$$m^{2,0}=\frac{-(\alpha_2+\alpha_3)\,\alpha_1}{(-\beta)^2(-\beta-1)},\quad m^{1,1}=\frac{\alpha_1\cdot(-\alpha_2)}{(-\beta)^2(-\beta-1)},\quad m^{0,2}=\frac{-\alpha_2\cdot(\alpha_1+\alpha_3)}{(-\beta)^2(-\beta-1)};\qquad (32)_{\mathrm{IIa}\gamma}$$

$$\gamma=+\sqrt{\frac{\alpha_1\,(-\beta+\alpha_1+\alpha_3)}{(\alpha_1+\alpha_3)(-\beta+\alpha_1)}},\quad 1-\gamma^2=\frac{\alpha_3\,(-\beta)}{(\alpha_1+\alpha_3)(-\beta+\alpha_1)}.\qquad (32')_{\mathrm{IIa}\gamma}$$

$\delta)$

$$\hat{X}=\frac{-\alpha_1}{-\beta},\quad \hat{Y}=\frac{\alpha_2}{-\beta};\quad \ldots\ldots\quad (31)_{\mathrm{IIa}\delta}$$

$$K_0''=\frac{\Gamma(-\alpha_1+1)}{\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(-\beta+1)};\quad \mathbf{R}=-\alpha_1+(\alpha_1+\alpha_3)X-\alpha_1 Y,\quad \mathbf{S}=-\alpha_2+\alpha_2X-$$

$$-(\alpha_2+\alpha_3)\,Y;\quad \hat{\mathbf{G}}=\frac{-\alpha_1\cdot\alpha_3}{(-\beta)(-\beta-1)},\quad \hat{\mathbf{H}}=\frac{\alpha_2\,\alpha_3}{(-\beta)(-\beta-1)};\quad \delta=\alpha_3\,(-\beta);$$

$$m^{2,0}=\frac{-\alpha_1\cdot(\alpha_2+\alpha_3)}{(-\beta)^2(-\beta-1)},\quad m^{1,1}=\frac{-\alpha_1\cdot\alpha_2}{(-\beta)^2(-\beta-1)},\quad m^{0,2}=\frac{-(\alpha_1+\alpha_3)\cdot\alpha_2}{(-\beta)^2(-\beta-1)};\qquad (32)_{\mathrm{IIa}\delta}$$

$$\gamma=+\sqrt{\frac{\alpha_2\,(-\beta+\alpha_2+\alpha_3)}{(\alpha_2+\alpha_3)(-\beta+\alpha_2)}},\quad 1-\gamma^2=\frac{\alpha_3\,(-\beta)}{(\alpha_2+\alpha_3)(-\beta+\alpha_2)}.\qquad (32')_{\mathrm{IIa}\delta}$$

Mathematics. — *Inequalities concerning polynomials in the complex domain.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 29, 1947.)

In this paper inequality theorems for polynomials will be obtained by means of one and the same underlying method which uses theorems on the location of the roots of polynomials.

The method can be illustrated by the following proof for S. BERNSTEINS theorem [1]: "If $P(z)$ and $Q(z)$ are polynomials satisfying $|P(z)|\leq|Q(z)|$, $Q(z)\neq 0$ for any z in the upper half-plane or on the real axis, then we have $|P'(z)|\leq|Q'(z)|$ for those values of z". Proof: If λ is a complex number, $|\lambda|>1$, then all the roots of $P(z)-\lambda\,Q(z)$ lie in the lower half-plane. Now, by the well-known GAUSS–LUCAS theorem, it follows that $P'(z)-\lambda\,Q'(z)$ has its roots in the same domain and consequently $P'(z)-\lambda\,Q'(z)\neq 0$ for z in the closed upper half plane. Since this is true for any λ whose modulus exceeds unity, the assertion follows.

The simple idea on which this proof is based yields some surprising results if we use some other theorems on the location of roots. In section 1 of this paper, we use the general form of the GAUSS–LUCAS theorem. In section 2 a theorem of SZEGÖ is shown to lead to a result which includes a theorem of SCHAAKE and VAN DER CORPUT and which leads to a simple proof of a conjecture of P. ERDÖS, recently proved by P. D. LAX. Section 3 is based on GRACE's Apolarity Theorem. In section 4, which stands apart from the other sections more or less, we consider an inequality of ZYGMUND for polynomials, in the special case of functions which have no roots inside the unit circle.

1. We first prove a direct generalisation of the BERNSTEIN theorem mentionned in the introduction.

Theorem 1. *Let R be a convex region in the z-plane and let B be its boundary [2]. Let $P(z)$ and $Q(z)$ be polynomials; suppose that the roots of $Q(z)$ belong to $R+B$, and that the degree of P does not exceed that of Q.*

Now if $|P(z)|\leq|Q(z)|$ for z on B, then we have $|P'(z)|\leq|Q'(z)|$ for z on B.

Proof. Let D denote the complement of $R+B$. Since $Q(z)\neq 0$ for $z\in D$, the inequality $|P|\leq|Q|$ for $z\in B$ implies that $|P|\leq|Q|$ for $z\in B+D$. Consequently, if $|\lambda|>1$, the roots of $P(z)-\lambda\,Q(z)$ belong to R. Now, by the GAUSS–LUCAS theorem, the roots of $P'(z)-\lambda\,Q'(z)$ also belong to R. From this the assertion follows.

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[1]) BERNSTEIN [1] p. 56. Bracketed numbers refer to the bibliography at the end.

[2]) B may contain the point $z=\infty$.

The above result was obtained by S. BERNSTEIN for the case that B is the unit circle [3]). His proof does not depend on a direct application of the GAUSS–LUCAS theorem and does not admit the generalization obtained here.

BERNSTEIN's result is a generalization of the well-known theorem: "If $|P(z)| \leq 1$ for $|z| \leq 1$, then $|P'(z)| \leq n$ for $|z| \leq 1$". This is obtained by specializing $Q(z) = z^n$. Analogous results may be obtained for general convex domains, in virtue of Theorem 1.

Without any difficulty we can prove the following generalisation of Theorem 1:

Theorem 2. *Let R be a convex region in the z-plane, B its boundary, and S a simply connected region in the w-plane. Let $P(z)$ and $Q(z)$ be polynomials, the degree of P not exceeding that of Q, and suppose that the roots of $Q(z)$ belong to $R+B$. Now if $w = P(z)/Q(z) \in S$ for any $z \in B$, then we have $P'(z)/Q'(z) \in S$ for those values of z.*

2. By *circular domain* we denote a domain in the z-plane whose image on the z-sphere is either a closed region or an open region bounded by a circle. For instance, the point sets $|z| \geq 1$, $|z| > 1$, $|z| < 1$, $Re\, z \geq 0$ are circular domains.

We shall use the following theorem of G. SZEGÖ [4]):

Theorem 3. *If the polynomial $P(z)$ of degree n has no roots in the circular domain C [5]), and if $\xi \in C$, then we have*

$$(\xi - z)\, P'(z) + n\, P(z) \neq 0 \text{ for } z \in C. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We directly infer

Theorem 4. *Let C be a circular domain in the z-plane, and S an arbitrary point set in the w-plane. If the polynomial $P(z)$ of degree n satisfies $P(z) = w \in S$ for any $z \in C$, then we have, for any $z \in C$ and any $\xi \in C$*

$$\frac{\xi}{n}\, P'(z) + P(z) - \frac{z\, P'(z)}{n} \in S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Proof. If the number λ does not belong to S, we have $P(z) \neq \lambda$ for $z \in C$. Applying theorem 3 to the polynomial $P(z) - \lambda$ we infer that $(\xi - z)P'(z) + n\,P(z) \neq n\lambda$ for $z \in C$, $\xi \in C$ and any λ which does not belong to S. This proves (2).

We notice that a special case of theorem 4 was proved by SCHAAKE and VAN DER CORPUT [6]), who assumed that C is the unit circle (an unessential restriction) but also that S is a convex domain. A number of old and new results concerning polynomials and trigonometric polynomials were derived from this special case by these authors. We now give an application where S is not convex.

Theorem 5 (ERDÖS–LAX [7])). *If the polynomial $P(z)$ of degree n satisfies $|P(z)| \leq 1$ for $|z| \leq 1$ and if $P(z)$ has no roots in $|z| \leq 1$, then $|P'(z)| \leq \frac{1}{2} n$ for $|z| \leq 1$ [8]).*

Proof. Take for C the region $|z| < 1$ and for S the set $0 < |w| < 1$. Now (2) expresses, if $|z| < 1$, that the interior of a circle with radius $P'(z)/n$ completely belongs to S. Since the maximum radius of such a circle is $\frac{1}{2}$, the result follows.

It is however neither difficult to prove this result by the SCHAAKE and VAN DER CORPUT theorem, taking for S the region $|w| < 1$, inferring that $|P'(z)/n| + |P(z) - zP'(z)/n| \leq 1$ for $|z| \leq 1$ and noticing that from $P(z) \neq 0$ ($|z| \leq 1$) it follows that $|P'(z)/n| \geq |P(z) - zP'(z)/n|$ [9]).

3. We shall now expose some consequences of J. H. GRACE's theorem on the roots of polynomials [10]).

Theorem 6 (Grace's Apolarity Theorem). *If $n \geq 1$, and*

$$P(z) = a_0 + \binom{n}{1} a_1\, z + \binom{n}{2} a_2\, z^2 + \ldots + \binom{n}{n} a_n\, z^n, \quad . \quad . \quad (3)$$

$$Q(z) = b_0 + \binom{n}{1} b_1\, z + \binom{n}{2} b_2\, z^2 + \ldots + \binom{n}{n} b_n\, z^n, \quad . \quad . \quad (4)$$

and if $P(z)$ has no roots in a circular domain C which contains all the roots of $Q(z)$, then we have

$$\{P, Q\} = a_0\, b_n - \binom{n}{1} a_1\, b_{n-1} + \binom{n}{2} a_2\, b_{n-2} + \ldots + (-1)^n \binom{n}{n} a_0\, b_n \neq 0 \quad (5)$$

We can put this in a different form by taking $Q(z) = (z - z_1) \ldots (z - z_n)$.

Theorem 6^a. *Let $f(z_1, z_2, \ldots, z_n)$ be a linear combination of the elementary symmetric functions of $z_1, \ldots, z_n$:*

$$f(z_1, \ldots, z_n) = a_0 + a_1 \Sigma z_1 + a_2 \Sigma z_1 z_2 + \ldots + a_\mu \Sigma z_1 \ldots z_\mu + a_n z_1 \ldots z_n \quad (6)$$

so that, if $P(z)$ is given by (3):

$$f(z, z, \ldots, z) = P(z).$$

Now if $f(z, \ldots, z)$ (considered as a polynomial of degree n [11])) has no roots in the circular domain C, then for $z_1 \in C, \ldots, z_n \in C$ we have $f(z_1, \ldots, z_n) \neq 0$.

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3) BERNSTEIN [2].

4) SZEGÖ [1], p. 33.

5) We adopt the convention that $z = \infty$ is a root of $P(z)$ if the coëfficient of z^n vanishes.

6) SCHAAKE and VAN DER CORPUT [1], p. 350, Satz 20.

7) LAX [1].

8) Of course, several alterations of this theorem are possible by replacing signs $\leq$ by $<$. The same remark applies to Theorems 8, 9 and 10.

9) In LAX [1], p. 511, a similar argument is used.

10) GRACE [1]; SZEGÖ [1]; PÓLYA–SZEGÖ [2], Abschnitt V, Aufg. 145.

11) Hence $a_n = 0$ would imply that $z = \infty$ is a root.

From this we deduce, in the same way as Theorem 4 was derived from Theorem 3:

Theorem 7. *Let C be a circular domain in the z-plane and S an arbitrary point-set in the w-plane. Suppose that $f(z_1, \dots, z_n)$ is of the type (6) and satisfies $f(z, \dots, z) = w \in S$ for any $z \in C$. Then we have, for $z_1 \in C, \dots, z_n \in C$:*

$$f(z_1, \dots, z_n) \in S.$$

Since $f(z, \dots, z, \xi) = P(z) + (\xi - z) P'(z)/n$, Theorem 4 is a special case of this one.

SCHAAKE and VAN DER CORPUT's paper again contains the result of Theorem 7 for the special case that S is convex. Their proof is based on Theorem 11 below.

In the following theorem, a direct consequence of Theorem 6, it is convenient to restrict ourselves to the case that C is the unit circle. Application of Theorem 6 to $P(z) - \alpha$ and $z^n Q(-\xi/z)$ leads to

Theorem 8. *Let S be a point-set, let $P(z)$ and $Q(z)$ be given by (3) and (4), and suppose that $Q(z) \neq 0$ for $|z| < 1$, $b_0 = 1$, and*

$$P(z) \in S \text{ for } |z| \leq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now putting

$$P\,Q(z) = a_0 b_0 + \binom{n}{1} a_1 b_1 z + \dots + \binom{n}{n} a_n b_n z^n, \quad . \quad . \quad (8)$$

we have

$$P\,Q(z) \in S \text{ for } |z| \leq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Conversely, if the numbers $1 = b_0, b_1, \dots, b_n$ are such that (9) holds for any S and for any polynomial $P(z)$ satisfying (7), we have $Q(z) \neq 0$ for $|z| < 1$. This immediately follows from a theorem of SZEGÖ [12]) which covers the case that S is the set $w \neq 0$.

This remark leads to the following consideration on FÉJER sums: The FÉJER sums of the polynomial $P(z)$, viz.

$$s_k(z) = \frac{1}{k+1} \left\{ (k+1) a_0 + k \binom{n}{1} a_1 z + (k-1) \binom{n}{2} a_2 z^2 + \dots + \binom{n}{k} a_k z^k \right\},$$

have the following well-known property: "If S is a convex domain and if $P(z) \in S$ for $|z| \leq 1$, then $s_k(z) \in S$ for $|z| \leq 1$". This need not hold for general point-sets S. That depends on the location of the roots of

$$(k+1)\,Q(z) = (k+1) + k \binom{n}{1} z + (k-1) \binom{n}{2} z^2 + \dots + \binom{n}{k} z^k,$$

which may have roots inside the unit circle (e.g. if $n = 3$, $k = 1$).

The convexity of S may however be dropped if $k \geq n - 1$, for then we have $(k+1)\,Q(z) = \{(k - n + 1)z + k + 1\}\,(z+1)^{n-1}$. The case $k = n - 1$ also follows from Theorem 4 since $s_{k-1}(z) = P(z) - zP'(z)/n$.

12) SZEGÖ [1], p. 50. It is sufficient to consider $P(z) = (z - \zeta)^n$, $|\zeta| > 1$.

As an application of Theorem 8 we give

Theorem 9. *Let S be a point-set, $P(z)$ a polynomial of degree n, and suppose that $P(z) \in S$ for $|z| \leq 1$. Then for $p > 1$, $|z| \leq 1$, $|\lambda| \leq 1$ we have*

$$\left[p^n P\left(\frac{z}{p}\right) - p^{-n} P(pz) - \lambda \left\{ P(pz) - P\left(\frac{z}{p}\right) \right\} \right] \Big/ (p^n - p^{-n}) \in S. \quad (10)$$

Proof. According to Theorem 8 it is sufficient to prove that for $p > 1$, $|\lambda| \leq 1$ the polynomial

$$(z+p)^n - (z+p^{-1})^n - \lambda\{(zp+1)^n - (zp^{-1}+1)^n\} \quad . \quad . \quad (11)$$

has no roots in $|z| < 1$.

For $|z| \leq 1$, we have $|z + p| < |z + p^{-1}|$, hence $\varphi_1(z) = (z+p)^n - (z+p^{-1})^n \neq 0$ for $|z| \leq 1$. If we put $\varphi_2(z) = (zp+1)^n - (zp^{-1}+1)^n$, then we have $|\varphi_1(z)| = |\varphi_2(z)|$ for $|z| = 1$. It follows that $|\varphi_1(z)| \leq |\varphi_2(z)|$ for $|z| \leq 1$, consequently the polynomial (11), equalling $\varphi_1(z) - \lambda \varphi_2(z)$, has no roots in $|z| < 1$ if $|\lambda| \leq 1$.

The limit case $p \to 1$ leads back to theorem 4. Another special case of Theorem 9 was proved by SCHAEFFER and SZEGÖ ([1]); there S represents the region $|Re\, w| \leq 1$.

The following consequence [13]) of Theorem 8 is symmetric in P and Q.

Theorem 10. *If $P(z)$, $Q(z)$ and $PQ(z)$ are given by (3), (4) and (8), respectively, and if $|P(z)| \leq 1$, $|Q(z)| \leq 1$ for $|z| \leq 1$, then we have $|PQ(z)| \leq 1 - ||b_0| - |a_0||$ for $|z| \leq 1$.*

Proof. Let λ satisfy $|\lambda| > 1$, then $Q(z) - \lambda \neq 0$ for $|z| < 1$. On applying Theorem 8 to $P(z)$ and $(Q(z) - \lambda)/(b_0 - \lambda)$ we obtain

$$|P\,Q(z) - \lambda a_0| \leq |b_0 - \lambda| \text{ for } |z| \leq 1. \quad . \quad . \quad . \quad . \quad (12)$$

An argument of continuity shows that this holds for $|\lambda| = 1$ also. We can choose a special λ_0 with modulus 1 such that $|b_0 - \lambda_0| = 1 - |b_0|$. It follows that

$$|P\,Q(z)| \leq |\lambda_0 a_0| + 1 - |b_0| = 1 - \{|b_0| - |a_0|\}.$$

By interchanging the roles of P and Q the result follows.

GRACE's theorem also supplies a proof for the following theorem of SCHAAKE and VAN DER CORPUT [14]), which they showed to lead to Theorem 7 (for S convex).

Theorem 11. (SCHAAKE–VAN DER CORPUT). *If $f(z_1, \dots, z_n)$ is of the type (6), and if we put*

$$\lambda_n(z_1, \dots, z_n) = \frac{1}{n} \sum_{\mu=0}^{n-1} \binom{n}{\mu}^{-1} \Sigma\, z_1 z_2 \dots z_\mu,$$

13) Communicated by Mr. T. A. SPRINGER.

14) SCHAAKE and VAN DER CORPUT [1] Satz 17, p. 343 and Satz 18, p. 345.

then we have the identity

$$f(z_1,\dots,z_n)=\sum_p \lambda_n\left(\frac{z_1}{p},\dots,\frac{z_n}{p}\right) f(p,p,\dots,p), \quad \dots \quad (13)$$

where p runs through the n-th roots of $z_1 z_2 \dots z_n$.

Furthermore

$$\sum_p \lambda_n\left(\frac{z_1}{p},\dots,\frac{z_n}{p}\right)=1, \quad \dots \quad (14)$$

and if $|z_1|=|z_2|=\dots=|z_n|=1$ we have

$$\lambda_n(z_1 p^{-1},\dots,z_n p^{-1})\geq 0 \quad \dots \quad (15)$$

Proof. The relations (13) and (14) are easily verified; the difficulty lies in proving that $\lambda_n \geq 0$ if all z_i have the modulus 1. Putting $z_i = p\zeta_i$, we have to establish that

$$\lambda_n(\zeta_1,\dots,\zeta_n)\geq 0 \text{ if } |\zeta_1|=\dots=|\zeta_n|=1,\ \zeta_1\zeta_2\dots\zeta_n=1.$$

Putting $b_\mu=\Sigma\zeta_1\zeta_2\dots\zeta_\mu$ we find $\overline{b_\mu}=b_{n-\mu}$, consequently λ_n is real. It remains to be shown that λ_n cannot be negative. Taking $P(z)=z+z^2+\dots+z^{n-1}+\delta z^n$ and $Q(z)=z^m-b_1z^{m-1}+\dots=(z-\zeta_1)\dots(z-\zeta_n)$, we obtain for the expression (5):

$$\{P,Q\}=\binom{n}{1}^{-1}b_1+\binom{n}{2}^{-1}b_2+\dots+\binom{n}{n-1}^{-1}b_{n-1}+\delta b_n=$$
$$=n\,\lambda_n(\zeta_1,\dots,\zeta_n)+\delta-1.$$

Now if $\delta>1$, $P(z)$ has no roots for $|z|\geq 1$, according to a theorem of KAKEYA [15]), so that Theorem 6 yields $\{P,Q\}\neq 0$. It follows that λ_n cannot be negative.

4. In this concluding section we shall obtain an integral inequality related to an inequality of ZYGMUND (formule (18) below), generalizing the ERDÖS–LAX theorem. We deduce it from the following result which depends on SCHAAKE and VAN DER CORPUT's theorem (Theorem 11 above).

Theorem 12. *If $f(z_1,\dots,z_n)$ is of the type (6), and if $\Phi(w)$ is a real and convex function of the complex variable w, i.e.*

$$\Phi(\alpha w_1+\beta w_2)\leq\alpha\Phi(w_1)+\beta\Phi(w_2) \text{ for } \alpha\geq 0,\beta\geq 0,\alpha+\beta=1,$$

then we have, for $|z_1|\leq 1,\dots,|z_n|\leq 1$

$$\int_0^{2\pi}\Phi\{f(z_1e^{i\theta},\dots,z_ne^{i\theta})\}\,d\theta\leq\int_0^{2\pi}\Phi\{f(e^{i\theta},\dots,e^{i\theta})\}\,d\theta. \quad (16)$$

Proof. Since $f(z_1e^{i\theta},\dots,z_ne^{i\theta})$ is a linear function of z_1, the left hand side of (16) is a convex function of z_1. Consequently, its maximum for $|z_1|\leq 1$ is attained at the boundary $|z_1|=1$. The same applies to $z_2,\dots,z_n$, and hence it is sufficient to prove (16) for the case

$$|z_1|=\dots=|z_n|=1.$$

15) Cf. PÓLYA–SZEGÖ [1], Abschn. III, Aufg. 22.

By theorem 11 we then have

$$f(z_1e^{i\theta},\dots,z_ne^{i\theta})=\sum_p\lambda_n(z_1p^{-1},\dots,z_np^{-1})\,f(pe^{i\theta},\dots,pe^{i\theta}),$$

where p runs through the n-th roots of $z_1\dots z_n$, and where the λ_n satisfy (14) and (15). Since $\Phi(w)$ is convex we have

$$\Phi\{f(z_1e^{i\theta},\dots,z_ne^{i\theta}\}\leq\sum_p\lambda_n(z_1p^{-1},\dots,z_np^{-1})\,\Phi\{f(pe^{i\theta},\dots,pe^{i\theta})\}.$$

On integrating, and using (14), we obtain (16).

Now let $P(z)$ be a polynomial of degree n and let $f(z_1,\dots,z_n)$ be such that $f(z,\dots,z)=P(z)$. Take $z_1=z_2\dots=z_{n-1}=1$, $z_n=e^{i\eta}$, where η is a real number, and $\Phi(w)=|w|^p$ $(p\geq 1)$. Since $f(z,\dots,z,\xi)=P(z)+(\xi-z)P'(z)/n$, Theorem 12 gives

$$\int_0^{2\pi}|P(e^{i\theta})-e^{i\theta}P'(e^{i\theta})/n+e^{i(\eta+\theta)}P'(e^{i\theta})/n|^p\,d\theta\leq\int_0^{2\pi}|P(e^{i\theta})|^p\,d\theta.$$

Putting $P(e^{i\theta})-e^{i\theta}P'(e^{i\theta})/n=A(\theta)$, $e^{i\theta}P'(e^{i\theta})/n=B(\theta)$, we obtain

$$\int_0^{2\pi}d\theta\int_0^{2\pi}|A(\theta)+B(\theta)e^{i\eta}|^p\,d\eta\leq 2\pi\int_0^{2\pi}|P(e^{i\theta})|^p\,d\theta. \quad \dots \quad (17)$$

ZYGMUND's inequality [16])

$$\int_0^{2\pi}\left|\frac{P'(e^{i\theta})}{n}\right|^p d\theta\leq\int_0^{2\pi}|P(e^{i\theta})|^p\,d\theta \qquad (p\geq 1) \quad \dots \quad (18)$$

can be derived from (17), by the formula (valid for any real value of p)

$$\int_0^{2\pi}|a+be^{i\eta}|^p\,d\eta\geq 2\pi\ \mathrm{Max}\ \{|a|^p,|b|^p\}. \quad \dots \quad (19)$$

Owing to the symmetry with respect to a and b it is sufficient to prove (19) for $a\geq b>0$. Then it follows by

$$2\pi|\varphi(0)|^2\leq\int_0^{2\pi}|\varphi(e^{i\eta})|^2\,d\eta, \text{ where } \varphi(z)=(a+bz)^{\frac{1}{2}p}.$$

Our present aim is to investigate how ZYGMUND's result can be refined if we suppose that $P(z)$ has no roots inside the unit circle. In that case we have, by Theorem 4, $A(\theta)+\xi B(\theta)\neq 0$ for $|\xi|<1$, so that

$$|B(\theta)|\leq|A(\theta)| \qquad (0\leq\theta\leq 2\pi). \quad \dots \quad (20)$$

For $|a|\geq|b|$ we have

$$\int_0^{2\pi}|a+be^{i\eta}|\,d\eta\geq|b|^p\int_0^{2\pi}|1+e^{i\eta}|^p\,d\eta. \qquad (p\geq 0). \quad \dots \quad (21)$$

It is sufficient to prove this for $b=1$, $a>1$. In that case (21) follows from $|a+e^{i\eta}|\geq|1+e^{i\eta}|$ (η real).

16) ZYGMUND [1].

From (17), (20) and (21) we infer

$$\int_0^{2\pi} |B(\theta)|^p \, d\theta \cdot \int_0^{2\pi} |1+e^{i\eta}|^p \, d\theta \leqq 2\pi \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta,$$

and we obtain

Theorem 13. *If the polynomial* $P(z)$ *of degree* n *has no roots for* $|z|<1$, *then we have, for* $p \geqq 1$,

$$\int_0^{2\pi} \left| \frac{P'(e^{i\theta})}{n} \right|^p d\theta \leqq C_p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta$$

where $C_p = 2\pi / \int_0^{2\pi} |1+e^{i\eta}|^p \, d\eta = 2^{-p} \sqrt{\pi}\, \Gamma(\frac{1}{2}p+1)/\Gamma(\frac{1}{2}p+\frac{1}{2})$.

It is easily seen that the sign of equality holds if $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$. It can also be shown that the sign $<$ holds otherwise.

The case $p=2$ was obtained by P. D. LAX (LAX [1]), whereas $p \to \infty$ leads to the ERDÖS–LAX theorem.

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REFERENCES.

S. BERNSTEIN. I. Leçons sur les propriétés extrémales, etc., (Collection Borel) Paris 1926.
——— II. Sur la limitation des dérivées des polynomes, C. R. Acad. Sc. Paris **190**, 338—340 (1930).
J. H. GRACE. I. The zeros of a polynomial, Proc. Cambr. Phil. Soc. **11**, 352—357 (1900—'02).
P. D. LAX. I. Proof of a conjecture of P. Erdös on the derivative of a polynomial, Bull. Am. Math. Soc. **50**, 509—513 (1944).
G. PÓLYA and G. SZEGÖ. I, II. Aufgaben und Lehrsätze aus der Analysis, vol. 1 and 2, Leipzig und Berlin 1925, New York 1945.
G. SCHAAKE and J. G. VAN DER CORPUT. I. Ungleichungen für Polynome und trigonometrische Polynome. Compositio Math. **2**, 321—361 (1935); Berichtigung ibid. **3**, 128 (1936).
A. C. SCHAEFFER and G. SZEGÖ. I. Inequalities for harmonic polynomials in two and three dimensions, Trans. Am. Math. Soc., **50**, 187—225 (1941).
G. SZEGÖ. I. Bemerkungen zu einem Satz von J. H. Grace, etc., Math. Zeitschr. **13**, 28—55 (1922).
A. ZYGMUND. I. A remark on conjugate series, Proc. Lond. Math. Soc. (2) **34**, 392—400 (1932).

Mathematics. — *Sur les intégrales multiples dans les corps évalués et algébriquement-fermés.* By F. LOONSTRA. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Nous supposons que le corps T satisfasse aux conditions suivantes:

1. T est évalué non-archimédien;
2. T est algébriquement-fermé, c'est à dire que chaque polynome de l'anneau $T[x]$ se compose de facteurs linâires;
3. T est complet.

Les éléments a avec $|a| \leqq 1$ s'appellent des éléments „entiers".

Nous considérons un nombre fini quelconque de suites de polynomes

$$\begin{array}{l} g_{11}(x), g_{12}(x), \ldots, g_{1n}(x), \ldots \\ g_{21}(x), g_{22}(x), \ldots, g_{2n}(x), \ldots \\ \ldots\ldots\ldots\ldots\ldots \\ g_{s1}(x), g_{s2}(x), \ldots, g_{sn}(x), \ldots, \end{array}$$

où

$$g_{ki}(x) = x^{n_{ki}} + c_{ki,1}\, x^{n_{ki,1}} + c_{ki,2}\, x^{n_{ki,2}} + \ldots + c_{ki,\mu}\, x^{n_{ki,\mu}} + c_{ki,\mu+1}$$
$$(k = 1, 2, \ldots, s;\; n_{ki} > n_{ki,1} > \ldots > n_{ki,\mu}),$$

avec des exposants croissants et de sorte qu'aucun des polynomes considérés ne possède des racines multiples. En outre nous supposons que les coëfficients soient entiers et $|c_{ki,\mu+1}| = 1$ $(k = 1, 2, \ldots, s)$ [1]).

En vertu de 2. le polynome $g_{ki}(x)$ a n_{ki} racines

$$\alpha_{ki,1}, \alpha_{ki,2}, \ldots, \alpha_{ki,n_{ki}},$$

pour lesquelles on a

$$|\alpha_{ki,1}| = |\alpha_{ki,2}| = \ldots = |\alpha_{ki,n_{ki}}| = |c_{ki,\mu+1}|^{\frac{1}{n_{ki}}} = 1.$$

Nous supposons ensuite $|n_{ki}| = 1$ et

$$n_{ki} \to \infty,\; n_{ki,l} \to \infty \; (l = 1, 2, \ldots, \mu),\; (n_{ki} - n_{ki,1}) \to \infty \text{ pour } i \to \infty.$$

Alors on a

$$\left| \frac{n_{ki} - n_{ki,l}}{n_{ki}} \right| = \left| 1 - \frac{n_{ki,l}}{n_{ki}} \right| \leqq \max\left(1, \left| \frac{n_{ki,l}}{n_{ki}} \right|\right) = \max(1, |n_{ki,l}|) = 1,$$

parce que les $n_{ki,l}$ sont des nombres naturels, c'est à dire $|n_{ki,l}| \leqq 1$ $(l = 1, 2, \ldots, \mu)$.

En vertu de la relation $|\alpha_{ki,l}| = 1$ pour chaque k à part les racines des polynomes $g_{ki}(x)$ se trouvent sur le „cercle d'unité".

[1]) Nous supprimerons dans la suite l'adjonction $k = 1, 2, \ldots, s$.