Mathematics. — *Semantical considerations on intuitionistic mathematics* [1]).
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A. TARSKI, in a series of papers [2]), has introduced a new method, which has proved of great value to axiomatics. This paper is intended to point out the great importance of this so-called semantical method for the elucidation of the foundations of intuitionistic mathematics.

The semantical method may be considered as a synthesis of the mathematical and the metamathematical (or proof theoretical) methods, which so far dominated axiomatics.

The mathematical method is characteristic of that study of mathematics, which does not aim at an elucidation of its foundations, but only at the enlargement of our mathematical knowledge, as well as of older axiomatics, of which HILBERT's "Grundlagen der Geometrie" constitute a classical example. It is directed towards the mathematical entities — numbers, progressions, functions, sets — themselves and towards the properties and mutual relations of these entities; consequently it coincides with BROUWER's 'first order mathematics' [3]). In order to render its results communicable, one has to use names for the mathematical entities and for their properties and relations; the total of these names will be called mathematical terminology.

The metamathematical method — BROUWER's 'second order mathemathics', HILBERT's 'Metamathematik' or proof theory — is directed, not towards the mathematical entities themselves or towards their properties and relations, but towards the elements of mathematical terminology and towards the statements, definitions and proofs, obtained by combining these elements. In order to formulate its results, one has to make use, not of the names of mathematical entities and of their properties and relations, but of names for the elements of mathematical terminology; the total of these names will be called metamathematical terminology. It was CARNAP [4]), who stressed the necessity of a sharp distinction between mathematical

[1]) This paper was written in July 1945, immediately after the liberation of our country; for external reasons, it could not be published earlier. I have preferred to publish it now in its original form, notwithstanding the publication, in the meantime, of S. C. KLEENE, "On the interpretation of intuitionistic number theory", Journal of symbolic logic **10**, 1945, and of L. E. J. BROUWER, "Richtlijnen der intuitionistische wiskunde", Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. **50** (1947).

[2]) A. TARSKI, "Der Wahrheitsbegriff in den formalisierten Sprachen", Studia philosophica **1**, 1935; "Ueber den Begriff der logischen Folgerung", Actes du Congrès Int. de Philos. scient., fasc. VII, Act. Scient. et industr. **394**, Paris 1936.

[3]) L. E. J. BROUWER, "Over de grondslagen der wiskunde", Amsterdam 1907, pp. 173—175.

[4]) R. CARNAP, "Logische Syntax der Sprache", Wien 1934.

terminology ("Objektsprache") and metamathematical terminology ("Syntaxsprache"). A great number of the problems of axiomatics — including the problem of consistency — belong to the field of metamathematics.

The semantical method deals with mathematical entities and their properties and relations as well as with the entities belonging to mathemathical terminology, that is to say, with the names of mathematical entities and of their properties and relations, and with mathematical statements, definitions and proofs. In order to deal with its results, we must have the disposal of a terminology, which embraces mathematical as well as metamathematical terminology (semantical terminology).

Even in the "ordinary" study of mathematics, one is not always able to evade the application of the semantical method. A very simple illustration is found in the theorem, stating the existence of an unambiguously determined decimal representation for any natural number. The special character of this method — which HILBERT and BERNAYS [5]) called 'set theoretic method' —, however, was not clearly conceived before the publication of TARSKI's papers. To the field of semantics belong, besides the concepts of *truth* and of *meaning*, such fundamental notions of axiomatics as *model* and *categoricity* (as defined by O. VEBLEN).

It should be noted, that the concept of *logical consequence* likewise belongs to the field of semantics; whereas the proof theoretical notion of consequence does not coincide with the idea, we connect with the word 'consequence' in the 'ordinary' study of mathematics — as is shown by GÖDEL's theorem —, the following adequate definition of the 'ordinary' concept of logical consequence was given by TARSKI in semantical terms: a statement X is a locical consequence of a classe K of statements, if and only if any model of the class K is as well a model of the statement X.

In applying the semantical method, TARSKI considers mathematical entities as well as their properties and relations as extant independently of mathematical thought, as this is usual in non-intuitionistic mathematics. With regard to those entities, which belong to mathematical terminology, he adopts a more constructive point of view. This should not be considered as an inconsistency: the main intention of semantical research is an investigation into the requirements to be imposed upon mathematical terminology.

A very important result of TARSKI's researches, which is closely connected with GÖDEL's theorem — I need not dwell upon questions of priority — and with the analysis of the so-called semantical antinomies, should be mentioned here: a complete mathematical terminology is impossible; or, if stated more explicitly: if for a certain field of mathematical research a terminology T_0 is given, there are

1) notions, which according to their intention belong to this field of research, but are not capable of being defined by means of T_0;

[5]) D. HILBERT und P. BERNAYS, "Grundlagen der Mathematik", 1. Bd, Berlin 1934.

2) true statements, which according to their intention belong to the field of research, the proofs of which are, however, not capable of being formulated by means of T_0.

Queer — but important, as it constitutes a refutation of a well-known assertion by KRONECKER — is the following result: there is a statement, belonging to the theory of real numbers, which is, however, not equivalent to any statement, belonging to the theory of natural numbers.

Returning to TARSKI's results concerning the incompleteness of any mathematical terminology T_0, I observe, that T_0 is supposed to agree in general structure with usual mathematical terminologies. This explains CHURCH's attempts [6]) to construct a complete mathematical terminology of divergent structure. Apart from this possibility of a complete mathematical terminology of a more or less unusual type — which is far from proved and which seems to me hardly probable —, we must, even if we accept a cantorist or a logicistic conception of mathematical existence, consider mathematical terminology, not as extant in a finished state, but as being in a constant state of growth. In order to formulate the results of mathematical research we need a series T_0, T_1, T_2, ... of terminologies of ever growing complication. For intuitionistic mathematics this need was indicated by HEYTING [7]).

I now come to my theme, that is, to the question, how far application of the semantical method may contribute to an elucidation of the principles of intuitionistic mathematics. Giving proofs for assertions to be made is, of course, out of the question within the modest scope of this paper, which is intended only to give some directives for further work on the question; I will, however, give an application 'per analogiam' of TARSKI's results in the consideration of intuitionistic mathematics.

In the first place a remark relating to a discussion, some years ago, between H. FREUDENTHAL and A. HEYTING on the occasion of the publication of a paper by the first [8]).

The problems concerning the interpretation of logical formulae can be solved only by applying the semantical method. In agreement with current use, which is also accepted by HEYTING, I consider a mathematical statement (or formula) or definition as belonging to mathematical terminology. A proof may then be considered as a series of statements or formulae, and consequently as belonging to mathematical terminology as well. Investigation into mathematical statements (formulae, definitions) and proofs therefore belongs to the field of metamathematics.

Mathematical method, on the contrary, is to the purpose, if questions concerning mathematical entities or their properties and relations are raised.

[6]) A. CHURCH, "The Richard paradox", Am. math. monthly **41**, 1934.

[7]) A. HEYTING, "Die formalen Regeln der intuitionistischen Logik", Sitzungsber. d. Preuss. Akad. d. Wiss., Phys.-math. Kl., 1930.

[8]) H. FREUDENTHAL, "Zur intuitionistischen Deutung logischer Formeln"; A. HEYTING, "Bemerkungen zu dem Aufsatz von Herrn Freudenthal ...", Compos. math. 4, 1936.

As soon, however, as we enter into the relations between mathematical entities (and their properties and relations) and mathematical constructions on the one side, and mathematical statements (formulae, definitions) and proofs on the other, we are to combine mathematical and metamathematical method, that is to say, we must apply the semantical method.

As an example I mention the interpretation of PEANO's axioms for arithmetics, which without any essential modification may be adopted by intuitionistic arithmetics; it should be observed that, as long as we do not allow definitions by recursion, only a very slight part of arithmetics can be derived from these axioms.

We now suppose an intuitive construction of the series of natural numbers. We then must show, that this series may be considered as a model for the axiom system; this may be done by applying TARSKI's methods, upon which, however, I will not dwell here.

Then we must justify the modes of inference, as adopted in intuitionistic mathematics. These modes of inference are best characterized by a method we owe to G. GENTZEN [9]); in this way we get away from the objections, which from the point of view of intuitionism may be raised against stating general logical laws. As a matter of fact, we do not state general logical laws; we only draw up inference schemes and these are justified, not in general, but only with regard to arithmetics.

For this purpose we must show: if the series of natural numbers is a model for a set of statements $X, Y, Z, \ldots$, while the statement U is obtained from the statements $X, Y, Z, \ldots$ by applying one of the inference schemes, then the series of natural numbers is a model for the statement U. This also may be done by applying TARSKI's method.

It is true that we must be prepared for technical and perhaps even for essential difficulties in realizing this programme.

Even of classical mathematics so far only rather elementary sections — e.g., the so-called logic of classes — have been submitted to elaborate semantical analysis.

From the point of view of intuitionism, only limited value may be attributed to the analysis of such elementary formalisms. The formalism in question should contain at least a not too trivial part of arithmetics.

So much about the semantical method and its application to the elucidation of intuitionistic mathematics in general. I now pass to a more special problem, namely the definition of the notion of spread, as given by BROUWER [10]):

[9]) G. GENTZEN, "Untersuchungen über das logische Schliessen", Math. Zs. **39**, 1934.

[10]) Quoted from BROUWER's Cambridge lectures on intuitionistic mathematics, which will be published by the Cambridge University Press. Originally both notions, of spread law and of spread, were introduced by BROUWER under the name of "Menge" in his "Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten", 1. Teil, Kon. Akad. v. Wetensch., Amsterdam, Verhandelingen 1e Sectie, **12** (1918). Comp. also "Intuitionistische Mengenlehre", these Proceedings **23** (1920), pp.949—954. Later on, in his lectures, for the notion of spread BROUWER replaced the term "Menge" by "Mengenspezies".

"A *spread law* is an instruction, according to which, if again and again an arbitrary natural number is chosen as "index", each of those choices has as its predeterminate effect, depending also on the preceding choices, that either a certain "figure" (viz. either no thing or a mathematical entity) is generated, or that the choice is "sterilized", in which case the figures generated are destroyed and generation of any further figures is prevented, so that all following choices are sterilized likewise. The only condition to be satisfied is that after each non-sterilized initial sequence of $n-1>0$ choices, one natural number at least is available which, if chosen as n_{th} index, generates a figure.

The infinite sequences of figures generated according to a spread law by infinitely proceeding sequences of choices are, by virtue of this genesis, together with all infinite sequences equal to one of them, the elements of a species. This species is called *a spread.*"

Such a spread law definition may be given the following symbolical form:

$$\left.\begin{array}{ll} \alpha_1 = M'_1 & p_1 = M_1(\tau\alpha_1) \\ \alpha_{k+1} = M'_{k+1}(\tau\alpha_1, \tau\alpha_2, \ldots \tau\alpha_k) & p_{k+1} = M_{k+1}(\tau\alpha_1, \tau\alpha_2, \ldots \tau\alpha_{k+1}) \end{array}\right\}$$

The relation $\prec$ between a spread and its elements may be defined as follows:

$$\left.\begin{array}{l} X \subset M' \underset{\text{Df}}{=} x_1 \,\varepsilon\, M'_1 \,\&\, (k)(x_{k+1} \,\varepsilon\, M'_{k+1}(x_1, x_2, \ldots x_k)) \\ P \prec M \underset{\text{Df}}{=} (Ex)[X \subset M' \,\&\, (k)(p_k = M_k(x_1, x_2, \ldots x_k)) \end{array}\right\}$$

We have applied the following notations:

X is a series, the terms x_1, x_2, ... of which are natural numbers;

P is a series, the terms p_1, p_2, ... of which are objects;

α_1, α_2, ... are classes, the elements of which are natural numbers;

$\tau\alpha$ is an element, arbitrarily chosen from α;

$M'_{k+1}(x_1, x_2, \ldots x_k)$ is a function, the argument values of which are natural numbers, whereas its function values are classes of natural numbers;

$M_{k+1}(x_1, x_2, \ldots x_{k+1})$ is a function, the argument values of which are natural numbers, whereas its function values are objects.

Any spread is determined by two progressions M' and M of functions; consequently the question arises, in which manner these progressions should be defined; it will be evident, that we should apply recursion procedures.

The adoption of the semantical point of view gives rise to two questions, namely

1. whether there is, for any formal definition, a corresponding spread in the sense of non-formalized intuitionistic mathematics;

2. whether there is, for any spread in the sense of non-formalized intuitionistic mathematics, a corresponding formal definition of the type, described before.

The first question may be answered in the affirmative without any hesitation.

In dealing with the second question, we should, however, be prepared to meet difficulties, which are analogous to the ones, underlying the negative results, obtained by TARSKI, GÖDEL, CHURCH and SKOLEM. If, in the definition of the progressions M' and M, we stick to recursions of a definite type, only part of the spread in the sense of non-formalized intuitionistic mathematics will be capable of being defined in the formal manner described above. In this connection we should ask, whether from the intuitionistic point of view only recursions of a certain definite type are to be admitted; in my opinion, we should rather admit an indefinite range of types of recursion.