Mathematics. — *On the figure of four projective spaces* $[n_1—1]$, $[n_2—1]$, $[n_3—1]$ *and* $[n_4—1]$ *in a* $[n—1]$, *where* $n_1 + n_2 + n_3 + n_4 = 2\,n$. II. By G. H. A. GROSHEIDE F.WZN. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of September 27, 1947.)

11. Using a common phrase we can summarize the contents of this section as follows:

(GP) : *We suppose that A, B, C and D are spaces in general position.*

An exact examination of the meaning of the expression "in general position" shows that our assumption contains the following suppositions.

If $[n_{i_1}-1]$, $[n_{i_2}-1]$, $[n_{i_3}-1]$, $[n_{i_4}-1]$ is an arbitrary permutation of the four spaces A, B, C, D for which

$$n - n_{i_1} - n_{i_2} = n_{i_3} + n_{i_4} - n \geqq 0$$

then

(GP_1) : the projective space of lowest dimension containing both $[n_{i_1}-1]$ and $[n_{i_2}—1]$ is $(n_{i_1}+n_{i_2}—1)$-dimensional.

(GP_2) : the projective space of highest dimension contained both in $[n_{i_3}—1]$ and in $[n_{i_4}—1]$ is $(n_{i_3}+n_{i_4}—n—1)$-dimensional.

(GP_3) : the $[n_{i_1}+n_{i_2}—1]$ and the $[n_{i_3}+n_{i_4}—n—1]$ introduced just now have no common points.

Hereat we remark that two spaces have a (-1)-dimensional space as intersection if they have no points in common. Besides we notice that the first supposition is equivalent with

(GP_1^*) : the projective spaces $[n_{i_1}-1]$ and $[n_{i_2}-1]$ have no common points.

The validity of (GP) can be expressed by a number of three inequalities, in keeping with the fact that the four spaces can be divided on three manners into pairs of two.

Putting $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, $i_4 = 4$ we arrive at

$$(GPA): \qquad (d^{n_4}\, c^{n-n_4})\,(c^{n_3+n_4-n}\, a^{n_1}\, b^{n_2}) \neq 0.$$

Putting $i_1 = 1$, $i_2 = 3$, $i_2 = 2$, $i_4 = 4$ we obtain

$$(GPB): \qquad (d^{n_4}\, b^{n-n_4})\,(b^{n_2+n_4-n}\, a^{n_1}\, c^{n_3}) \neq 0.$$

Finally we have

if $T = n_1 < n—n_4$ for $i_1 = 1$, $i_2 = 4$, $i_3 = 2$, $i_3 = 3$

$$(c^{n_3}\, b^{n-n_3})\,(b^{n_2+n_3-n}\, a^{n_1}\, d^{n_4}) \neq 0,$$

if $T = n—n_4 < n_1$ for $i_1 = 2$, $i_2 = 3$, $i_3 = 1$, $i_4 = 4$

$$(d^{n_4}\, a^{n-n_4})\,(a^{n_1+n_4-n}\, b^{n_2}\, c^{n_3}) \neq 0,$$

if $T = n_1 = n—n_4$ on account of (GP_1^*)

$$(d^{n_4}\, a^{n_1})\,(b^{n_2}\, c^{n_3}) \neq 0,$$

and thus irrespective the value of T

$$(GPC): \qquad I_T = (d^{n_4}\, a^T\, b^{n-n_4-T})\,(c^{n_3}\, a^{n_1-T}\, b^{n-n_1-n_3+T}) \neq 0.$$

Since all points of a line transversal of A, B, C and D belong to the $[n_{i_1}+n_{i_2}—1]$ that joins $[n_{i_1}-1]$ and $[n_{i_2}-1]$, from (GP_3) it follows that such a line (if present) has no common point with the $[n_{i_3}+n_{i_4}-n—1]$ defined by $[n_{i_3}—1]$ and $[n_{i_4}—1]$. Hence we are entitled to pronounce the

Lemma. *If l is a line tranversal of A, B, C and D, then the intersection points of l with these four spaces are mutual different.*

12. Let now be given an arbitrary point P_a

$$\alpha_1\,\{y^{(1)}\} + \alpha_2\,\{y^{(2)}\} + \ldots + \alpha_{n_1}\,\{y^{(n_1)}\} = \sum_{i=1}^{n_1} \alpha_i\,\{y^{(i)}\}$$

of A and likewise an arbitrary point P_b

$$\beta_1\,\{z^{(1)}\} + \beta_2\,\{z^{(2)}\} + \ldots + \beta_{n_2}\,\{z^{(n_2)}\} = \sum_{j=1}^{n_2} \beta_j\,\{z^{(j)}\}$$

of B (according to (GP_1^*) necessarily distinct from P_a). Then the straight line $P_a P_b$

$$\{x\} = \lambda \sum_i \alpha_i\,\{y^{(i)}\} + \mu \sum_j \beta_j\,\{z^{(j)}\}$$

joining P_a and P_b meets C in a point P_c if and only if the equations

$$\lambda \sum_i \alpha_i\,(v'_{(k)}\, y^{(i)}) + \mu \sum_j \beta_j\,(v'_{(k)}\, z^{(j)}) = 0 \qquad (k = 1, 2, \ldots, n—n_3)$$

have a common solution (λ_c, μ_c). In the same manner there exists an intersection point P_d of $P_a P_b$ with D if and only if the equations

$$\lambda \sum_i \alpha_i\,(w'_{(l)}\, y^{(i)}) + \mu \sum_j \beta_j\,(w'_{(l)}\, z^{(j)}) = 0 \qquad (l = 1, 2, \ldots, n—n_4)$$

have a common solution (λ_d, μ_d). Thus a necessary and sufficient condition for the presence of a line transversal of A, B, C and D is the resolvability of the equations

$$\left.\begin{aligned} \lambda_d \sum_i \alpha_i\,(w'_{(l)}\, y^{(i)}) + \mu_d \sum_j \beta_j\,(w'_{(l)}\, z^{(j)}) &= 0 \\ \lambda_c \sum_i \alpha_i\,(v'_{(k)}\, y^{(i)}) + \mu_c \sum_j \beta_j\,(v'_{(k)}\, z^{(j)}) &= 0 \\ (l = 1, 2, \ldots, n—n_4;\; k = 1, 2, \ldots, n—n_3)& \end{aligned}\right\} \quad \ldots \quad (9)$$

with indeterminates $\alpha_1, \alpha_2, \ldots, \alpha_{n_1};\ \beta_1, \beta_2, \ldots, \beta_{n_2};\ \lambda_c, \mu_c;\ \lambda_d, \mu_d$.

On account of the Lemma proved in the preceding section an arbitrary solution of (9) will deliver two points P_c and P_d, which neither coincide with P_a or P_b. This means that there exist no solutions of (9) with $\lambda_c, \mu_c, \lambda_d$ or $\mu_d = 0$ and so we may put $\lambda_c = \mu_c = 1$; $\lambda_d = S\,\mu_d$, where S is a new indeterminate that supersedes (λ_d, μ_d).

Now we consider

$$\left.\begin{aligned} S\sum_i \alpha_i\,(w'_{(l)}\,y^{(i)}) + \sum_j \beta_j\,(w'_{(l)}\,z^{(j)}) = 0 \\ \sum_i \alpha_i\,(v'_{(k)}\,y^{(i)}) + \sum_j \beta_j\,(v'_{(k)}\,z^{(j)}) = 0 \\ (l=1,2,\ldots,n-n_4;\; k=1,2,\ldots,n-n_3) \end{aligned}\right\} \quad . \;.\;.\;.\; (9^*)$$

as a system of $2n-n_3-n_4 = n_1+n_2$ linear homogeneous equations in the n_1+n_2 variables $(\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \beta_1, \beta_2, \ldots, \beta_{n_2})$ and observe that a solution as desired exists if and only if the determinant on the coefficients of (9*) vanishes. This imposes on S the condition

$$\begin{matrix} n-n_4\{ \\ n-n_3\{ \end{matrix}\left|\begin{array}{c:c} S\,(w'_{(l)}\,y^{(i)}) & (w'_{(l)}\,z^{(j)}) \\ \hdashline (v'_{(k)}\,y^{(i)}) & (v'_{(k)}\,z^{(j)}) \end{array}\right| = 0, \quad . \;.\;.\;.\; (10)$$

$$\qquad\qquad n_1 \qquad\qquad n_2$$

or if we introduce complex-symbols (as allowed!)

$$\begin{matrix} n-n_4\{ \\ n-n_3\{ \end{matrix}\left|\begin{array}{c:c} S\,(d'\,a) & (d'\,b) \\ \hdashline (c'\,a) & (c'\,b) \end{array}\right| = 0.$$

$$\qquad\qquad n_1 \qquad\qquad n_2$$

Simultaneous expansion with respect to the first $n-n_4$ rows gives

$$\sum_{p=0}^{T} (-1)^{(n_1-p)(n-n_4-p)} \binom{n_1}{p} \binom{n_2}{n-n_4-p} \times \Big|\underbrace{S\,(d'\,a)}_{p} \vdots \underbrace{(d'\,b)}_{n-n_4-p}\Big| \times$$

$$\times \Big|\underbrace{(c'\,a)}_{n_1-p} \vdots \underbrace{(c'\,b)}_{n-n_1-n_3+p}\Big| = \sum_{p=0}^{T} (-1)^{(n_1-p)(n-n_4-p)} \binom{n_1}{p} \binom{n_2}{n-n_4-p} \times$$

$$\times (n-n_4)!\,(n-n_3)!\,(d'\,a)^p\,(d'\,b)^{n-n_4-p}\,(c'\,a)^{n_1-p}\,(c'\,b)^{n-n_1-n_3+p}\,S^p = 0,$$

where T is the integer introduced in section 10.

We simplify the coefficients putting

$$g^{n_3 n_4}_{n_1 n_2, p} = (-1)^{(n_1-p)(n-n_4-p)} \binom{n_1}{p} \binom{n_2}{n-n_4-p}$$

and then after multiplication with $\dfrac{n_4!}{(n-n_4)!} \cdot \dfrac{n_3!}{(n-n_3)!}$ we obtain

$$\sum_{p=0}^{T} g^{n_3 n_4}_{n_1 n_2, p}\,(d^{n_4}\,a^p\,b^{n-n_4-p})\,(c^{n_3}\,a^{n_1-p}\,b^{n-n_1-n_3+p})\,S^p = 0. \quad .\;.\; (11)$$

According to (GPC) this represents a non vanishing equation in S of

degree T, the roots of which we call $S_1, S_2, \ldots, S_T$. Since from (GP_1^*) it follows that there exists no solution of (9*) different from $(0, 0, \ldots, 0)$ with $\beta_1 = \beta_2 = \ldots = \beta_{n_2} = 0$ and since (GP_3) shows that a solution with $\alpha_1 = \alpha_2 = \ldots = \alpha_{n_1} = 0$ is not present, all T roots S_i of (11) are useful for us.

13. We suppose in this and the following sections that the T roots S_i of (11) are mutual different. Then putting in (9*) $S = S_i$ we obtain a system of equations with a single solution $(\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \beta_1, \beta_2, \ldots, \beta_{n_2})_{(i}$ that furnishes two points $P_a^{(i)}$ and $P_b^{(i)}$ which inversely determine S_i. Hence with every root S_i there corresponds a single line transversal l_i and in this case A, B, C and D possess T different line transversals [7]).

As the introduction of S in section 12 shows, the value of S_i is just equal to the cross ratio $(P_d^{(i)}\,P_c^{(i)}\,P_b^{(i)}\,P_a^{(i)})$ of the four intersection points on l_i. Now the considerations that led to (11) remain valid if we replace the assumption (2) by the fainter supposition

$$n_1 + n_2 \leqq n \quad ; \quad n_2 + n_4 \geqq n.$$

Thus on account of

$$n_1 + n_3 \leqq n \quad ; \quad n_3 + n_4 \geqq n$$

after changing the spaces B and C we can follow the same way. Consequently the cross ratios

$$(P_d^{(i)}\,P_b^{(i)}\,P_c^{(i)}\,P_a^{(i)}) = 1 - S_i \qquad (i = 1, 2, \ldots, T)$$

shall be roots of the equation

$$\sum_{q=0}^{T} g^{n_2 n_4}_{n_1 n_3, q}\,(d^{n_4}\,a^q\,c^{n-n_4-q})\,(b^{n_2}a^{n_1-q}\,c^{n-n_1-n_2+q})\,x^q = 0. \quad .\;.\; (12)$$

If we substitute in (11) $S = 1 - x$ we must arrive at an equation

$$\sum_{p=0}^{T} g^{n_3 n_4}_{n_1 n_2, p}\,(d^{n_4}\,a^p\,b^{n-n_4-p})\,(c^{n_3}\,a^{n_1-p}\,b^{n-n_1-n_3+p})\,(1-x)^p =$$

$$= \sum_{p=0}^{T} \sum_{q=0}^{p} (-1)^q \binom{p}{q} g^{n_3 n_4}_{n_1 n_2, p}\,(d^{n_4}\,a^p\,b^{n-n_4-p})\,(c^{n_3}\,a^{n_1-p}\,b^{n-n_1-n_3+p})\,x^q = 0$$

with the same roots as (12). Since the coefficients of x^T

$$(-1)^{(n_1-T)(n-n_4-T)} \binom{n_1}{T} \binom{n_3}{n-n_4-T} (d^{n_4}\,a^T\,c^{n-n_4-T})\,(b^{n_2}\,a^{n_1-T}\,c^{n-n_1-n_2+T})$$

and

$$(-1)^{T+(n_1-T)(n-n_4-T)} \binom{T}{T} \binom{n_1}{T} \binom{n_2}{n-n_4-T} (d^{n_4}a^T b^{n-n_4-T})(c^{n_3}\,a^{n_1-T} b^{n-n_1-n_3+T})$$

[7]) Compare Math. Enc. III 2, 2 A, p. 815.

respectively, for $T = n - n_4 \leqq n_1$ are equal, apart from a factor

$$\varepsilon = (-1)^{n-n_4+n_2(n-n_2)+n_3(n_1+n_4-n)}$$

in this case the corresponding coefficients in both equations differ only by a factor ε. As $\sum_{p=0}^{T} \sum_{q=0}^{p} = \sum_{q=0}^{T} \sum_{p=q}^{T}$ we can deduce from this fact the identities

$$\left.\begin{aligned} & g^{n_2 n_4}_{n_1 n_3, q} (d^{n_4} a^q c^{n-n_4-q}) (b^{n_2} a^{n_1-q} c^{n-n_1-n_2+q}) \equiv \\ & \equiv \varepsilon (-1)^q \sum_{p=q}^{n-n_4} \binom{p}{q} g^{n_3 n_4}_{n_1 n_2, p} (d^{n_4} a^p b^{n-n_4-p}) (c^{n_3} a^{n_1-p} b^{n-n_1-n_3+p}) \\ & \qquad (q = 0, 1, 2, \ldots, n - n_4 \leqq n_1). \end{aligned}\right\} \quad (13)$$

Since $n_1 + n_3 \leqq n$; $n_3 + n_4 \geqq n$ by means of considerations similar with those of section 12 we obtain

$$\left.\begin{aligned} \Delta &= \frac{n_2!}{(n-n_2)!} \frac{n_3!}{(n-n_3)!} \begin{vmatrix} x(c'a) & (c'd) \\ (b'a) & (b'd) \end{vmatrix} = \\ &= \sum_{q=0}^{n_1} g^{n_2 n_3}_{n_1 n_4, q} (c^{n_3} a^q d^{n-n_3-q}) (b^{n_2} a^{n_1-q} d^{n-n_1-n_2+q}) x^q \end{aligned}\right\} \quad . \quad . \quad (14)$$

This formula is true independent from the values of n_1 and n_4 and thus among others for $T = n - n_4 < n_1$. Then, however, after the substitution $x = 1$, Δ has become a determinant of rank $\leqq n$ and thus $x = 1$ is at least a $(n_1 + n_4 - n)$-fold root of $\Delta = 0$. Therefore in this case there exist also the identities

$$\left.\begin{aligned} & \sum_{q=i}^{n_1} \binom{q}{i} g^{n_2 n_3}_{n_1 n_4, q} (c^{n_3} a^q d^{n-n_3-q}) (b^{n_2} a^{n_1-q} d^{n-n_1-n_2+q}) \equiv 0 \\ & \qquad (i = 0, 1, \ldots, n_1 + n_4 - n - 1). \end{aligned}\right\} \quad . \quad . \quad (15)$$

In the interest of an application in section 16 we notice that the results of section 12 hold irrespective the value of $n_2 + n_4$, if we replace in (11) the zero that indicates the minimum value of p by L and add

$$L = 0 \qquad \text{for} \quad n_2 + n_4 \geqq n$$
$$L = n - n_2 - n_4 \qquad \text{for} \quad n_2 + n_4 \leqq n.$$

14. If we choose a $[n_3 - 1]$ through the T points

$$\{y^{(i)}\} + \{z^{(i)}\} \qquad (i = 1, 2, \ldots, T)$$

as C and a $[n_4 - 1]$ through the points

$$\omega_i \{y^{(i)}\} + \{z^{(i)}\} \qquad (i = 1, 2, \ldots, T)$$

as D, where ω_i are arbitrary real or complex numbers, then the equation in S corresponding to the four spaces A, B, C and D has the roots ω_i. Hence it is impossible that there exists a relation between the absolute invariants that appear as coefficients in the S-equation after dividing the left side of (11) through $I_0 = (d^{n_4} b^{n-n_4}) (c^{n_3} a^{n_1} b^{n-n_1-n_3})$. Therefore an integrity basis composed of less than $T + 1$ invariants as ours cannot be present. In the general case (IV) the invariants

$$I_p = (d^{n_4} a^p b^{n-n_4-p}) (c^{n_3} a^{n_1-p} b^{n-n_1-n_3+p}) \qquad (p = 0, 1, \ldots, T)$$

form a *smallest integrity basis*. In the third case (III) I_{n_1} becomes the product of $(d^{n_4} a^{n_1})$ and $(c^{n_3} b^{n_2})$ and so a smallest integrity basis consists of $I_0, I_1, \ldots, I_{n_1-1}$ and these invariants.

In the second case (II) likewise I_0 becomes reducible and a smallest basis is composed by

$$I_1, I_2, \ldots, I_{n_1-1}, (d^{n_4} a^{n_1}), (d^{n_4} b^{n_2}), (c^{n_3} a^{n_1}), (c^{n_3} b^{n_2}).$$

In the case of four medials there appear further the invariants $(a^{n_1} b^{n_2})$ and $(c^{n_3} d^{n_4})$. After adding them to the basis one of the invariants $I_1, I_2, \ldots, I_{n_1-1}$ can be omitted [8]). This follows from (13) for $n_1 = n_2 = n_3 = n_4$; $q = 0$ or from

$$(a^{n_1} b^{n_1}) (c^{n_1} d^{n_1}) \equiv \sum_{p=0}^{n_1} (-1)^p \binom{n_1}{p}^2 I_p.$$

15. The points $P_a^{(i)}$ $(i = 1, 2, \ldots, T)$ are not contained in a $(T-2)$-dimensional subspace A' of A. For if this happens, we can choose the points $\{y^{(i)}\}$ so that the first $T-1$ of them occur in A', and then the $n_1 + n_2$ linear homogeneous equations

$$\left.\begin{aligned} & S \sum_{i=1}^{T-1} \alpha_i (w'_{(l)} y^{(i)}) + \sum_{j=1}^{n_2} \beta_j (w'_{(l)} z^{(j)}) = 0 \\ & \sum_{i=1}^{T-1} \alpha_i (v'_{(k)} y^{(i)}) + \sum_{j=1}^{n_2} \beta_j (v'_{(k)} z^{(j)}) = 0 \\ & (l = 1, 2, \ldots, n - n_4;\ k = 1, 2, \ldots, n - n_3) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

in the $T - 1 + n_2 < n_1 + n_2$ indeterminates $(\alpha_1, \alpha_2, \ldots, \alpha_{T-1}, \beta_1, \beta_2, \ldots, \beta_{n_2})$ shall become a common solution different from $(0, 0, \ldots, 0)$ through each substitution $S = S_i$ $(i = 1, 2, \ldots, T)$. This requires that the determinant on the coefficients of every number of $T - 1 + n_2$ of the equations (16) which is a polynomial in S of a degree lower than T, vanishes for T different values S_i of S, and thus is equal to zero irrespective the value of S. However for $S = 1$ at least one of these determinants differs from zero, since on account of (GP_3) the space joining A and B has no common points with the intersection of C and D. Hence there exists no $[T-2]$ containing $P_a^{(i)}$ and likewise no $[T-2]$ containing $P_b^{(i)}$ $(i = 1, 2, \ldots, T)$.

Now we pass to a new system of coordinates, the symplex of which we

[8]) TURNBULL II, p. 61; BRUINS, p. 444—445.

indicate by $Q_1 Q_2 \ldots Q_n$ and the unit point of which we call E. The n points Q_i are defined in this manner

T	points Q_i coincide with the points $P_a^{(i)}$
T	points Q_i coincide with the points $P_b^{(i)}$
$n_3 + n_4—n$	points Q_i are contained both in C and in D
$n_2 + n_4—n$	points Q_i are contained both in B and in D
$n_1—T$	points Q_i are contained both in A and in D
$n—n_4—T$	points Q_i are contained both in B and in C.

In order to determine E we agree that the projection of this point from the opposite $Q_{i_1} Q_{i_2} \ldots Q_{i_{n-2}}$ on $P_a^{(j)} P_b^{(j)}$ coincides with $P_c^{(j)}$ for each value of j ($= 1, 2, \ldots, T$). Then the coordinates of the T points $P_d^{(j)}$ and thus the equations of A, B, C and D are defined unequivocally. Nevertheless there are yet ∞^{n-T-1} points, that can perform the duties of the unit point E.

From this we deduce that there exist ∞^{n-T-1} collineations for which the figure consisting of the four spaces A, B, C and D remains invariant.

16. The considerations of the preceding section show that if the roots S_i are mutual different, their values define unequivocally the projective geometrical type of the figure. A second number of four spaces in general position $[n_1—1]^*$, $[n_2—1]^*$, $[n_3—1]^*$ and $[n_4—1]^*$ for which the roots S_i^* have the same values as the roots S_i can be carried over in the original system by a projective transformation. This is not always possible when two or more roots S_i are equal. For after the substitution $S = S_j$, where S_j is a multiple root of (11) the equations (9*) may be both once and more than once dependent. In this case, that we don't exclude further, we must consider the minors of the determinant (10). Now the minor that appears after dropping the rows containing

$$\{w'_{(l_1)}\}, \{w'_{(l_2)}\}, \ldots, \{w'_{(l_{p_0})}\};\ \{v'_{(k_1)}\}, \{v'_{(k_2)}\}, \ldots, \{v'_{(k_{q_0})}\}$$

and the columns containing

$$\{y^{(i_1)}\}, \{y^{(i_2)}\}, \ldots, \{y^{(i_{r_0})}\};\ \{z^{(j_1)}\}, \{z^{(j_2)}\}, \ldots, \{z^{(j_{s_0})}\}$$

where $p_0 + q_0 = r_0 + s_0 = N \leq T—1$, is just the S-determinant corresponding with the spaces

$$A': \quad (y^{(i_{r_0+1})} \ldots y^{(i_{n_1})}\, \pi^{n-n_1+r_0}) = 0,$$
$$B': \quad (z^{(j_{s_0+1})} \ldots z^{(j_{n_2})}\, \pi^{n-n_2+s_0}) = 0,$$
$$C': \quad (v'_{(k_{q_0+1})} \ldots v'_{(k_{n-n_3})}\, \pi'^{n_3+q_0}) = 0,$$
$$D': \quad (w'_{(l_{p_0+1})} \ldots w'_{(l_{n-n_4})}\, \pi'^{n_4+p_0}) = 0.$$

We suppose that A' is the intersection of A with the $[n—r_0—1]$: $(\alpha^{n-r_0} \pi^{r_0}) = 0$, that B' is the intersection of B with the $[n—s_0—1]$: $(\beta^{n-s_0} \pi^{s_0}) = 0$, that C' arises from joining C with the $[q_0—1]$: $(\gamma^{q_0} \pi^{n-q_0}) = 0$ and that D' arises from joining D with the $[p_0—1]$: $(\delta^{p_0} \pi^{n-p_0}) = 0$. Then the equations of our four spaces are

$$A': \quad (\alpha^{n-r_0} a^{r_0})(a^{n_1-r_0} \pi^{n-n_1+r_0}) = 0 \qquad (n_1' = n_1—r_0),$$
$$B': \quad (\beta^{n-s_0} b^{s_0})(b^{n_2-s_0} \pi^{n-n_2+s_0}) = 0 \qquad (n_2' = n_2—s_0),$$
$$C': \quad (\gamma^{q_0} c^{n_3} \pi^{n-n_3-q_0}) = 0 \qquad (n_3' = n_3+q_0),$$
$$D': \quad (\delta^{p_0} d^{n_4} \pi^{n-n_4-p_0}) = 0 \qquad (n_4' = n_4+p_0),$$

and apart from a constant number the value of the considered minor becomes

$$\left.\begin{array}{l} \sum\limits_{t=L'}^{T'} g^{n_3', n_4'}_{n_1', n_2', t} (\alpha^{n-r_0} a^{r_0})(\beta^{n-s_0} b^{s_0})(\delta^{p_0} d^{n_4} a^t b^{n-n_4-p_0-t}) \times \\ \qquad \times (\gamma^{q_0} c^{n_3} a^{n_1-t-r_0} b^{n-n_1-n_3+t+r_0-q_0}) S^t \end{array}\right\} \quad . \quad . \quad (17)$$

where
$$\begin{array}{lll} T' = n_1 - r_0 & \text{for} & p_0 - r_0 \leqq n - n_1 - n_4 \\ T' = n - n_4 - p_0 & \text{for} & p_0 - r_0 \geqq n - n_1 - n_4 \\ L' = 0 & \text{for} & s_0 - p_0 \leqq n_2 + n_4 - n \\ L' = n - n_2 - n_4 + s_0 - p_0 & \text{for} & s_0 - p_0 \geqq n_2 + n_4 - n. \end{array}$$

If S_i is a common root of the equations

$$\left.\begin{array}{l} \sum\limits_t g^{n_3+q, n_4+p}_{n_1-r, n_2-s, t} (\pi_{(1)}^{n-r} a^r)(\pi_{(2)}^{n-s} b^s)(\pi_{(3)}^{p} d^{n_4} a^t b^{n-n_4-p-t}) \times \\ \qquad \times (\pi_{(4)}^{q} c^{n_3} a^{n_1-t-r} b^{n-n_1-n_3-q+t+r}) S^t \equiv 0 \left\{\begin{array}{l} \pi_{(1)}, \pi_{(2)} \\ \pi_{(3)}, \pi_{(4)} \end{array}\right\} \\ (p + q = r + s = N; p, q, r, s = 0, 1, 2, \ldots, N \leqq T - 1) \end{array}\right\} \quad . \quad . \quad (18)$$

then after the substitution $S = S_i$ in (17) we obtain zero.

Evidently the resolvability of (18) is a sufficient condition for the existence of a root S_i that causes a diminishing of the rank of (10) to a number $< n_1 + n_2—N$. Moreover this condition is necessary, since it is always possible to choose the points $\{y^{(1)}\}$, $\{y^{(2)}\}$, ..., $\{y^{(n_1-r)}\}$ in such a manner that they are contained in an arbitrary space $[n—r—1]$: $(\alpha_{(1)}^{n-r} \pi^r) = 0$, and so on. We conclude that a complete projective classification of the numbers of four spaces, for which the S-equation has one or more multiple roots is obtained, if besides the absolute invariants are taken in consideration the concomitants

$$(\pi_{(1)}^{n-r} a^r)(\pi_{(2)}^{n-s} b^s)(\pi_{(3)}^{p} d^{n_4} a^t b^{n-n_4-p-t})(\pi_{(4)}^{q} c^{n_3} a^{n_1-t-r} b^{n-n_1-n_3-q+t+r})$$
$$(0 \leqq p + q = r + s = N \leqq T - 1; p, q, r, s = 0, 1, 2, \ldots, N).$$

With every number of four spaces A, B, C, D is connected an expression as introduced by SEGRE in the theory of the elementary divisors and inversely such an expression from projective geometrical standpoint gives a complete summary of the properties of the figure [9]).

[9]) For $n_1 = n_2 = n_3 = n_4 = 3$ compare BOTTEMA p. 34.