Mathematics. — *On conformal differential geometry. Theory of plane curves.* By W. VAN DER WOUDE.

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Introduction.

In the development of conformal differential geometry several different lines of research were followed. Now it is remarkable that in all theories just in the beginning the introduction to the theory of plane curves is not quite satisfying [1]).

As to the choice of a system of coordinates it is natural to choose as first axis the osculating circle. Than the first problem is the construction of a second axis (circle). But this is not the way followed by BLASCHKE–THOMSEN, nor by HAANTJES [2]) nor in the publication of HAANTJES–SMITS [3]), representing three different methods.

In this paper we start from the osculating circle. The usual coordinate-system is constructed in § 5. In the notation we follow BLASCHKE–THOMSEN.

§ 1. *Normalised tetracyclical coordinates.*

Be $\vec{x}$ a vector whose homogeneous components (x_0, x_1, x_2, x_3) are the tetracyclical coordinates of an oriented circle, provided that

$$(\vec{x}, \vec{x}) = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0.$$

The coordinates can be normalised by the condition

$$(\vec{x}, \vec{x}) = 1.$$

This can be done in two different ways, each of them belonging to a definite orientation of the circle. If

$$(\vec{x}, \vec{x}) = 0$$

$\vec{x}$ represents a point.

The invariant $(\vec{x}, \vec{y})$ of two circles each with a definite orientation

$$(\vec{x}, \vec{y}) \overset{\text{def}}{=} -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$$

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[1]) The method used in the beautiful work of BLASCHKE–THOMSEN: Differential geometrie III, is really somewhat more general than the method used here because a general system of circles is used. But in application to the theory of plane curves it is more or less troublesome. Especially the construction of invariant coordinatesystems is not quite satisfying and the same can be said of other methods.

[2]) J. HAANTJES: conformal geometry I, II; Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 814—824 (1941); **45**, 249—255 (1942).

[3]) J. HAANTJES and C. SMITS. De differentiaalmeetkunde van Moebius in het platte vlak. Nieuw Archief voor Wiskunde XXI, p. 34—47 (1943).

is the cosinus of the angle φ between them. By the orientation of the circles it is possible to determine cos. φ uniquely. By introducing a definite order of the two points of intersection it is also possible to fix sin. φ. Then φ is fixed mod. 2π [4]).

If $\vec{x}$ represents a point and $\vec{y}$ a circle the equation

$$(\vec{x}, \vec{y}) = 0$$

expresses that $\vec{x}$ is a point of the circle $\vec{y}$.

§ 2. *The normalized osculating circle of a plane curve.*

Be

$$\vec{x} = \vec{x}(t)$$

with the condition $(\vec{x}, \vec{x}) = 0$, the parametric equation of a plane curve. We assume in the following that all functions are real and that they can be differentiated as often as may be necessary. Then the normalized osculating circle $\vec{y}$ in a point $\vec{x}$ is expressed by

$$(\vec{y}, \vec{y}) = 1\,; \ (\vec{y}, \vec{x}) = (\vec{y}, \dot{\vec{x}}) = (\vec{y}, \ddot{\vec{x}}) = 0\,; \ \left(\dot{x} = \frac{dx}{dt}\right).$$

From this it follows that

$$(\vec{x}, \dot{\vec{y}}) = (\dot{\vec{x}}, \dot{\vec{y}}) = (\vec{x}, \ddot{\vec{y}}) = 0.$$

Hence $\dot{\vec{y}}$ satisfies the equations

$$(\vec{y}, \dot{\vec{y}}) = (\vec{x}, \vec{y}) = (\vec{x}, \dot{\vec{y}}) = 0$$

and this implies that $\vec{x}$ and $\dot{\vec{y}}$ differ only by a constant scalar factor. It is not necessary to except the case that $\vec{x}$, $\vec{y}$ and $\dot{\vec{x}}$ are linearly dependent because by multiplication with y it becomes clear that this case is impossible.

Starting from the osculating circle $\vec{y}$ we have already

$$(\vec{y}, \vec{y}) = 1, \ (\vec{y}, \dot{\vec{y}}) = (\vec{y}, \ddot{\vec{y}}) = (\vec{y}, \dddot{\vec{y}}) = 0 \quad \ldots \ldots \quad (1)$$

$$(\dot{\vec{y}}, \dot{\vec{y}}) = (\dot{\vec{y}}, \ddot{\vec{y}}) = 0 \quad \ldots \ldots \ldots \quad (2)$$

where $\dot{\vec{y}}$ is the point of osculation on the curve $\vec{y}(t)$.

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[4]) Cf. Differentialgeometrie III § 15.

§ 3. *The parameters $d\sigma$ and $d\sigma^{+}$.*

If a new parameter s is introduced instead of t we have

$$\frac{d\vec{y}}{ds}=\frac{d\vec{y}}{dt}\frac{dt}{ds}\quad\left(\frac{d\vec{y}}{dt}=\dot{\vec{y}}\right)$$

$$\frac{d^2\vec{y}}{ds^2}=\frac{d^2\vec{y}}{dt^2}\left(\frac{dt}{ds}\right)^2+\frac{d\vec{y}}{dt}\frac{d^2t}{ds^2}$$

and this implies according to (2):

$$\left(\frac{d^2\vec{y}}{ds^2},\frac{d^2\vec{y}}{ds^2}\right)=\left(\frac{d^2\vec{y}}{dt^2},\frac{d^2\vec{y}}{dt^2}\right)\left(\frac{dt}{ds}\right)^4$$

$$\left(\frac{d^2\vec{y}}{ds^2},\frac{d^2\vec{y}}{ds^2}\right)ds^4=\left(\frac{d^2\vec{y}}{dt^2},\frac{d^2\vec{y}}{dt^2}\right)dt^4.$$

This proves that the differential form (the positive value is meant)

$$d\sigma=\sqrt[4]{\left|\left(\frac{d^2\vec{y}}{dt^2},\frac{d^2\vec{y}}{dt^2}\right)\right|}\,dt\quad\ldots\ldots\ldots\quad(3)$$

is independent of the choice of the parameter t.

We remind that the method of tetracyclical coordinates is intimately connected with the wellknown transformation, carrying the points of a sphere into the points of the plane considered and the points outside of the sphere into the circles of that plane.

Now we have found that $\frac{d\vec{y}}{dt}=\vec{x}$ is a point of the plane, i.e. the image of a point of the fundamental sphere. Consequently $\frac{d^2\vec{y}}{dt^2}=\frac{d\vec{x}}{dt}$ is the image of a point on a tangent of the sphere, that is a point on the sphere or outside of the sphere. Hence $\frac{d^2\vec{y}}{dt^2}$ is in our plane a point or a circle and $\left(\frac{d^2\vec{y}}{dt^2},\frac{d^2\vec{y}}{dt^2}\right)\geq 0$. That implies that in (3) the bars may be dropped:

$$d\sigma=+\sqrt[4]{\left(\frac{d^2\vec{y}}{dt^2},\frac{d^2\vec{y}}{dt^2}\right)}\,dt.$$

Generally

$$\left(\frac{d^2\vec{y}}{dt^2},\frac{d^2\vec{y}}{dt^2}\right)\neq 0.$$

From

$$\left(\frac{d^2\vec{y}}{dt^2}, \frac{d^2\vec{y}}{dt^2}\right) = 0$$

it would follow after differentiation of (2)

$$\left(\frac{d\vec{y}}{dt}, \frac{d^3\vec{y}}{dt^3}\right) = \left(\vec{y}, \frac{d^4\vec{y}}{dt^4}\right) = 0;$$

that means: the osculating circle would have in $\vec{\dot{y}}$ a contact of third order with the enveloping curve. This case will be excepted.

We now assume that from the beginning the parameter σ were used. Then the equations

$$(\vec{\ddot{y}}, \vec{\ddot{y}}) = 1, \ (\vec{y}, \vec{\dddot{y}}) = (\ddot{y}, \dddot{y}) = 0 \qquad \left(\vec{\dot{y}} = \frac{d\vec{y}}{d\sigma}\right). \quad . \quad . \quad . \quad (4)$$

hold next to (1) and (2). From now on $\vec{\dot{y}} = \frac{d\vec{y}}{d\sigma}$.

Two normalized circles $\vec{y}$ and $\vec{\ddot{y}}$ are fixed. From (1) it follows that they intersect orthogonally. One of the points of intersection is $M(\vec{\dot{y}})$, the other be denoted by $W(\vec{w})$. For $\vec{w}$ we have

$$(\vec{y}, \vec{w}) = (\vec{\ddot{y}}, \vec{w}) = (\vec{w}, \vec{w}) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and w can be normalized such that

$$(\vec{\dot{y}}, \vec{w}) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Here is a table of scalar products written in the form of a determinant

	$\vec{y}$	$\vec{\ddot{y}}$	$\vec{\dot{y}}$	$\vec{w}$
$\vec{y}$	1	0	0	0
$\vec{\ddot{y}}$	0	1	0	0
$\vec{\dot{y}}$	0	0	0	1
$\vec{w}$	0	0	1	0

. . . . (7)

This determinant is -1, and this implies that the four vectors $\vec{y}, \vec{\ddot{y}}, \vec{\dot{y}}, \vec{w}$ are linearly independent and every other vector can be linearly expressed in them. According to a wellknown theorem the determinant is equal to $-(\vec{y}, \vec{\ddot{y}}, \vec{\dot{y}}, \vec{w})^2$.

Hence

$$\text{Det}\,|\vec{y}, \ddot{\vec{y}}, \dot{\vec{y}}, \vec{w}| = \pm 1$$

In fact both values may occur. Suppose the value is $+1$. Then we apply a transformation leaving $(\vec{y}, \vec{y})$ invariant, i.e. a transformation of the group M_6. Its determinant can have the value ± 1; we apply a transformation having the value -1.

After this transformation (7) takes the value -1. But scalar products being invariant $(\dot{\vec{y}}, \vec{w})$ remains $+1$.

The differential $d\sigma$ is invariant for all transformations of M_6. We now define an other differential form $d\sigma^+$ by the equation

$$d\sigma^{+3} = |\vec{y}, \frac{d^2 \vec{y}}{dt^2}, \frac{d\vec{y}}{dt}, \vec{w}|\, dt^3$$

recurring to the arbitrary parameter t. This form is independent of the choice of the parameter t. The same holds for

$$d\sigma^+ = \sqrt[3]{|\vec{y}, \frac{d^2 \vec{y}}{dt^2}, \frac{d\vec{y}}{dt}, \vec{w}|}\, dt. \quad \ldots\ldots \quad (8)$$

If a transformation of M_6 is applied to the righthand side of (8) with fixed t, the determinant remains invariant or changes its sign according to the value of the transformationdeterminant being $+1$ or -1.

Consequently $d\sigma^+$ is invariant with respect to transformations of the parameter and the transformations of $M_6{}^+$, the subgroup of M_6, with determinant $+1$.

If in (8) t is replaced by σ we get the relation between $d\sigma$ and $d\sigma^+$

$$d\sigma^+ = |\vec{y}, \ddot{\vec{y}}, \dot{\vec{y}}, \vec{w}|\, d\sigma \left(\dot{\vec{y}} = \frac{d\vec{y}}{d\sigma} \right).$$

If we start from $d\sigma^+ = d\sigma$ this relation is invariant for every transformation of $M_6{}^+$. That implies that all formulae hold if $\dot{\vec{y}}, \ddot{\vec{y}}$ ... denote derivatives of $\vec{y}$ with respect to $d\sigma^+$. In the sequel σ^+ will be used as parameter and only transformations of $M_6{}^+$ will be used. Then we have allways

$$|\vec{y}, \ddot{\vec{y}}, \dot{\vec{y}}, \vec{w}| = +1. \quad \ldots\ldots \quad (9)$$

§ 4. Frenet's *formulae in conformal geometry.*

In this section the formulae of Frenet will be derived. We suppose that

$$(\dddot{\vec{y}}, \vec{w}) = -b. \quad \ldots\ldots \quad (10a)$$

hence (c.f. (6))

$$(\ddot{\vec{y}}, w) = b \quad \ldots\ldots \quad (10b)$$

If now we write

$$\vec{\dddot{y}} = \alpha \vec{y} + \beta \vec{\ddot{y}} + \gamma \vec{\dot{y}} + \delta \vec{w}$$

the coefficients α, β, γ, δ can be determined immediately by scalar multiplication of this equation with $\vec{y}$, $\vec{\ddot{y}}$, $\vec{\dot{y}}$, and $\vec{w}$, using the table (7). In the same way the coefficients in

$$\vec{w} = \lambda \vec{y} + \mu \vec{\ddot{y}} + \nu \vec{\dot{y}} + \varrho \vec{w}$$

can be determined. Thus we get

$$\left.\begin{aligned} \vec{\dddot{y}} &= \qquad\qquad -b\vec{\dot{y}} - \vec{w} \\ \vec{\dot{w}} &= -\vec{y} + b\vec{\ddot{y}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (11)$$

and they are the formulae of FRENET in conformal geometry. We add the formula

$$\vec{\ddddot{y}} = \vec{y} - 2b\vec{\ddot{y}} - \dot{b}\vec{\dot{y}}$$

to be used in § 6.

In the usual way it can now be proved that, if b is given as a function of σ^+ the functions $\vec{y}(\sigma^+)$, and from this $\vec{\ddot{y}}$, $\vec{\dot{y}}$, and $\vec{w}$ and their relations, can be determined, uniquely to within transformations of M_6^+, in such a way that (11) is satisfied and that the circles $\vec{y}(\sigma^+)$ are all osculating circles of the curve enveloping them.

This implies that b and its derivatives form a complete set of conformal invariants of the curve $\vec{\dot{y}}(\sigma^+)$.

§ 5. *Construction of the preferred coordinate system.*

If $\vec{y}_1$ and $\vec{y}_2$ are two normalized circles with no real common points, in the linear circlemanifold $\alpha \vec{y}_1 + \beta \vec{y}_2$ there exist two points $\vec{p}$ and $\vec{q}$, the limiting points.

Now we ask for the two circles $\vec{z}$ satisfying the condition that the two double ratios

$$D(\vec{p}, \vec{q}, \vec{y}_1, \vec{z}) = D(\vec{p}, \vec{q}, \vec{z}, \vec{y}_2)$$

are equal. In order to determine $\vec{p}$ and $\vec{q}$ we solve μ from the equation

$$(\vec{y}_1 + \mu \vec{y}_2, \vec{y}_1 + \mu \vec{y}_2) = 0$$

$$1 + \mu (\vec{y}_1 + \vec{y}_2) + \mu^2 = 0$$

If μ_1 and μ_2 are the solutions we have

$$\vec{p} = \vec{y}_1 + \mu_1 \vec{y}_2, \quad \vec{q} = \vec{y}_1 + \mu_2 \vec{y}_2.$$

Be now $$\vec{z} = \vec{y}_1 + \lambda \vec{y}_2$$

then we have $$\frac{\mu_1}{\mu_2} : \frac{\mu_1 - \lambda}{\mu_2 - \lambda} = \frac{\mu_1 - \lambda}{\mu_2 - \lambda}$$

$$\lambda^2 = \mu_1 \mu_2 = 1$$

$$\vec{z} = \vec{y}_1 \pm \vec{y}_2.$$

The geometrical signification can be obtained as follows. If $\vec{z}$ is written in the normal form

$$\vec{z} = \frac{\vec{y}_1 \pm \vec{y}_2}{\sqrt{\{1 + 2(\vec{y}_1, \vec{y}_2)\}}}$$

it follows that

$$(\vec{z}, \vec{y}_1) = \frac{1 \pm (\vec{y}_1, \vec{y}_2)}{\sqrt{\{1 + 2(\vec{y}_1, \vec{y}_2)\}}} = (\vec{z}, \vec{y}_2).$$

Hence the two circles $\vec{z}$ are those circles of the linear manifold fixed by $\vec{y}_1$ and $\vec{y}_2$, that have the same invariant with $\vec{y}_1$ and $\vec{y}_2$ [5]).

The foregoing consideration be applied now to two neighbouring osculating circles of the curve. They may be chosen as near that one of them lies entirely inside the other. They be denoted by $\vec{y} + \Delta_1\vec{y}$ and $\vec{y} + \Delta_2\vec{y}$. Of the two circles $\vec{z}$ derived as above we chose one at random and denote this one by $\vec{z}$. From this circle and $\vec{y}$ we construct a linear circlemanifold. Hence this manifold is fixed by $\vec{y}$ and $\Delta_1\vec{y} + \Delta_2\vec{y}$ or $\Delta_1\vec{y} - \Delta_2\vec{y}$. If the + sign is chosen we have

$$\Delta_i \vec{y} = \dot{\vec{y}} \Delta_i \sigma^+ + \frac{\ddot{\vec{y}}}{1 \cdot 2} \Delta \sigma^{+2} + \dots \qquad (i = 1, 2)$$

and this gives rise to the following consideration. Let $\Delta_1\sigma^+$ and $\Delta_2\sigma^+$ tend to zero in such a way that $\dfrac{\Delta_1\sigma^+}{\Delta_2\sigma^+}$ is constant then the linear manifold tends in general to a limit, determined by $\vec{y}$ and $\dot{\vec{y}}$. But there is one exception. If

$$\frac{\Delta_1 \sigma^+}{\Delta_2 \sigma^+} = -1$$

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[5]) Till here we did not use the condition, that $\vec{y}_1$ and $\vec{y}_2$ have no real points in common. Hence the deduction does not loose sense if this condition is dropped. Then the circles $\vec{z}$ bisect the angles between $\vec{y}_1$ and $\vec{y}_2$.

we get instead the manifold determined by $\vec{y}$ and $\ddot{\vec{y}}$ with $\dot{\vec{y}}$ as one of the points of intersection.

Briefly said, we consider two osculating circles $\vec{y} + \Delta_1\vec{y}$ and $\vec{y} + \Delta_2\vec{y}$ with $\Delta_1\sigma^+ = \pm\Delta_2\sigma^+$ and (in a suitable way for each choice of the sign $\pm$) one of the two circles $\vec{z}$ having the same invariant with $\vec{y} + \Delta_1\vec{y}$ and $\vec{y} + \Delta_2\vec{y}$. If $\Delta_1\vec{y}$ and $\Delta_2\vec{y}$ tend to zero, $\vec{z}$ tends to a circle $\ddot{\vec{y}}$, forming with $\vec{y}$ the axes of our invariant coordinatesystem. That fixes this coordinatesystem in a constructive way.

§ 6. *Comparison with the method of* HAANTJES *and* SMITS.

In this § we compare our method with the method of H. and S. They start from the coordinates of GAUSS. Be (in our notation) $\vec{x}$ a point, then these coordinates can be introduced by

$$z = \frac{x_2 + i x_3}{x_0 + x_1} \qquad \qquad (12)$$

Now let z describe a curve $z(t)$ and be $z', z'', \ldots$ the derivatives with respect to t. Then the socalled derivative of SCHWARZ

$$\{z, t\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 \qquad \qquad (13)$$

is invariant for all transformations of M_6^+. H. and S. fixed their parameter (t) by the condition

$$I\{z, t\} = 1,$$

where $I\ \{z, t\}$ denotes the imaginary part of $\{z, t\}$.

Now in our notation we let some point describe the same curve by putting in (12)

$$\vec{x} = \dot{\vec{y}} = \frac{d\vec{y}}{d\sigma^+}$$

and we ask for the relation between the parameter t (of H. and S.) and our σ^+. In order to show that σ^+ and t coincide, we have only to prove that

$$I\{z, \sigma^+\} = 1.$$

From (12), (13), (14) it follows that

$$\left.\begin{aligned} \{z, \sigma^+\} &= \frac{\dddot{z}}{\dot{z}} - \frac{3}{2}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 = \frac{1}{q^2}, \frac{1}{(q\dot{p} - p\dot{q})^2} \times \\ &\times [2q\dot{q}\,(q\dot{p} - p\dot{q})\,(q\ddot{p} - p\ddot{q}) - 2\,q\ddot{q})^2\,(q\dot{p} - p\dot{q})^2 + \\ &+ q^2\,\{(\dot{q}\ddot{p} - \dot{p}\ddot{q} + q\dddot{p} - p\dddot{q})\,(q\dot{p} - p\dot{q}) - \tfrac{3}{2}\,(q\ddot{p} - p\ddot{q})^2\}] \end{aligned}\right\} \qquad (14)$$

where we have put

$$p = \dot{y}_2 + i\,\dot{y}_3, \quad q = \dot{y}_0 + \dot{y}_1.$$

The expression (14) can be simplified by the following transformation T of M_6^+ for which $\{z, \sigma^+\}$ is invariant:

$$\vec{y} \rightarrow \text{the straight line } (0, 0, 0, 1)$$
$$\vec{\ddot{y}} \rightarrow \text{the straight line } (0, 0, 1, 0)$$
$$\vec{\dot{y}} \quad \text{the point } (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0).$$

Then $\vec{w}$ is the point $(-1, 1, 0, 0)$. According to FRENET's formulae the following equations can be added

$$\vec{\dddot{y}} \rightarrow (-\tfrac{1}{2}b + 1,\ -\tfrac{1}{2}b - 1, 0, 0)$$
$$\vec{\ddddot{y}} \rightarrow (-\tfrac{1}{2}\dot{b},\ -\tfrac{1}{2}\dot{b},\ -2b, 1)$$

and this leads to

$$\{z, \sigma^+\} = b + i.$$

Hence the parameter t may be considered as induced by σ^+. As was to be expected also the real part of $\{z, \sigma^+\}$ is invariant, that is equal to b.

The second base point of the invariant linear circlemanifold considered by H. and S. is now, after performing the transformation T, the point $z = \infty$ or in our notation the point $(-1, 1, 0, 0)$. It always coincides with our point W. Our circle $\vec{\ddot{y}}$ is the normal circle of H. and S.