedge \mathrm{R}\,[y]_{|} \supset_{||} x \leqq y_{|} \stackrel{\mathrm{df}}{\equiv} \forall_n\, {}_{|}\mathrm{N}\,[n] \supset_{|} \mathrm{d}_n(x) \leqq \mathrm{d}_n(y) + 1_{||||} $$

Here also the properties ${}_{||}x = y_{|} \supset {}_{|}x \leqq y_{||}$,

hence ${}_{|}x \leqq x_{|}$ and ${}_{||}x \leqq y_{|} \wedge {}_{|}y \leqq x_{||} \supset {}_{|}x = y_{||}$

are trivial, whereas ${}_{|||}x \leqq y_{|} \wedge {}_{|}y \leqq z_{||} \supset {}_{|}x \leqq z_{||}$ is easy to prove. The maximum Max (x, y) of two real numbers x and y is defined by

$$ {}_{|}\mathrm{d}_n\,(\mathbf{Max}\,(x, y)) \stackrel{\mathrm{df}}{=} \mathrm{Max}\,(\mathrm{d}_n(x), \mathrm{d}_n(y))_{|} $$

and has the properties

$$ \mathrm{Max}\,(x, y) = \mathrm{Max}\,(y, x), \qquad x \leqq \mathrm{Max}\,(x, y) $$

and

$$ {}_{|||}x \leqq z_{|} \wedge_{|} y \leqq z_{||} \supset_{|} \mathrm{Max}\,(x, y) \leqq z_{||} $$

[27]) BROUWER [5] I p. 253. A λ-interval is an interval of length 2^{-n} for some natural n, the endpoints of which are integral multiples of 2^{-n-1}. A real number ("point of the linear continuum") is a sequence of λ-intervals, each of which is contained with its endpoints in the interior of the preceding one.

[28]) According to BROUWER's definition this "velocity" may be augmented, but not diminished arbitrarily. Therefore his definition contains an unrestricted existence statement which we have avoided.

Also the absolute value $|x|$ of x can now be defined as Max $(x, -x)$, and has the ordinary properties. We can, however, *not* prove affirmatively that

$$_{\|}|x| = x_{|} \vee_{|} |x| = -x_{\|}$$

Moreover we can *not* define affirmatively the relations $<$ and $\neq$. This is to be expected, as e.g. the relation $x \neq y$ means that a natural n *exists*, such that $|x - y| \geqq 2^{-n}$. This however implies an unrestricted existence-statement; this is connected with the fact that the relation $x \neq y$ can not lead to any affirmative result. For instance the statement $_{|}y \neq 0_{|}$ gives us no means to determine even the first approximation of y^{-1}, as long as no rational lower boundary for y is known. For the same reason the statement $y \neq 0$ for an *empirical* quantity y (measured, not counted!) for which no lower boundary is known, can not be used for any empirical conclusion concerning y^{-1}.

We have therefore to "disperse" relations like $<$ and $\neq$ into a sequence of relations, say $\overset{n}{<}$ and $\overset{n}{\neq}$ [29]), defined by

$$_{|}x \overset{n}{<} y_{|} \overset{\mathrm{df}}{\equiv} {}_{|}2^{-n} \leqq y - x_{|}$$
$$_{|}x \overset{n}{\neq} y_{|} \overset{\mathrm{df}}{\equiv} {}_{|}2^{-n} \leqq |y - x|_{|}$$

We can then define xy^{-1} if $y \overset{n}{\neq} 0$ and prove the usual properties, in particular its independence of the value of n.

This "dispersion" of a statement into a "fine-structure" of other ones is rather characteristic for affirmative mathematics. As a further example we mention the limes-relation, e.g. $\lim\limits_{n \to \infty} a_n = a$. It states e.g. that for every natural e a natural $n(e)$ exists, such that

$$_{|}\mathrm{N}\,[n] \supset_{|} |a - a_{n(e)+n}| \leqq 2^{-e}{}_{\|}$$

Evidently this definition contains an unrestricted existential statement. In affirmative mathematics we can therefore only define limes-relations with a *given* [30]) function $n(e)$ where $_{|}\mathrm{N}\,[e] \supset \mathrm{N}[n(e)]_{|}$ i.e. with a given "velocity of convergence". Hence we can only define

$$_{|}\lim_{n(e)} a_n = a_{|} \overset{\mathrm{df}}{\equiv} \forall_e\, {}_{|}\mathrm{N}\,[e] \supset \forall_{n|}\, \mathrm{N}\,[n] \supset_{|} |a - a_{n(e)+n}| \leqq 2^{-e}{}_{\|\|}$$

In an analogous way definitions like that of continuity, etc. have to be altered, as well, of course, as the theorems in which they are used. Generally speaking the "dispersion" consists only in a more *explicit*

[29]) From which they can only be reobtained by means of an unrestricted existence-statement, e.g.

$$_{|}x < y_{|} \overset{\mathrm{df}}{\equiv} {}_{|}\exists_{\xi}\, {}_{|}\mathrm{N}\,[\xi] \wedge {}_{|}x \overset{\xi}{<} y_{\|\|}$$

[30]) Or bounded, which comes to the same.

statement of theorems: in ordinary mathematics, if we have to *prove* a certain relation $\lim a_n = a$, we prove the existence of a certain corresponding $n(e)$, and if we *use* such a relation, we use again the existence of such an $n(e)$, so that the "fine structure" is only a less elliptic form of the theorem. But of course there are many theorems which don't admit of an affirmative form at all [31]).

It need hardly be said that the results of affirmative mathematics express theorems which are correct according to the intuitionist's standard. It also is quite obvious that the strong interpretation, like the weak one, covers a part only of intuitionistic mathematics. I shall not go into further details now, as I hope to do so on another occasion.

Ch. 4. *Some remarks on formalism.*

In every human action — in particular in "acts of intercourse" by which human beings influence each other — we can recognize and distinguish: an *emotional* element, associated with the affections of joy and grief, and an *indicative* element, associated with the sensations of recognition and distinction, though in certain actions one or the other of these elements may heavily prevail [32]).

In science — as opposed to poetry, mysticism, etc. — and in particular in mathematics, we strive to eliminate the emotional elements as far as possible. This is often done by associating with the objects under investigation certain signs or marks, i.e. other objects which can more easily be recognized individually and discerned from each other. This process is called *formalization* of science, and often is very useful, as recognition and distinction of the objects may be possible or easy *mediately* by means of the signs, if it were difficult or even impossible without them.

This stripping off of emotional elements, however, never can be done consequently, as long as we have to do with human beings: a man cannot be separated from his emotions, and every little child knows to "read between the lines" before it can read at all. A consequent formalization can only be obtained by replacing the human beings by machines, carrying out the formal parts of their actions for them. This analogy between a formalism and a mechanism which I scetched in 1932 [33]) — later it was independently investigated in detail by A. M. TURING [1] — rests upon the fact that a machine has properties comparable with the possibility of recognition and distinction, but not or hardly such as are comparable with joy, grief, love, hate, rage, etc.

The principal difference between the Dutch "group of Significists" [34]),

[31]) Cf. e.g. D. VAN DANTZIG [3], [4].

[32]) G. MANNOURY, [2], [5] p. 292.

[33]) D. VAN DANTZIG [1].

[34]) The leading members of the group were L. E. J. BROUWER, G. MANNOURY and the late F. VAN EEDEN, J. VAN GINNEKEN, and J. I. DE HAAN. The group worked mainly

and some other groups of scientists (e.g. the "logical empirists") lies in the fact that the latter consider science etc. as a system of words or symbols, and the former as a kind of human activity.

The main disadvantage of the standpoint of these latter groups lies in their inconsequence. Their attitude with regard to words and symbols sometimes implies a kind of "existential absolutism" which they otherwise always try to avoid. If science is not interested in stars and animals, but only in their observations, or rather in *descriptions* of these observations, no reason can be seen, why it should be interested in words or symbols, instead of in their observations, or rather in descriptions of these observations, or rather in descriptions of these descriptions, etc. The significists on the contrary center their attention around scientific (and other) human activities, among wihch formalization may or may not occur.

Moreover such a formalization or mechanization never is *complete*: we may always discover new regularities in the *produce* of a machine, which can be formalized with the aid of a new mechanism or formalism, but not with the old one. Therefore also TURING's "universal machine" (as he himself shows) is not universal. In 1932 I illustrated this by the example of a linotype, by which every ordinary letter or combination of letters may be printed. It can, however, not ascertain whether the rhyme-scheme of a poem is $ABBA$, though one could very well imagine a "poets-controll-machine" verifying this property.

Some modern logicians sometimes forget this very restricted range of any formal system — restricted in as far as it is formal! — Some of them go as far as defining "knowledge" as "an interpreted system", "a calculus supplemented by an interpretation". This is an almost grotesque over-straining of the (in several cases and for restricted purposes undeniable) usefulness of formalization. It is rather analogous with defining "art" as "a catalogue, supplemented by a museum", and entirely opposed to our view [35]). We do *not* consider as an ideal mathematician the man who knows by heart PEANO's "Formulaire" or RUSSELL and WHITEHEAD's "Principia", but the man who discovers *new* properties, with or without a calculus, in or outside a formal system describing the old ones.

We also do not see mathematics as a "tautologie immense" [36]). On the contrary, we don't know of the existence of any tautology at all: saying twice the same thing is not saying twice the same thing. More precisely: if two things (or actions, statements, symbols, etc.) are recognized as being the "same" in certain respects, nevertheless as being "two", i.e. discerned as being different in other respects (and be it only in space or time), it may always become desirable to fix one's attention on the difference, even if they were treated (and formalized!) as "identical"

in the jears 1917—1922. Its most active member since is G. MANNOURY, who edited the "significal dialogues" and also gave short descriptions of its history and its work [9].

[35]) D. VAN DANTZIG [5].

[36]) H. POINCARÉ [1] p. 10.

before. Therefore the paradox we wrote down above is not a contradiction: the second "same" is not the *same* "same" as the first one, at least for one who knows to "read between the lines".

Inasfar as progress of science consists of the discovery of new regularities of the *formal system*, the preceding formalization will be very useful, but it may be (even if one is willing to replace the old formalism by a new one) an impediment to the discovery of such new properties of the objects under investigation, which require finer distinctions ("fine structure") of relations hitherto regarded (and formalized!) as "identical" [37]).

It is to a large extent by such "finer distictions" and broader generalizations that progress of science proceeds, as numerous examples show. *After* they have been made, formalization may become useful again. Formalization therefore covers a small part of science only, in particular a part which to a certain extent is "ready" or "closed" at the moment, and therefore formalism is running *behind* actual science [38]).

If mathematics is not regarded as a formal tautology, no reason remains at all to claim any kind of "absoluteness" for it, either with regard to "certainty" or to "exactness". For, "recognizing A as B" may be called by other investigators or in a later stage of development: "not seeing the important difference between A and B", and "clearly distinguishing between A and B" is the same as "not yet having discovered the hidden resemblance between A and B". And, whether the processes involved are called "mental" ("splitting up a moment of life ...") or "physical" ("writing down a dash"), or both, and whether the mental processes are said to "accompany" the physical ones or vice-versa, is entirely irrelevant — if the distinction has any meaning at all! —. In any case, both formalists and intuitionists try to reduce mathematics to a system of actions which can be split up into a finite number of elements ("elementary steps"), for which the only relevant condition is that they can, practically almost without ambiguity, be recognized individually and distinguished from each other.

I have dwelled somewhat longer upon these generalities because I thereby hope to bridge the gulf between classical mathematics and logical empirism at one side and the apparently so distant shore of intuitionism and significs.

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[37]) The first case is implied in the second one, viz insofar as the observed objects are the signs of a calculus.

[38]) This, of course, does not deny the fact that, on the other hand, by the greater surveyability and the closer scrutinizing it allows, formalization often works as a usefull heuristic method for finding finer distinctions and new analogies.

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