Mathematics. — *Extension of* PEARSON'*s Probability Distributions to two Variables.* II. By M. J. VAN UVEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 25, 1947.)

§ 2. Classification of the probability density functions.

The nature of the solution of eq. (1) depends mainly on the structure of the denominators **G** and **H**, especially on their having common factors or not.

So the density functions φ will be classified according to the degree and the mutual divisibility of **G** and **H**. Thus we can distinguish:

I. **G** and **H** have *no* common factors, with the subdivision:

a) Both **G** and **H** indecomposable; b) **G** indecomposable, $\mathbf{H}=\mathbf{BC}$ (resp. $\mathbf{G}=\mathbf{AC}$, **H** indecomp.); c) **G** indecomp., $\mathbf{H}=\mathbf{B}^2$ (resp. $\mathbf{G}=\mathbf{A}^2$, **H** indecomp.); d) $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=\mathbf{BD}$; e) $\mathbf{G}=\mathbf{A}^2$, $\mathbf{H}=\mathbf{BD}$ (resp. $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=\mathbf{B}^2$); f) $\mathbf{G}=\mathbf{A}^2$, $\mathbf{H}=\mathbf{B}^2$; g) **G** indecomp., $\mathbf{H}=\mathbf{B}$ (resp., $\mathbf{G}=\mathbf{A}$, **H** indecomp.); h) $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=\mathbf{B}$ (resp. $\mathbf{G}=\mathbf{A}$, $\mathbf{H}=\mathbf{BC}$); i) $\mathbf{G}=\mathbf{A}^2$, $\mathbf{H}=\mathbf{B}$ (resp. $\mathbf{G}=\mathbf{A}$, $\mathbf{H}=\mathbf{B}^2$); j) $\mathbf{G}=\mathbf{A}$, $\mathbf{H}=\mathbf{B}$; k) **G** indecomp., $\mathbf{H}=1$ (resp. $\mathbf{G}=1$, **H** indecomp.); l) $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=1$ (resp. $\mathbf{G}=1$, $\mathbf{H}=\mathbf{BC}$); m) $\mathbf{G}=\mathbf{A}^2$, $\mathbf{H}=1$ (resp. $\mathbf{G}=1$, $\mathbf{H}=\mathbf{B}^2$); n) $\mathbf{G}=\mathbf{A}$, $\mathbf{H}=1$ (resp. $\mathbf{G}=1$, $\mathbf{H}=\mathbf{B}$).

II. Both **G** and **H** quadratic with one common factor:

a) $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=\mathbf{BC}$; b) $\mathbf{G}=\mathbf{A}^2$, $\mathbf{H}=\mathbf{AB}$ (resp. $\mathbf{G}=\mathbf{AB}$, $\mathbf{H}=\mathbf{B}^2$).

III. **G** and **H** quadratic and identical:

a) $\mathbf{G}\equiv\mathbf{H}$ indecomp.; b) $\mathbf{G}\equiv\mathbf{H}=\mathbf{AC}$; c) $\mathbf{G}\equiv\mathbf{H}=\mathbf{A}^2$.

IV. **G** quadratic, **H** linear factor of **G** (resp. **H** quadratic, **G** linear factor of **H**)

a) $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=\mathbf{C}$ (resp. $\mathbf{G}=\mathbf{C}$, $\mathbf{H}=\mathbf{BC}$); b) $\mathbf{G}=\mathbf{A}^2$, $\mathbf{H}=\mathbf{A}$ (resp. $\mathbf{G}=\mathbf{B}$, $\mathbf{H}=\mathbf{B}^2$).

V. **G** and **H** linear and identical: $\mathbf{G}\equiv\mathbf{H}=\mathbf{C}$.

VI. Both **G** and **H** of degree zero: $\mathbf{G}\equiv\mathbf{H}=1$.

The above types will be submitted to condition (2):

$$\mathbf{H}\frac{\partial\mathbf{P}}{\partial y}-\mathbf{G}\frac{\partial\mathbf{Q}}{\partial x}=\frac{\mathbf{H}}{\mathbf{G}}\mathbf{P}\frac{\partial\mathbf{G}}{\partial y}-\frac{\mathbf{G}}{\mathbf{H}}\mathbf{Q}\frac{\partial\mathbf{H}}{\partial x},$$

of which the left member is a whole function. The demand that also the right member shall be whole therefore restricts the possibilities for **G**, **H**, **P**, **Q**.

In many cases it is simpler to integrate $\frac{\partial\log\varphi}{\partial x}=\frac{\mathbf{P}}{\mathbf{G}}$ directly over x and to determine the additive function of y from the shape $\frac{\mathbf{Q}}{\mathbf{H}}$ of $\frac{\partial\log\varphi}{\partial y}$.

Type I. As **G** and **H** have no common factors, condition (2) can only be satisfied by

$$\frac{\partial\mathbf{G}}{\partial y}=0\ (\text{or } \mathbf{G}=\mathbf{G}_1),\qquad \frac{\partial\mathbf{H}}{\partial x}=0\ (\text{or } \mathbf{H}=\mathbf{H}_2),$$

whence $\mathbf{H}\frac{\partial\mathbf{P}}{\partial y}-\mathbf{G}\frac{\partial\mathbf{Q}}{\partial x}=p_2\mathbf{H}-q_1\mathbf{G}=0.$

As $\frac{\mathbf{G}}{\mathbf{H}}=\frac{p_2}{q_1}=$ const. has been excluded, there remains

$$p_2=\frac{\partial\mathbf{P}}{\partial y}=0\ (\text{or } \mathbf{P}=\mathbf{P}_1),\qquad q_1=\frac{\partial\mathbf{Q}}{\partial x}=0\ (\text{or } \mathbf{Q}=\mathbf{Q}_2).$$

So we obtain

$$\frac{\partial\log\varphi}{\partial x}=\frac{\mathbf{P}_1}{\mathbf{G}_1},\qquad \frac{\partial\log\varphi}{\partial y}=\frac{\mathbf{Q}_2}{\mathbf{H}_2},$$

whence

$$\log\varphi=\int\frac{\mathbf{P}_1}{\mathbf{G}_1}dx+\int\frac{\mathbf{Q}_2}{\mathbf{H}_2}dy=\log\varphi_1(x)+\log\varphi_2(y)$$

and

$$\varphi(x,y)=\varphi_1(x)\cdot\varphi_2(y)\quad\ldots\ldots\ldots\quad(20)$$

Here x and y are mutually independent (their correlation coefficient is zero). According to the subdivision a) ... n) φ_1 and φ_2 are of various *Pearsonian* types.

Type II a) $\mathbf{G}=\mathbf{AC}$, $\mathbf{H}=\mathbf{BC}$.

Here condition (2) is reduced to

$$\mathbf{B}\left(\mathbf{C}\frac{\partial\mathbf{P}}{\partial y}-\mathbf{P}\frac{\partial\mathbf{C}}{\partial y}\right)-\mathbf{A}\left(\mathbf{C}\frac{\partial\mathbf{Q}}{\partial x}-\mathbf{Q}\frac{\partial\mathbf{C}}{\partial x}\right)=\frac{\mathbf{BC}}{\mathbf{A}}\mathbf{P}\frac{\partial\mathbf{A}}{\partial y}-\frac{\mathbf{AC}}{\mathbf{B}}\mathbf{Q}\frac{\partial\mathbf{B}}{\partial x}.$$

Since the left member is a whole function, the right member must also be whole.

Therefore:

$$\frac{\partial\mathbf{A}}{\partial y}=0,\ \text{or}\ \mathbf{A}=\mathbf{A}_1\equiv a_0+a_1x\ \text{and}\ \frac{\partial\mathbf{B}}{\partial x}=0,\ \text{or}\ \mathbf{B}=\mathbf{B}_2\equiv b_0+b_2y.$$

We now get

$$p_2\mathbf{C}-c_2\mathbf{P}\equiv\lambda\mathbf{A}_1,\qquad q_1\mathbf{C}-c_1\mathbf{Q}\equiv\lambda\mathbf{B}_2,$$

whence

$$\mathbf{P}\equiv\frac{p_2}{c_2}\mathbf{C}-\frac{\lambda}{c_2}\mathbf{A}_1,\qquad \mathbf{Q}\equiv\frac{q_1}{c_1}\mathbf{C}-\frac{\lambda}{c_1}\mathbf{B}_2.$$

So putting

$$\mu_1=\frac{p_2}{a_1 c_2},\qquad \mu_2=\frac{q_1}{b_2 c_1},\qquad \mu_3=\frac{-\lambda}{c_1 c_2},$$

$$\frac{\partial\log\varphi}{\partial x}=\frac{\mathbf{P}}{\mathbf{A}_1\mathbf{C}}=\mu_1\frac{a_1}{\mathbf{A}_1}+\mu_3\frac{c_1}{\mathbf{C}},\quad \frac{\partial\log\varphi}{\partial y}=\frac{\mathbf{Q}}{\mathbf{B}_2\mathbf{C}}=\mu_2\frac{b_2}{\mathbf{B}_2}+\mu_3\frac{c_2}{\mathbf{C}},$$

whence

$$\log\varphi=\mu_1\log\mathbf{A}_1+\mu_2\log\mathbf{B}_2+\mu_3\log\mathbf{C}+\log K_0,\ \text{or}$$

Type II a) $\mathbf{G}=\mathbf{A}_1\mathbf{C},\ \mathbf{H}=\mathbf{B}_2\mathbf{C}:\qquad \varphi=K_0\,\mathbf{A}_1^{\mu_1}\mathbf{B}_2^{\mu_2}\mathbf{C}^{\mu_3}$ . . $(20)_{\mathrm{II}a}$[1]

Type II b) $\mathbf{G}=\mathbf{A}^2,\ \mathbf{H}=\mathbf{AB}$.

For condition (2) we can now write

$$\mathbf{B}\frac{\partial\mathbf{P}}{\partial y}-\mathbf{A}\frac{\partial\mathbf{Q}}{\partial x}+\mathbf{Q}\frac{\partial\mathbf{A}}{\partial x}=2\frac{\mathbf{B}}{\mathbf{A}}\mathbf{P}\frac{\partial\mathbf{A}}{\partial y}-\frac{\mathbf{A}}{\mathbf{B}}\mathbf{Q}\frac{\partial\mathbf{B}}{\partial x}.$$

Consequently

$$\frac{\partial\mathbf{A}}{\partial y}=0,\ \text{or}\ \mathbf{A}=\mathbf{A}_1\equiv a_0+a_1x,\quad \frac{\partial\mathbf{B}}{\partial x}=0,\quad \text{or}\ \mathbf{B}=\mathbf{B}_2\equiv b_0+b_2y,$$

and $\mathbf{Q}\equiv q_0+q_1x+q_2y\equiv\frac{q_1}{a_1}\mathbf{A}_1-\frac{p_2}{a_1}\mathbf{B}_2$, or $q_0=\frac{a_0q_1-b_0p_2}{a_1},\ q_2=-\frac{b_2p_2}{a_1}$.

Writing $\mathbf{P}\equiv p_0+p_1x+p_2y$ in the form $\mathbf{P}\equiv\mu_1a_1\mathbf{A}_1+a_1\mathbf{D}_2$ (whence $p_2=a_1d_2$) and putting $\mu_2=\frac{q_1}{a_1b_2}$ (whence $\mathbf{Q}=\mu_2b_2\mathbf{A}_1-d_2\mathbf{B}_2$), we obtain

$$\frac{\partial\log\varphi}{\partial x}=\frac{\mathbf{P}}{\mathbf{A}_1{}^2}=\mu_1\frac{d\log\mathbf{A}_1}{dx}-\frac{\partial}{\partial x}\left(\frac{\mathbf{D}_2}{\mathbf{A}_1}\right),$$

$$\frac{\partial\log\varphi}{\partial y}=\frac{\mathbf{Q}}{\mathbf{A}_1\mathbf{B}_2}=\frac{\mu_2b_2}{\mathbf{B}_2}-\frac{d_2}{\mathbf{A}_1}=\mu_2\frac{d\log\mathbf{B}_2}{dy}-\frac{\partial}{\partial y}\left(\frac{\mathbf{D}_2}{\mathbf{A}_1}\right).$$

So

$$\log\varphi=\mu_1\log\mathbf{A}_1+\mu_2\log\mathbf{B}_2-\frac{\mathbf{D}_2}{\mathbf{A}_1}+\log K_0,\ \text{or}$$

Type II b) $\mathbf{G}=\mathbf{A}_1{}^2,\mathbf{H}=\mathbf{A}_1\mathbf{B}_2:\qquad \varphi=K_0\,\mathbf{A}_1^{\mu_1}\mathbf{B}_2^{\mu_2}e^{-\frac{\mathbf{D}_2}{\mathbf{A}_1}}$ . . $(20)_{\mathrm{II}b}$

To $\mathbf{D}_2$ any multiple of $\mathbf{A}_1$ (say $\varrho\mathbf{A}_1$) may be added (by which K_0 is multiplied by e^{ϱ}).

Type III a): $\mathbf{G}\equiv\mathbf{H}$ indecomposable.

Condition (2) now becomes

$$(p_2-q_1)\,\mathbf{G}=\mathbf{P}\frac{\partial\mathbf{G}}{\partial y}-\mathbf{Q}\frac{\partial\mathbf{G}}{\partial x}.$$

[1]) This density function has been derived by L. N. G. FILON and L. ISSERLISS, and published by K. PEARSON in his paper: Notes on Skew Frequency Surfaces. Biometrika vol. V (1923), p. 224.

1. If $p_2-q_1\neq 0$, the conic $\mathbf{G}=0$ passes through the point of intersection of $\frac{\partial\mathbf{G}}{\partial x}=0$ and $\frac{\partial\mathbf{G}}{\partial y}=0$, so through its own centre. Then, however, it degenerates into a pair of straight lines, which implies that $\mathbf{G}$ is decomposable, contrary to the assumption. The only admissible solution, therefore, is:

2. $p_2=q_1$, and $\dfrac{\mathbf{P}}{\frac{\partial\mathbf{G}}{\partial x}}=\dfrac{\mathbf{Q}}{\frac{\partial\mathbf{G}}{\partial y}}=\mu$ (const.), or $\mathbf{P}=\mu\frac{\partial\mathbf{G}}{\partial x},\ \mathbf{Q}=\mu\frac{\partial\mathbf{G}}{\partial y}$;

this gives $\qquad\log\varphi=\mu\log\mathbf{G}+\log K_0,\quad$ or

Type III a): $\mathbf{G}\equiv\mathbf{H}$ indecomposable:

$$\varphi=K_0\,\mathbf{G}^{\mu}.\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\qquad(20_{\mathrm{III}a}\ {}^2)$$

This type can be subdivided into two classes (see § 3).

Type III b α): $\mathbf{G}\equiv\mathbf{H}=\mathbf{AC}$ ($\mathbf{A}$ and $\mathbf{C}$ real).

Here condition (2) leads to

$$p_2-q_1=\frac{a_2\mathbf{P}-a_1\mathbf{Q}}{\mathbf{A}}+\frac{c_2\mathbf{P}-c_1\mathbf{Q}}{\mathbf{C}},$$

whence

$$a_2\mathbf{P}-a_1\mathbf{Q}=\varrho\mathbf{A},\qquad c_2\mathbf{P}-c_1\mathbf{Q}=\sigma\mathbf{C}\qquad(\varrho+\sigma=p_2-q_1).$$

Putting

$$\mu_1=\frac{\sigma}{a_1c_2-a_2c_1},\qquad \mu_3=\frac{-\varrho}{a_1c_2-a_2c_1},$$

we find

$$\mathbf{P}=\mu_1a_1\mathbf{C}+\mu_3c_1\mathbf{A},\quad \mathbf{Q}=\mu_1a_2\mathbf{C}+\mu_3c_2\mathbf{A}.$$

So

$$\frac{\partial\log\varphi}{\partial x}=\frac{\mathbf{P}}{\mathbf{AC}}=\mu_1\frac{a_1}{\mathbf{A}}+\mu_3\frac{c_1}{\mathbf{C}},\qquad \frac{\partial\log\varphi}{\partial y}=\frac{\mathbf{Q}}{\mathbf{AC}}=\mu_1\frac{a_2}{\mathbf{A}}+\mu_3\frac{c_2}{\mathbf{C}},$$

$$\log\varphi=\mu_1\log\mathbf{A}+\mu_3\log\mathbf{C}+\log K_0.$$

Hence

Type III b α)

$$\mathbf{G}\equiv\mathbf{H}=\mathbf{AC}:\qquad \varphi=\mathbf{K}_0\mathbf{A}^{\mu_1}\mathbf{C}^{\mu_3}.\ \ .\ \ .\ \ .\qquad(20)_{\mathrm{III}b}$$

Type III b β): $\mathbf{G}\equiv\mathbf{H}=\mathbf{AC}$ ($\mathbf{A}$ and $\mathbf{C}$ complex).

Starting from $\mathbf{G}\equiv g_{00}+2g_{01}x+2g_{02}y+g_{11}x^2+2g_{12}xy+g_{22}y^2$, we put

$$\triangle\equiv\begin{vmatrix}g_{00}, & g_{01}, & g_{02}\\ g_{10}, & g_{11}, & g_{12}\\ g_{20}, & g_{21}, & g_{22}\end{vmatrix},\quad \triangle_{ij}=\frac{\partial\triangle}{\partial g_{ij}},\quad \mathbf{J}_2\equiv-\triangle_{00}y^2+2\triangle_{02}y-\triangle_{22}.\ (21)$$

[2]) This type has been studied by K. PEARSON: On Non-skew Frequency Surfaces, Biometrika vol. V (1923) p. 231.

Since $\mathbf{G} = \mathbf{AC}$, we have

$\triangle = 0$, so $g_{11}\triangle = \triangle_{00}\triangle_{22} - \triangle_{02}{}^2 = 0$, and $J_2 = -\triangle_{00}\left(y - \frac{\triangle_{02}}{\triangle_{00}}\right)^2$.

For $\mathbf{G}$ we can write $\mathbf{G} = g_{11}(x-x_1)(x-x_2)$, where

$$x_1 = \frac{1}{g_{11}}\{-(g_{01}+g_{12}y) + \sqrt{J_2}\}, \quad x_2 = \frac{1}{g_{11}}\{-(g_{01}+g_{12}y) - \sqrt{J_2}\}.$$

$\mathbf{A}$ and $\mathbf{C}$ being complex, J_2 is negative, so $\triangle_{00} > 0$. Putting $J_2 = -I_2{}^2$,

we have $\quad I_2 = \sqrt{\triangle_{00}} \cdot \left(y - \frac{\triangle_{02}}{\triangle_{00}}\right)$ and $g_{11}(x_1 - x_2) = 2 i I_2$.

Thus

$$g_{11}(x-x_1) = (g_{01} + g_{11}x + g_{12}y) - iI_2 = \varrho e^{-i\theta},$$

$$g_{11}(x-x_2) = (g_{01} + g_{11}x + g_{12}y) + iI_2 = \varrho e^{+i\theta},$$

$$\frac{x-x_1}{x-x_2} = e^{-2i\theta}, \qquad \cot\theta = \frac{g_{01}+g_{11}x+g_{12}y}{I_2}.$$

Integrating

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{G}} = \frac{p_0 + p_1 x + p_2 y}{g_{11}(x-x_1)(x-x_2)} \text{ over } x, \text{ we find}$$

$$\log\varphi = \frac{p_1}{2g_{11}}\log \mathbf{G} + \frac{(g_{11}p_0 - g_{01}p_1) - (g_{12}p_1 - g_{11}p_2)y}{2ig_{11}\sqrt{\triangle_{00}}\cdot\left(y - \frac{\triangle_{02}}{\triangle_{00}}\right)} \cdot \log\left(\frac{x-x_1}{x-x_2}\right) + f(y).$$

As $\frac{\partial \log\varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{H}} = \frac{\mathbf{Q}}{\mathbf{G}}$ must be algebraic, the factor of $\log\left(\frac{x-x_1}{x-x_2}\right)$ must be a constant, hence also

$$\omega(y) \equiv \frac{(g_{11}p_0 - g_{01}p_1) - (g_{12}p_1 - g_{11}p_2)y}{y - \frac{\triangle_{02}}{\triangle_{00}}} = \text{constant, namely}$$

$$\omega = -(g_{12}p_1 - g_{11}p_2).$$

So we arrive at

$$\log\varphi = \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{i(g_{12}p_1 - g_{11}p_2)}{2g_{11}\sqrt{\triangle_{00}}}\log\left(\frac{x-x_1}{x-x_2}\right) + f(y),$$

and

$$\frac{\partial\log\varphi}{\partial y} = \frac{p_1}{2g_{11}}\cdot\frac{\frac{\partial \mathbf{G}}{\partial y}}{\mathbf{G}} + \frac{i(g_{12}p_1-g_{11}p_2)}{2g_{11}\sqrt{\triangle_{00}}}\cdot\frac{\partial\log\left(\frac{x-x_1}{x-x_2}\right)}{\partial y} + f'(y).$$

Since

$$\frac{\partial\log\left(\frac{x-x_1}{x-x_2}\right)}{\partial y} = \frac{\frac{\partial\{g_{11}(x-x_1)\}}{\partial y}}{g_{11}(x-x_1)} - \frac{\frac{\partial\{g_{11}(x-x_2)\}}{\partial y}}{g_{11}(x-x_2)} =$$

$$= \frac{g_{12} - i\sqrt{\triangle_{00}}}{g_{11}(x-x_1)} - \frac{g_{12}+i\sqrt{\triangle_{00}}}{g_{11}(x-x_2)} = \frac{(g_{12}-i\sqrt{\triangle_{00}})(x-x_2) - (g_{12}+i\sqrt{\triangle_{00}})(x-x_1)}{\mathbf{G}},$$

$f'(y)$ is zero. Hence

$$\log\varphi = \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{i(g_{12}p_1 - g_{11}p_2)}{2g_{11}\sqrt{\triangle_{00}}} \times -2i\theta + \log K_0 =$$

$$= \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{g_{12}p_1 - g_{11}p_2}{g_{11}\sqrt{\triangle_{00}}}\operatorname{arc\,cot}\frac{g_{01}+g_{11}x+g_{12}y}{I_2} + \log K_0 =$$

$$= \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{g_{11}p_2 - g_{12}p_1}{g_{11}\sqrt{\triangle_{00}}}\arctan\frac{g_{01}+g_{11}x+g_{12}y}{I_2} + \log K_0',$$

or, putting

$$\mu = \frac{p_1}{2g_{11}}, \qquad \lambda = \frac{g_{11}p_2 - g_{12}p_1}{g_{11}\sqrt{\triangle_{00}}},$$

$$\log\varphi = \mu\log\mathbf{G} + \lambda\arctan\frac{g_{01}+g_{11}x+g_{12}y}{\sqrt{\triangle_{00}}\cdot\left(y-\frac{\triangle_{02}}{\triangle_{00}}\right)} + \log K_0',$$

whence

Type III b β) $\qquad \mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$ ($\mathbf{A}$ and $\mathbf{C}$ complex):

$$\varphi = K_0'\,\mathbf{G}^{\mu}\,e^{\lambda\arctan\frac{g_{01}+g_{11}x+g_{12}y}{\sqrt{\triangle_{00}}\cdot\left(y-\frac{\triangle_{02}}{\triangle_{00}}\right)}} \quad . \quad . \quad . \quad . \quad (20)_{\mathrm{III\,b}\,\beta}$$

Type III c) $\qquad \mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$.

Here condition (2) furnishes $\quad p_2 - q_1 = \frac{2(a_2\mathbf{P} - a_1\mathbf{Q})}{\mathbf{A}}$.

So

$$\mathbf{Q} \equiv q_0 + q_1 x + q_2 y = \frac{2a_2\mathbf{P} - (p_2 - q_1)\mathbf{A}}{2a_1}, \text{ whence } q_1 = \frac{a_2 p_1}{a_1} - \frac{p_2 - q_1}{a_1},$$

or $\qquad q_1 = \frac{2a_2p_1}{a_1} - p_2, \quad p_2 - q_1 = \frac{2a_2p_1}{a_1} - 2q_1.$

Putting for $\mathbf{P} \equiv p_0 + p_1x + p_2y$ $\;\mathbf{P} = \varrho\mathbf{A} + a_1\mathbf{D}_2$, whence $p_1 = \varrho a_1$, $p_2 = \varrho a_2 + a_1 d_2$, and

$$q_1 = 2\varrho a_2 - (\varrho a_2 + a_1 d_2) = \varrho a_2 - a_1 d_2, \quad p_2 - q_1 = 2a_1 d_2,$$

we get $\qquad \mathbf{Q} = \frac{a_2}{a_1}\varrho\mathbf{A} + a_2\mathbf{D}_2 - d_2\mathbf{A},$

and with $\mu = \frac{\varrho}{a_1} = \frac{p_1}{a_1{}^2}$,

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}^2} = \frac{\mu a_1}{\mathbf{A}} + \frac{a_1 \mathbf{D}_2}{\mathbf{A}^2}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{A}^2} = \frac{\mu a_2}{\mathbf{A}} + \frac{a_2 \mathbf{D}_2}{\mathbf{A}^2} - \frac{d_1}{\mathbf{A}},$$

whence

$$\log \varphi = \mu \log \mathbf{A} - \frac{\mathbf{D}_2}{\mathbf{A}} + \log K_0.$$

Therefore

Type III c) $\quad \mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$: $\quad \varphi = K_0 \mathbf{A}^\mu e^{-\frac{\mathbf{D}_2}{\mathbf{A}}}$ . . . . . . $(20)_{\mathrm{III}c}$

To $\mathbf{D}_2$ any multiple of $\mathbf{A}$ may be added.

Type IV a) $\quad \mathbf{G} = \mathbf{AC}, \mathbf{H} = \mathbf{C}$.

Condition (2) becomes

$$\mathbf{C}\frac{\partial \mathbf{P}}{\partial y} - \mathbf{AC}\frac{\partial \mathbf{Q}}{\partial x} = \frac{\mathbf{PC}}{\mathbf{A}}\frac{\partial \mathbf{A}}{\partial y} + \mathbf{P}\frac{\partial \mathbf{C}}{\partial y} - \mathbf{AQ}\frac{\partial \mathbf{C}}{\partial x},$$

whence $\quad \frac{\partial \mathbf{A}}{\partial y} = 0, \quad$ or $\quad \mathbf{A} = \mathbf{A}_1 \equiv a_0 + a_1 x$.

So (2) leads to $\quad p_2\mathbf{C} - c_2\mathbf{P} = \mathbf{A}_1(q_1\mathbf{C} - c_1\mathbf{Q})$.

Consequently $\quad q_1\mathbf{C} - c_1\mathbf{Q} = \varrho$ (const.), or

$$\mathbf{Q} = \frac{q_1}{c_1}\mathbf{C} - \frac{\varrho}{c_1} \quad \text{and} \quad \mathbf{P} = \frac{p_2}{c_2}\mathbf{C} - \frac{\varrho}{c_2}\mathbf{A}_1.$$

This gives

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}_1\mathbf{C}} = \frac{p_2}{c_2}\cdot\frac{1}{\mathbf{A}_1} - \frac{\varrho}{c_2}\cdot\frac{1}{\mathbf{C}}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{C}} = \frac{q_1}{c_1} - \frac{\varrho}{c_1}\cdot\frac{1}{\mathbf{C}}.$$

So

$$\log \varphi = \frac{p_2}{a_1 c_2}\log \mathbf{A}_1 - \frac{\varrho}{c_1 c_2}\log \mathbf{C} + \mathrm{f}(y), \ \frac{\partial \log \varphi}{\partial y} = -\frac{\varrho}{c_1}\frac{1}{\mathbf{C}} + \mathrm{f}'(y),$$

whence $\quad \mathrm{f}'(y) = \frac{q_1}{a_1} = -\lambda_2, \ \mathrm{f}(y) = -\lambda_2 y + \log K_0$

and, with $\quad \mu_1 = \frac{p_2}{a_1 c_2}, \quad \mu_3 = -\frac{\varrho}{c_1 c_2}$,

$$\log \varphi = \mu_1 \log \mathbf{A}_1 + \mu_3 \log \mathbf{C} - \lambda_2 y + \log K_0.$$

Therefore

Type IV a) $\quad \mathbf{G} = \mathbf{A}_1\mathbf{C}, \mathbf{H} = \mathbf{C}$: $\quad \varphi = K_0 \mathbf{A}_1{}^{\mu_1} \mathbf{C}^{\mu_3} e^{-\lambda_2 y}$. . . $(20)_{\mathrm{IV}a}$

Type IV b) $\quad \mathbf{G} = \mathbf{A}^2, \mathbf{H} = \mathbf{A}$.

Here condition (2) is reduced to

$$\frac{\partial \mathbf{P}}{\partial y} - \mathbf{A}\frac{\partial \mathbf{Q}}{\partial x} = 2\frac{\mathbf{P}}{\mathbf{A}}\cdot\frac{\partial \mathbf{A}}{\partial y} - \mathbf{Q}\frac{\partial \mathbf{A}}{\partial x}.$$

So $\frac{\partial \mathbf{A}}{\partial y} = 0$, or $\mathbf{A} = \mathbf{A}_1 \equiv a_0 + a_1 x$, and $p_2 - q_1\mathbf{A}_1 = -a_1\mathbf{Q}$, whence $\frac{\partial \mathbf{Q}}{\partial y} = 0$, or $\mathbf{Q} = \mathbf{Q}_1 \equiv q_0 + q_1 x$, and $p_2 - q_1(a_0 + a_1 x) = -a_1(q_0 + q_1 x)$, or $p_2 - q_1 a_0 + a_1 q_0 = 0$.

Writing $\mathbf{P} \equiv p_0 + p_1 x + p_2 y$ in the form

$$\mathbf{P} = \varrho\mathbf{A}_1 + a_1\mathbf{D}_2,$$

we get $p_0 = \varrho a_0 + a_1 d_0$, $p_1 = \varrho a_1$, $p_2 = a_1 d_2$.

Further

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}_1{}^2} = \frac{\varrho}{\mathbf{A}_1} + a_1\frac{\mathbf{D}_2}{\mathbf{A}_1{}^2}, \ \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}_1}{\mathbf{A}_1} = \frac{-p_2 + q_1\mathbf{A}_1}{a_1\mathbf{A}_1} = -\frac{d_2}{\mathbf{A}_1} + \frac{q_1}{a_1}.$$

Hence

$$\log \varphi = \frac{\varrho}{a_1}\log \mathbf{A}_1 - \frac{\mathbf{D}_2}{\mathbf{A}_1} + \mathrm{f}(y), \ \frac{\partial \log \varphi}{\partial y} = -\frac{d_2}{\mathbf{A}_1} + \mathrm{f}'(y),$$

so $\quad \mathrm{f}'(y) = \frac{q_1}{a_1} = -\lambda_2, \quad$ or $\quad \mathrm{f}(y) = -\lambda_2 y + \log K_0$.

Consequently, putting $\mu_1 = \frac{\varrho}{a_1}$,

$$\log \varphi = \mu_1 \log \mathbf{A}_1 - \frac{\mathbf{D}_2}{\mathbf{A}_1} - \lambda_2 y + \log K_0, \quad \text{and}$$

Type IV b) $\quad \mathbf{G} = \mathbf{A}_1{}^2, \mathbf{H} = \mathbf{A}_1$: $\quad \varphi = K_0\mathbf{A}_1{}^{\mu_1} e^{-\frac{\mathbf{D}_2}{\mathbf{A}_1} - \lambda_2 y}$. . . $(20)_{\mathrm{IV}b}$

Type V. $\quad \mathbf{G} \equiv \mathbf{H} = \mathbf{C}$.

Condition (2) becomes

$$\mathbf{C}\frac{\partial \mathbf{P}}{\partial y} - \mathbf{C}\frac{\partial \mathbf{Q}}{\partial x} = \mathbf{P}\frac{\partial \mathbf{C}}{\partial y} - \mathbf{Q}\frac{\partial \mathbf{C}}{\partial x}, \ \text{or} \ (p_2 - q_1)\mathbf{C} = c_2\mathbf{P} - c_1\mathbf{Q}.$$

Putting $\mathbf{P} = -\lambda_1\mathbf{C} + \varrho_1$, $\mathbf{Q} = -\lambda_2\mathbf{C} + \varrho_2$, we get

$(p_2 - q_1)\,\mathbf{C} = (-\lambda_1 c_2 + \lambda_2 c_1)\,\mathbf{C} + (c_2\varrho_1 - c_1\varrho_2)$, whence $\varrho_1 = \mu c_1$, $\varrho_2 = \mu c_2$.

So

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{C}} = -\lambda_1 + \mu\frac{c_1}{\mathbf{C}}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{C}} = -\lambda_2 + \mu\frac{c_2}{\mathbf{C}}.$$

This gives

$$\log \varphi = -\lambda_1 x + \mu \log \mathbf{C} + \mathrm{f}(y), \ \frac{\partial \log \varphi}{\partial y} = \frac{\mu c_2}{\mathbf{C}} + \mathrm{f}'(y), \ \text{whence } \mathrm{f}'(y) = -\lambda_2,$$

or $\quad \mathrm{f}(y) = -\lambda_2 y + \log K_0$.

Consequently

$$\log \varphi = -\lambda_1 x + \mu \log \mathbf{C} - \lambda_2 y + \log K_0,$$

or

Type V. $\mathbf{G} \equiv \mathbf{H} = \mathbf{C}$: $\quad \varphi = K_0\, \mathbf{C}^{\mu}\, e^{-\lambda_1 x - \lambda_2 y}$ . . . . $(20)_{\mathrm{V}}$

Type VI. $\mathbf{G} \equiv \mathbf{H} = 1$.

Here $\dfrac{\partial \log \varphi}{\partial x} = p_0 + p_1 x + p_2 y$, $\log \varphi = p_0 x + \tfrac{1}{2} p_1 x^2 + p_2 xy + \mathfrak{f}(y)$,

$$\frac{\partial \log \varphi}{\partial y} = p_2 x + \mathfrak{f}'(y) \equiv q_0 + q_1 x + q_2 y.$$

Consequently

$$q_1 = p_2,\ \mathfrak{f}'(y) = q_0 + q_2 y,\ \mathfrak{f}(y) = q_0 y + \tfrac{1}{2} q_2 y^2 + \log K_0,$$

and

$$\log \varphi = \log K_0 + p_0 x + q_0 y + \tfrac{1}{2} p_1 x^2 + p_2 xy + \tfrac{1}{2} q_2 y^2 =$$
$$= \log K_0' - (\psi_{00} + 2\psi_{01} x + 2\psi_{02} y + \psi_{11} x^2 + 2\psi_{12} xy + \psi_{22} y^2),$$

whence

Type VI. $\mathbf{G} \equiv \mathbf{H} = 1$:

$\varphi = K_0' e^{-\Psi}$, where $\Psi \equiv \psi_{00} + 2\psi_{01} x + 2\psi_{02} y + \psi_{11} x^2 + 2\psi_{12} xy + \psi_{22} y^2$. $(20)_{\mathrm{VI}}$

Summary of the different types:

I $\mathbf{G}$ and $\mathbf{H}$ have no common factors: $\varphi = \varphi_1(x), \varphi_2(y)$, where φ_1 and φ_2 are *Pearsonian* types.

II a) $\mathbf{G} = \mathbf{A}_1\mathbf{C}$, $\mathbf{H} = \mathbf{B}_2\mathbf{C}$, $\varphi = K_0 \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} \mathbf{C}^{\mu_3}$.

II b) $\mathbf{G} = \mathbf{A}_1^2$, $\mathbf{H} = \mathbf{A}_1\mathbf{B}_2$, $\varphi = K_0' \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} e^{-\frac{\mathbf{D}}{\mathbf{A}_1}}$,
resp. $\mathbf{G} = \mathbf{A}_1\mathbf{B}_2$, $\mathbf{H} = \mathbf{B}_2^2$, $\varphi = K_0' \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} e^{-\frac{\mathbf{D}}{\mathbf{B}_2}}$.

III a) $\mathbf{G} \equiv \mathbf{H}$ indecomposable, $\varphi = K_0 \mathbf{G}^{\mu}$.

III b α) $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$, $\mathbf{A}$ and $\mathbf{C}$ real, $\varphi = K_0 \mathbf{A}^{\mu_1} \mathbf{C}^{\mu_3}$.

III b β) $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$, $\mathbf{A}$ and $\mathbf{C}$ complex, $\varphi = K_0' \mathbf{G}^{\mu} e^{\lambda \arctan \frac{g_{10} + g_{11} x + g_{12} y}{\left(y - \frac{\Delta_{20}}{\Delta_{00}}\right)\sqrt{\Delta_{00}}}}$.

III c) $\mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$, $\varphi = K_0 \mathbf{A}^{\mu} e^{-\frac{\mathbf{D}}{\mathbf{A}}}$.

IV a) $\mathbf{G} = \mathbf{A}_1\mathbf{C}$, $\mathbf{H} = \mathbf{C}$, $\varphi = K_0\, e^{-\lambda_2 y} \mathbf{A}_1^{\mu_1} \mathbf{C}^{\mu_3}$,
resp. $\mathbf{G} = \mathbf{C}$, $\mathbf{H} = \mathbf{B}_2\mathbf{C}$, $\varphi = K_0\, e^{-\lambda_1 x} \mathbf{B}_2^{\mu_2} \mathbf{C}^{\mu_3}$.

IV b) $\mathbf{G} = \mathbf{A}_1^2$, $\mathbf{H} = \mathbf{A}_1$, $\varphi = K_0' \mathbf{A}_1^{\mu_1} e^{-\lambda_2 y - \frac{\mathbf{D}}{\mathbf{A}_1}}$,
resp. $\mathbf{G} = \mathbf{B}_2$, $\mathbf{H} = \mathbf{B}_2^2$, $\varphi = K_0' \mathbf{B}_2^{\mu_2} e^{-\lambda_1 x - \frac{\mathbf{D}}{\mathbf{B}_2}}$.

V $\mathbf{G} \equiv \mathbf{H} = \mathbf{C}$, $\varphi = K_0\, e^{-\lambda_1 x - \lambda_2 y} \mathbf{C}^{\mu}$.

VI $\mathbf{G} \equiv \mathbf{H} = 1$, $\varphi = K_0' e^{-\Psi}$,
$\Psi \equiv \psi_{00} + 2\psi_{01} x + 2\psi_{02} y + \psi_{11} x^2 + 2\psi_{12} xy + \psi_{22} y^2$.

As we shall see in the next section, only part of the functions here found have natural boundaries and so can be considered as probability functions in a proper sense.

§ 3. Standard forms for the probability density functions.

By a suitable choice of zero-point and scale we can reduce the general expressions for φ in § 2 to simple standard forms with a minimal number of parameters.

As to type I: $\varphi = \varphi_1(x) \cdot \varphi_2(y)$, we may refer to the standardization of PEARSON's types for one variable.

Type II a). $\mathbf{G} = \mathbf{A}_1\mathbf{C}$, $\mathbf{H} = \mathbf{B}_2\mathbf{C}$, $\varphi = K_0 \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} \mathbf{C}^{\mu_3}$.

Putting $\mathbf{A}_1 \equiv a_0 + a_1 x = k_1 X$, $\mathbf{B}_2 \equiv b_0 + b_2 y = k_2 Y$ $\quad (k_1 > 0, k_2 > 0)$, we introduce new co-ordinates X and Y, acting only in the first quadrant.

With the abbreviations

$$l_0 = c_0 - \frac{c_1 a_0}{a_1} - \frac{c_2 b_0}{b_2}, \quad l_1 = \frac{c_1}{a_1} k_1, \quad l_2 = \frac{c_2}{a_2} k_2 \quad . \ . \ . \ (22)$$

the form $\mathbf{C}$ passes into

$$\mathbf{L}(X, Y) \equiv l_0 + l_1 X + l_2 Y \quad . \ . \ . \ . \ . \ . \ (23)$$

So we get

$$\varphi = K_0' X^{\mu_1} Y^{\mu_2} (l_0 + l_1 X + l_2 Y)^{\mu_3}, \quad . \ . \ . \ . \ . \ (24)$$

or with

$$\alpha_1 = \mu_1 + 1, \quad \alpha_2 = \mu_2 + 1, \quad \alpha_3 = \mu_3 + 1, \quad . \ . \ . \ . \ (25)$$
$$\varphi = K_0' X^{\alpha_1 - 1} Y^{\alpha_2 - 1} (l_0 + l_1 X + l_2 Y)^{\alpha_3 - 1} \quad . \ . \ . \ (24\text{ bis})$$

For $\mathbf{R}$ and $\mathbf{S}$ we now find

$$\mathbf{R} = \alpha_1 l_0 + (\alpha_1 + \alpha_3) l_1 X + \alpha_1 l_2 Y, \quad \mathbf{S} = \alpha_2 l_0 + \alpha_2 l_1 X + (\alpha_2 + \alpha_3) l_2 Y. \quad (26)$$

Also the form $\mathbf{L}(X, Y)$ must be positive.

Provided that $l_0 \neq 0$, $l_1 \neq 0$, $l_2 \neq 0$, the lines $X = 0$, $Y = 0$ and $\mathbf{L} = 0$ form with the line I at infinity a complete fourside, of which the sides can act as bounding lines of the probability domain.

We require that on the boundary lines $\mathbf{G}\varphi$ and $\mathbf{H}\varphi$ shall be zero.

The boundary triangle can consist

α) of $X=0$, $Y=0$ and $\mathbf{L}=0$, provided that $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$;
β) ,, $X=0$, $Y=0$,, I , ,, ,, $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 < 0$;
γ) ,, $X=0$, $\mathbf{L}=0$,, I , ,, ,, $\alpha_1 > 0$, $\alpha_3 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 < 0$;
δ) ,, $Y=0$, $\mathbf{L}=0$,, I , ,, ,, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 < 0$.

Only the 4 triangles formed by the fourside can constitute the contour of the probability domain. These triangles must lie in the first quadrant.

We have moreover

$$\hat{X} = \frac{-\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \cdot \frac{l_0}{l_1}, \quad \hat{Y} = \frac{-\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} \cdot \frac{l_0}{l_2} \quad . \ . \ . \ (27)$$

and

$$\mathbf{L}(\hat{X}, \hat{Y}) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \cdot l_0 \quad . \ . \ . \ . \ . \ . \ (28)$$

We choose k_1 and k_2 in such a way, that l_0, l_1 and l_2 have the same absolute value, by which all the coefficients of $\mathbf{L}$ can be made ± 1.

Thus we obtain the following cases:

α) Contour $X=0$, $Y=0$, $\mathbf{L}=0$; $\alpha_1>0$, $\alpha_2>0$, $\alpha_3>0$; hence (see (27) and (28))

$$l_0=+1,\quad l_1=-1,\quad l_2=-1,$$

and

$$\mathbf{L}_\alpha = 1-X-Y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)_\alpha$$

β) Contour $X=0$, $Y=0, I$; $\alpha_1>0$, $\alpha_2>0$, $\alpha_1+\alpha_2+\alpha_3<0$; hence

$$l_0=+1,\quad l_1=+1,\quad l_2=+1$$

and

$$\mathbf{L}_\beta = 1+X+Y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)_\beta$$

γ) Contour $X=0$, $\mathbf{L}=0, I$; $\alpha_1>0$, $\alpha_3>0$, $\alpha_1+\alpha_2+\alpha_3<0$; hence

$$l_0=-1,\quad l_1=-1,\quad l_2=+1$$

and

$$\mathbf{L}_\gamma = -1-X+Y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)_\gamma$$

δ) Contour $Y=0$, $\mathbf{L}=0, I$; $\alpha_2>0$, $\alpha_3>0$, $\alpha_1+\alpha_2+\alpha_3<0$; hence

$$l_0=-1,\quad l_1=+1,\quad l_2=-1$$

and

$$\mathbf{L}_\delta = -1+X-Y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)_\delta$$

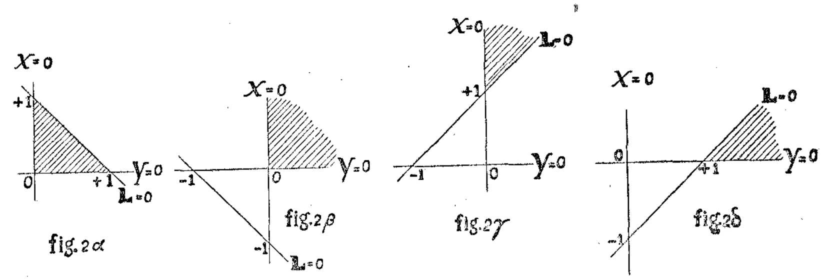

fig.2α fig.2β fig.2γ fig.2δ

The cases $l_0=0$, $l_1=0$ and $l_2=0$ lead to functions wanting artificial boundaries.

Consequently

$$\left.\begin{array}{ll}
\alpha) & \varphi = K_0'\, X^{\alpha_1-1}\, Y^{\alpha_2-1}(1-X-Y)^{\alpha_3-1}, \ \alpha_1>0,\ \alpha_2>0,\ \alpha_3>0, \\
\beta) & \varphi = K_0'\, X^{\alpha_1-1}\, Y^{\alpha_2-1}(1+X+Y)^{\alpha_3-1}, \ \alpha_1>0,\ \alpha_2>0,\ \alpha_3<-\alpha_1-\alpha_2, \\
\gamma) & \varphi = K_0''\, X^{\alpha_1-1}\, Y^{\alpha_2-1}(-1-X+Y)^{\alpha_3-1}, \ \alpha_1>0,\ \alpha_3>0,\ \alpha_2<-\alpha_1-\alpha_3, \\
\delta) & \varphi = K_0''\, X^{\alpha_1-1}\, Y^{\alpha_2-1}(-1+X-Y)^{\alpha_3-1}, \ \alpha_2>0,\ \alpha_3>0,\ \alpha_1<-\alpha_2-\alpha_3.
\end{array}\right\} (30)_{\text{II}}$$

Treating these cases separately, we get, putting

$$\alpha_1+\alpha_2+\alpha_3=\beta,$$

α)

$$\hat{X}=\frac{\alpha_1}{\beta},\quad \hat{Y}=\frac{\alpha_2}{\beta}, \quad . \; . \; . \; . \; . \; . \; . \quad (31)_{\text{II}\alpha}$$

$$K_0'=\frac{\Gamma(\beta)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)};\ \mathrm{R}=\alpha_1-(\alpha_1+\alpha_3)X-\alpha_1 Y,\ \mathrm{S}=\alpha_2-\alpha_2 X-(\alpha_2+\alpha_3)Y;$$

$$\hat{\mathrm{G}}=\frac{\alpha_1\,\alpha_3}{\beta(\beta+1)},\quad \hat{\mathrm{H}}=\frac{\alpha_2\,\alpha_3}{\beta(\beta+1)};\quad \delta\equiv r_1\,s_2-r_2\,s_1=\alpha_3\,\beta.$$

Eqq. (10′) and (10″) now give

$$\left.\begin{array}{l}
m^{2,0}=\dfrac{-s_2\hat{\mathrm{G}}}{\delta}=\dfrac{\alpha_1(\alpha_2+\alpha_3)}{\beta^2(\beta+1)},\quad m^{1,1}=\dfrac{+r_2\,\hat{\mathrm{H}}}{\delta}=\dfrac{+s_1\,\hat{\mathrm{G}}}{\delta}=\dfrac{-\alpha_1\,\alpha_2}{\beta^2(\beta+1)}, \\
\qquad\qquad m^{0,2}=\dfrac{-s_2\,\hat{\mathrm{H}}}{\delta}=\dfrac{\alpha_2(\alpha_1+\alpha_3)}{\beta^2(\beta+1)};
\end{array}\right\} (32)_{\text{II}\alpha}$$

$$\gamma=-\sqrt{\frac{\alpha_1\,\alpha_2}{(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}},\quad 1-\gamma^2=\frac{\alpha_3(\alpha_1+\alpha_2+\alpha_3)}{(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}. \quad (32')_{\text{II}\alpha}$$

The probability distribution here considered is that of the estimates of the *a priori* probabilities (round the *a posteriori* probabilities $f_1=\frac{\alpha_1}{\beta}$, $f_2=\frac{\alpha_2}{\beta}$, $f_3=1-f_1-f_2=\frac{\alpha_3}{\beta}$ as most probable values) with the trinomial distribution

$$dW(p_1,p_2)=C_0\,p_1^{\alpha_1}\,p_2^{\alpha_2}\,p_3^{\alpha_3} \qquad (p_3=1-p_1-p_2).$$

It is the extension of PEARSON's type I for both variables.

β)

$$\hat{X}=\frac{\alpha_1}{-\beta},\quad \hat{Y}=\frac{\alpha_2}{-\beta}; \quad . \; . \; . \; . \; . \; . \; . \quad (31)_{\text{II}\beta}$$

$$K_0'=\frac{\Gamma(-\alpha_3+1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(-\beta+1)};\ \mathrm{R}=\alpha_1+(\alpha_1+\alpha_3)X+\alpha_1 Y,\ \mathrm{S}=\alpha_2+\alpha_2 X+$$

$$+(\alpha_2+\alpha_3)\,Y;\ \hat{\mathrm{G}}=\frac{\alpha_1\cdot-\alpha_3}{(-\beta)(-\beta-1)},\ \hat{\mathrm{H}}=\frac{\alpha_2\cdot-\alpha_3}{(-\beta)(-\beta-1)};\ \delta=(-\alpha_3)(-\beta);$$

$$m^{2,0}=\frac{-(\alpha_2+\alpha_3)\cdot\alpha_1}{(-\beta)^2(-\beta-1)},\ m^{1,1}=\frac{+\alpha_1\,\alpha_2}{(-\beta)^2(-\beta-1)},\ m^{0,2}=\frac{-(\alpha_1+\alpha_3)\cdot\alpha_2}{(-\beta)^2(-\beta-1)}; \quad (32)_{\text{II}\beta}$$

$$\gamma=+\sqrt{\frac{\alpha_1\,\alpha_2}{(-\alpha_1-\alpha_3)(-\alpha_2-\alpha_3)}},\quad 1-\gamma^2=\frac{(-\alpha_3)(-\beta)}{(-\alpha_1-\alpha_3)(-\alpha_2-\alpha_3)}. \quad (32')_{\text{II}\beta}$$

γ)

$$\hat{X}=\frac{\alpha_1}{-\beta},\quad \hat{Y}=\frac{-\alpha_2}{-\beta}; \quad . \; . \; . \; . \; . \; . \; . \quad (31)_{\text{II}\gamma}$$

$$K_0''=\frac{\Gamma(-\alpha_2+1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_3)\,\Gamma(-\beta+1)};\ \mathrm{R}=-\alpha_1-(\alpha_1+\alpha_3)X+\alpha_1 Y,\ \mathrm{S}=-\alpha_2-\alpha_2 X+$$

$$+(\alpha_2+\alpha_3)\,Y;\ \hat{\mathrm{G}}=\frac{\alpha_1\,\alpha_3}{(-\beta)(-\beta-1)},\ \hat{\mathrm{H}}=\frac{-\alpha_2\,\alpha_3}{(-\beta)(-\beta-1)};\ \delta=\alpha_3(-\beta);$$

$$m^{2,0} = \frac{-(\alpha_2 + \alpha_3)\,\alpha_1}{(-\beta)^2(-\beta-1)}, \quad m^{1,1} = \frac{\alpha_1 \cdot (-\alpha_2)}{(-\beta)^2(-\beta-1)}, \quad m^{0,2} = \frac{-\alpha_2 \cdot (\alpha_1 + \alpha_3)}{(-\beta)^2(-\beta-1)}; \quad (32)_{\mathrm{II}a\gamma}$$

$$\gamma = +\sqrt{\frac{\alpha_1(-\beta+\alpha_1+\alpha_3)}{(\alpha_1+\alpha_3)(-\beta+\alpha_1)}}, \quad 1-\gamma^2 = \frac{\alpha_3(-\beta)}{(\alpha_1+\alpha_3)(-\beta+\alpha_1)}. \quad (32')_{\mathrm{II}a\gamma}$$

δ)

$$\hat{X} = \frac{-\alpha_1}{-\beta}, \quad \hat{Y} = \frac{\alpha_2}{-\beta}; \quad \ldots\ldots \quad (31)_{\mathrm{II}a\delta}$$

$$K_0'' = \frac{\Gamma(-\alpha_1+1)}{\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(-\beta+1)}; \quad \mathbf{R} = -\alpha_1 + (\alpha_1+\alpha_3)X - \alpha_1 Y, \quad \mathbf{S} = -\alpha_2 + \alpha_2 X -$$

$$-(\alpha_2+\alpha_3)\,Y; \quad \hat{\mathbf{G}} = \frac{-\alpha_1 \cdot \alpha_3}{(-\beta)(-\beta-1)}, \quad \hat{\mathbf{H}} = \frac{\alpha_2\,\alpha_3}{(-\beta)(-\beta-1)}; \quad \delta = \alpha_3(-\beta);$$

$$m^{2,0} = \frac{-\alpha_1 \cdot (\alpha_2+\alpha_3)}{(-\beta)^2(-\beta-1)}, \quad m^{1,1} = \frac{-\alpha_1 \cdot \alpha_2}{(-\beta)^2(-\beta-1)}, \quad m^{0,2} = \frac{-(\alpha_1+\alpha_3)\cdot\alpha_2}{(-\beta)^2(-\beta-1)}; \quad (32)_{\mathrm{II}a\delta}$$

$$\gamma = +\sqrt{\frac{\alpha_2(-\beta+\alpha_2+\alpha_3)}{(\alpha_2+\alpha_3)(-\beta+\alpha_2)}}, \quad 1-\gamma^2 = \frac{\alpha_3(-\beta)}{(\alpha_2+\alpha_3)(-\beta+\alpha_2)}. \quad (32')_{\mathrm{II}a\delta}$$

Mathematics. — *Inequalities concerning polynomials in the complex domain.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 29, 1947.)

In this paper inequality theorems for polynomials will be obtained by means of one and the same underlying method which uses theorems on the location of the roots of polynomials.

The method can be illustrated by the following proof for S. BERNSTEINS theorem [1]: "If $P(z)$ and $Q(z)$ are polynomials satisfying $|P(z)| \leq |Q(z)|$, $Q(z) \neq 0$ for any z in the upper half-plane or on the real axis, then we have $|P'(z)| \leq |Q'(z)|$ for those values of z". Proof: If λ is a complex number, $|\lambda| > 1$, then all the roots of $P(z) - \lambda Q(z)$ lie in the lower half-plane. Now, by the well-known GAUSS–LUCAS theorem, it follows that $P'(z) - \lambda Q'(z)$ has its roots in the same domain and consequently $P'(z) - \lambda Q'(z) \neq 0$ for z in the closed upper half plane. Since this is true for any λ whose modulus exceeds unity, the assertion follows.

The simple idea on which this proof is based yields some surprising results if we use some other theorems on the location of roots. In section 1 of this paper, we use the general form of the GAUSS–LUCAS theorem. In section 2 a theorem of SZEGÖ is shown to lead to a result which includes a theorem of SCHAAKE and VAN DER CORPUT and which leads to a simple proof of a conjecture of P. ERDÖS, recently proved by P. D. LAX. Section 3 is based on GRACE's Apolarity Theorem. In section 4, which stands apart from the other sections more or less, we consider an inequality of ZYGMUND for polynomials, in the special case of functions which have no roots inside the unit circle.

1. We first prove a direct generalisation of the BERNSTEIN theorem mentionned in the introduction.

Theorem 1. *Let R be a convex region in the z-plane and let B be its boundary* [2]*. Let $P(z)$ and $Q(z)$ be polynomials; suppose that the roots of $Q(z)$ belong to $R+B$, and that the degree of P does not exceed that of Q.*

Now if $|P(z)| \leq |Q(z)|$ for z on B, then we have $|P'(z)| \leq |Q'(z)|$ for z on B.

Proof. Let D denote the complement of $R+B$. Since $Q(z) \neq 0$ for $z \in D$, the inequality $|P| \leq |Q|$ for $z \in B$ implies that $|P| \leq |Q|$ for $z \in B + D$. Consequently, if $|\lambda| > 1$, the roots of $P(z) - \lambda Q(z)$ belong to R. Now, by the GAUSS–LUCAS theorem, the roots of $P'(z) - \lambda Q'(z)$ also belong to R. From this the assertion follows.

[1]) BERNSTEIN [1] p. 56. Bracketed numbers refer to the bibliography at the end.

[2]) B may contain the point $z = \infty$.