Aerodynamics. — *On the influence of gravity upon the expansion of a gas.* I. By J. M. BURGERS. (Mededeling no. 53 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

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1. *Statement of the problem.* — We consider a vertical column of gas, which before the instant $t = 0$ is limited by a horizontal plane wall at its upper end, whereas downwards the column extends indefinitely. Lateral motion of the gas is prevented (it may be assumed that the gas is enclosed in an infinite vertical cylinder with perfectly smooth walls, or that the lateral dimensions of the column are infinite). The gas originally is everywhere at rest. The pressure of the gas has a certain finite value p_0 at the level where it is in contact with the boundary plane; downward the pressure rises in consequence of the weight of the gas, according to the law valid for an atmosphere in adiabatic (isentropic) equilibrium. Above the boundary plane is vacuum extending towards infinity.

At the instant $t = 0$ the boundary plane is suddenly taken away, so that the gas can expand. It is asked to find the motion of the gas, taking account of its weight.

The gas shall obey the law $p/\varrho = RT$ (p = pressure; ϱ = density; T = absolute temperature; R = gas constant per unit of mass) and shall have constant specific heat $c_v = R/(k-1)$; $c_p = kR/(k-1)$. In working out the equations the case $k = {}^5/_3$ is taken, as this value of k (like the value ${}^7/_5$) makes possible a solution of the principal equations in finite terms. The scale of the field depends upon two parameters: c_0, the velocity of sound in the gas at the level just below the boundary plane in the original equilibrium state of the gas (for convenience a will be written for $3c_0$), and g, the acceleration of gravity, which is supposed to be independent of the height; the time scale is fixed by a/g, the scale of lengths by a^2/g.

Viscosity, heat conduction and radiation are neglected. It is found, however, that a shock wave appears in a particular point of the field at the instant $t = \sqrt{2} \cdot a/g$. Within the shock wave viscosity produces a sudden rise of entropy. The continuation of the solution beyond this instant is extremely difficult and we must restrict to a few indications concerning the first stages of the propagation of this shock wave, in which the change of entropy still is small.

2. *Equations of motion for the gas.* — We take the axis of z vertically upwards, z being counted from the upper level of the gas in the original equilibrium condition, *i.e.* from the level in contact with the boundary plane.

So long as the changes of state of the elements of volume of the gas are adiabatic and isentropic, the state of the gas in every element can be characterised by a single variable, for which we take the velocity of sound c.

The equations of motion can be brought into the form [1]):

$$\left\{ \frac{\partial}{\partial t} + (u \pm c) \frac{\partial}{\partial z} \right\} \left(u \pm \frac{2}{k-1} c\right) = -g \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(first equation: + signs; second equation: — signs), where u is the velocity of motion of an element of volume.

The gas was assumed to be at rest in the region $z < 0$ for $t < 0$. Both equations of the system (1) then reduce to the same form and by integration give:

$$c^2 = c_0^2 - (k-1)\, gz = c_0^2 - \tfrac{2}{3}\, gz \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From this formula the increase of the temperature, of the pressure and of the density of the gas downward from the level $z = 0$ can be deduced. The temperature *e.g.* is given by $T = T_0 - (k-1)gz/kR = T_0 - gz/c_p$, which is the well known relation for an atmosphere in isentropic equilibrium.

After the boundary plane at $z = 0$ has been taken away we expect that the expanding gas will not rise indefinitely (as it does in the case where gravity is not operating), but that finally a new state of equilibrium will be approached. It can also be expected that for large negative values of z the state of the gas will not change very much and that in the limit formula (2) will remain valid. Hence if a new equilibrium state should indeed be approached (possibly after a period of oscillatory motion), we must expect that the column of gas in the final state will not extend beyond an upper level determined by $c = 0$, from which:

$$z_m = 3c_0^2/2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

3. *Solution of the equations by means of* RIEMANN's *method.* — We introduce a change of variables:

$$\tau = t; \qquad \zeta = z + \tfrac{1}{2} g t^2; \qquad w = u + gt \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equations become transformed into (with $k = {}^5/_3$):

$$\left\{ \frac{\partial}{\partial \tau} + (w \pm c) \frac{\partial}{\partial \zeta} \right\} (w \pm 3c) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Here g no longer appears explicitly: equations (5) are identical with those for the expansion of a gas not subjected to gravity. We next introduce:

$$\vartheta = w + 3c; \qquad \eta = w - 3c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which:

$$w = \tfrac{1}{2}(\vartheta + \eta); \qquad u = \tfrac{1}{2}(\vartheta + \eta) - gt; \qquad c = \tfrac{1}{6}(\vartheta - \eta) \quad . \quad . \quad (7)$$

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[1]) Compare J. M. BURGERS, Some problems of the motion of interstellar gas clouds, these Proceedings **49**, 593, eqs. (7), (1946).

Equations (5) express that ϑ is constant along a curve of a ζ, τ-plane for which $d\zeta/d\tau = w + c$; and that η is constant along a curve for which $d\zeta/d\tau = w - c$. Hence the curves of the ζ, τ-plane determined by the equations:

$$\text{(I)}\ d\zeta/d\tau = w + c; \qquad \text{(II)}\ d\zeta/d\tau = w - c,$$

or in other form by:

$$\text{(I)}\ \vartheta = constant; \qquad \text{(II)}\ \eta = constant,$$

are the two sets of characteristics of the system (5).

As indicated by RIEMANN [2]) we take η and ϑ as independent variables, describing an η, ϑ-plane (fig. 1), in which ζ and τ will be considered as

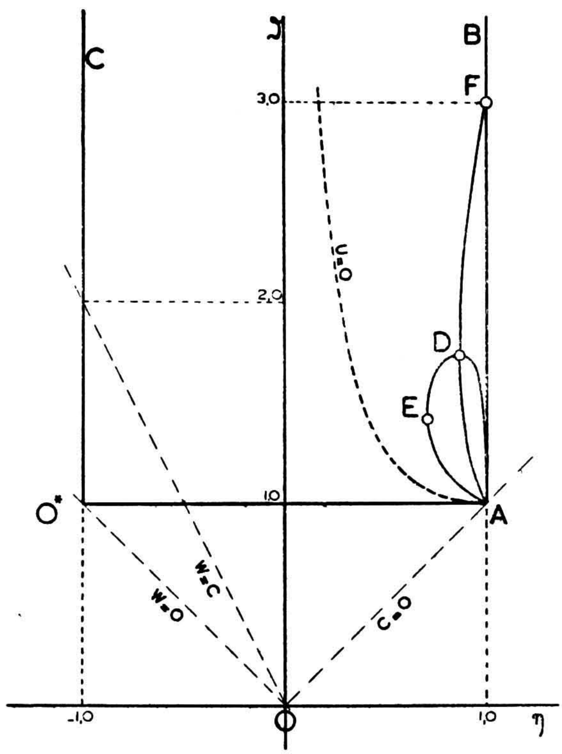

Fig. 1. η, ϑ-plane. The unit for the scales along both axes is a.

the dependent variables. The transformation of the equations of motion is obtained in the most simple way by observing that the relations just mentioned are equivalent to:

$$\left.\begin{aligned} \partial\zeta/\partial\eta &= (w + c) \cdot (\partial\tau/\partial\eta) \text{ for constant } \vartheta \\ \partial\zeta/\partial\vartheta &= (w - c) \cdot (\partial\tau/\partial\vartheta) \text{ for constant } \eta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

[2]) See B. RIEMANN–H. WEBER, Die partiellen Differentialgleichungen der mathematischen Physik (Braunschweig 1901), Bd. 2, p. 499 seqq.

Eliminating ζ by cross differentiation we arrive at:

$$\frac{\partial^2 \tau}{\partial \eta \, \partial \vartheta} - \frac{2}{\eta - \vartheta} \left(\frac{\partial \tau}{\partial \eta} - \frac{\partial \tau}{\partial \vartheta} \right) = 0 \quad \ldots \ldots \quad (9)$$

We further consider the boundaries appearing in the η, ϑ-plane. One boundary is obtained by observing that from $t = 0$ onward the first wave, initiating the expansion, penetrates into the gas in the direction of $-z$, with a velocity of propagation c given by (2). In a z, t-diagram (see fig. 2)

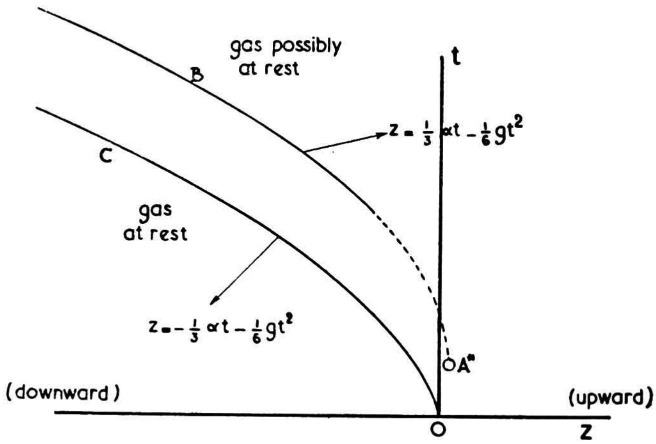

Fig. 2. z, t-plane.

the path OC of this wave is determined by $dz/dt = -c$, from which:

$$z = -\tfrac{1}{3}\, a t - \tfrac{1}{6}\, g t^2 \quad \ldots \ldots \ldots \quad (10)$$

Inserting (10) into (2) we find that along this path $c = c_0 + \frac{1}{3}\, gt =$ $= \frac{1}{3}\,(a + gt)$. As $u = 0$ along the path of this wave, we have $w = gt$, and we deduce from equations (6):

$$\vartheta = a + 2g\tau \text{ (with } \tau \geqslant 0); \quad \eta = -a \quad \ldots \ldots \quad (11)$$

These formulae determine the straight boundary O^*C *in* the η, ϑ-plane as indicated in fig. 1, along which boundary $\tau = (\vartheta - a)/2g$.

To obtain a second boundary we observe that the first stages of the expansion (appearing near $z = 0$ for t only slightly above 0) will be practically the same as in the case where gravity is absent. In that case [3]):

$$u = \tfrac{1}{4}\, a + \tfrac{3}{4}\, z/t; \qquad c = \tfrac{1}{4}\, a - \tfrac{1}{4}\, z/t.$$

Here the directions $z/t = \textit{constant}$ represent the characteristics for which $dz/dt = u - c$. In the expanding gas near $z = 0$ the value of c will at most be equal to $c_0 = a/3$; as on the other hand c cannot decrease below

[3]) Compare J. M. BURGERS *l.c.*, eqs. (6) at p. 592, changing the sign of c. The solution given page 594, eqs. (8), although likewise reverting to that given by eqs. (6) when t goes to zero, refers to a set of initial conditions differing from those introduced in the present problem.

zero, the admissible values of z/t are confined between $-a/3$ and $+a$. As the difference between w and u can be neglected for very small values of t, the characteristics $\eta = constant$ must be tangent for $t \to 0$ to the characteristics $z/t = constant$ of the equations from which g is absent. At the same time we find:

$$\vartheta = w + 3c \cong u + 3c = a;$$

$$\eta = w - 3c \cong u - 3c = -\tfrac{1}{2}a + \tfrac{3}{2}z/t \quad (\leqslant + a).$$

These equations determine the boundary O^*A in fig. 1, along which boundary $\tau = 0$.

With the aid of the boundary values of τ along O^*C and O^*A it is possible to obtain an expression for τ valid in the region $\vartheta \geq a$; $-a \leq \eta \leq +a$ by means of RIEMANN's method of integration. The auxiliary function v introduced by RIEMANN must satisfy the equation:

$$\frac{\partial^2 v}{\partial\eta\,\partial\vartheta} + \frac{\partial}{\partial\eta}\left(\frac{2v}{\eta-\vartheta}\right) - \frac{\partial}{\partial\vartheta}\left(\frac{2v}{\eta-\vartheta}\right) = 0,$$

and the supplementary conditions:

$$\frac{\partial v}{\partial\eta} = \frac{2v}{\eta-\vartheta} \text{ for } \vartheta = \vartheta_P; \quad \frac{\partial v}{\partial\vartheta} = -\frac{2v}{\eta-\vartheta} \text{ for } \eta = \eta_P,$$

η_P, ϑ_P being the coordinates of any point where it is desired to find the value of τ. The appropriate solution for v is [4]):

$$v = (\vartheta-\eta)^2 - \frac{2(\vartheta-\eta)(\vartheta-\vartheta_P)(\eta-\eta_P)}{\vartheta_P-\eta_P} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

4. *Continuation.* — Making use of RIEMANN's auxiliary function the following expression is obtained for τ:

$$\tau = \frac{\vartheta^2-a^2}{g(\vartheta-\eta)^2}\left\{\vartheta - \frac{\vartheta^2-a^2}{2(\vartheta-\eta)}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Having regard to (8) the corresponding expression for ζ is found to be:

$$\zeta = \frac{\vartheta^2-a^2}{g(\vartheta-\eta)}\left\{\frac{5\,\vartheta^2-a^2}{4(\vartheta-\eta)} - \frac{\vartheta(\vartheta^2-a^2)}{2(\vartheta-\eta)^2} - \frac{2\vartheta}{3}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

where the integration constant has been adjusted so that along O^*C (*i.e.* for $\eta = -a$): $\zeta = (\vartheta^2 - 4\vartheta a + 3a^2)/12\,g$, which leads to the value of z given by (10).

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4) See RIEMANN–WEBER [l.c. footnote 2) above], p. 510. — The circumstance that a solution of the equations for v can be expressed in finite form if k satisfies the condition that $(k+1)/2(k-1)$ is a whole number, was remarked by G. DARBOUX, Leçons sur la théorie des surfaces (Paris 1889), tome II, p. 65. Compare J. HADAMARD, Leçons sur la propagation des ondes (Paris 1903), p. 168 and also GOSSOT et LIOUVILLE, Ballistique intérieure (Paris 1922), p. 142.

It is useful to mention the equations:

$$2g\,(\vartheta-\eta)^4\,(\partial\tau/\partial\eta) = \vartheta^4 - 4\,\vartheta^3\,\eta + 2\,\vartheta^2\,a^2 + 4\,\vartheta\,\eta\,a^2 - 3\,a^4 \quad . \quad . \quad (15a)$$

$$2g\,(\vartheta-\eta)^4\,(\partial\tau/\partial\vartheta) = \vartheta^4 - 4\,\vartheta^3\,\eta + 6\,\vartheta^2\,\eta^2 - 4\,\vartheta\,\eta\,a^2 - 2\,\eta^2\,a^2 + 3\,a^4 \quad (15b)$$

The values of $\partial\zeta/\partial\eta$ and $\partial\zeta/\partial\vartheta$ are obtained from these expressions with the aid of (8). It follows from (15*a*) and (15*b*) that the derivatives $\partial\tau/\partial\eta$ and $\partial\zeta/\partial\eta$ simultaneously become zero:

a) along the line O^*A where $\vartheta = a$ (for all values of η);
b) in the points of the curve:

$$\eta = (\vartheta^2 + 3\,a^2)/4\,\vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This curve starts from A ($\vartheta = a$; $\eta = a$) and ends in F ($\vartheta = 3a$; $\eta = a$); the minimum value of η on this curve is found in the point D ($\vartheta = \sqrt{3}\cdot a$; $\eta = \frac{1}{2}\sqrt{3}\cdot a$).

The derivatives $\partial\tau/\partial\vartheta$ and $\partial\zeta/\partial\vartheta$ simultaneously become zero in the points of the curve:

$$(6\,\vartheta^2 - 2\,a^2)\,\eta^2 - 4\,(\vartheta^3 + \vartheta\,a^2)\,\eta + \vartheta^4 + 3\,a^4 = 0 \quad . \quad . \quad . \quad (17)$$

which starts from A and returns to the same point after having passed through D, where ϑ reaches a maximum. The minimum value of η on this curve is attained in E ($\vartheta = \sqrt{2}\cdot a$; $\eta = \frac{1}{2}\sqrt{2}\cdot a$).

The η, ϑ-plane is also of use in the calculation of the path of an individual element of volume of the gas. The motion of such an element is determined by:

$$dz/dt = u = \tfrac{1}{2}\,(\vartheta + \eta) - gt \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

from which, making use of (4):

$$d\zeta/d\tau = w = \tfrac{1}{2}\,(\vartheta + \eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

If the path of an element is represented in the ϑ, η-plane and $d\vartheta/d\eta$ refers to the curve thus defined, we have:

$$\frac{d\zeta}{d\tau} = \frac{(\partial\zeta/\partial\eta) + (\partial\zeta/\partial\vartheta)\,(d\vartheta/d\eta)}{(\partial\tau/\partial\eta) + (\partial\tau/\partial\vartheta)\,(d\vartheta/d\eta)} \quad . \quad . \quad . \quad . \quad . \quad (18b)$$

Making use of eqs. (8) and of (15*a*), (15*b*), equations (18*a*) and (18*b*) can be applied to deduce the following differential equation for the representation in the ϑ, η-plane of the path of an element:

$$\frac{d\vartheta}{d\eta} = \frac{\vartheta^4 - 4\,\vartheta^3\,\eta + 2\,\vartheta^2\,a^2 + 4\,\vartheta\,\eta\,a^2 - 3\,a^4}{\vartheta^4 - 4\,\vartheta^3\,\eta + 6\,\vartheta^2\,\eta^2 - 4\,\vartheta\,\eta\,a^2 - 2\,a^2\,\eta^2 + 3\,a^4} \quad . \quad . \quad (19)$$

This equation admits the integral:

$$\tfrac{1}{5}\,\vartheta^5 - \vartheta^4\,\eta + 2\,\vartheta^3\,\eta^2 - 2\,\vartheta^2\,\eta\,a^2 + \vartheta\,(3\,a^4 - 2\,\eta^2\,a^2) + 3\,\eta\,a^4 = \tfrac{1}{5}\,S\,a^5 \quad (20)$$

S being a constant. — From what has been observed in connection with (15*a*), (15*b*) it follows that the curves determined by (16) and (17) mark the points where the curves (20) have a horizontal or a vertical tangent respectively.

We further mention that with the aid of (13) the following expression for u can be deduced:

$$u = \tfrac{1}{2}\,(a^2 - \eta^2)\,(a^2 - 2\,\vartheta\eta + \eta^2)/(\vartheta - \eta)^3 \quad . \quad . \quad . \quad . \quad (21)$$

Hence u becomes zero:

a) along the boundaries O^*C ($\eta = -a$) and AB ($\eta = +a$);
b) in the points of the curve:

$$\vartheta = (\eta^2 + a^2)/2\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which has been indicated in fig. 1 by a dotted line.

5. *Provisional interpretation.* — Having arrived at a formal solution of the equations referring to the η, ϑ-plane, we attempt to interprete this solution in terms referring to the z, t-plane or "physical plane" (see fig. 2). It will be found that the correspondence between the two planes is not a simple one, as a certain overlapping occurs in the z, t-plane in the region corresponding to the neighbourhood of the point A of the η, ϑ-plane. We shall, however, first give attention to that part of the η, ϑ-plane between the lines O^*C and AB for which ϑ is large, as no difficulties are found here.

It has been mentioned that the line O^*C corresponds to the first wave penetrating downward into the gas and initiating the motion of the consecutive layers. This path has been indicated in fig. 2.

Along the boundary O^*A in the η, ϑ-plane we have $\vartheta = a$, $\tau = 0$, $\zeta = 0$. In the z, t-plane the single point O ($z = 0$, $t = 0$) corresponds to this line. The point A itself ($\vartheta = a$, $\eta = a$) is to be excluded, as the expressions (13) and (14) become indeterminate here.

Along the boundary AB in the η, ϑ-plane we have $\eta = +a$, $\vartheta > a$; hence $\tau = t = (\vartheta + a)/2g$; $\zeta = (\vartheta^2 + 4\vartheta a + 3a^2)/12g$, and:

$$z = (-\vartheta^2 + 2\,\vartheta a + 3\,a^2)/24\,g = \tfrac{1}{3}\,at - \tfrac{1}{6}\,gt^2 \quad . \quad . \quad . \quad (23)$$

This curve likewise has been indicated in fig. 2 (curve A^*B starting from $z = a^2/6g = 3c_0{}^2/2g$; $t = a/g$).

The curves OC and A^*B are both characteristics, as η is constant along these curves. Along OC we have $u = 0$, $c = \tfrac{1}{3}\,(a + gt) = \sqrt{c_0{}^2 - \tfrac{2}{3} gz}$, as mentioned before. Making use of (7) and (23) we find that exactly the same relations hold along A^*B. Now it is a known property of equations of the hyperbolic class that along characteristics solutions of different type can be joined together. Hence it seems possible to suppose that not only the region of the z, t-plane below the parabola OC, but also that above the parabola A^*B — at least for large values of t — represents a state of rest of the gas in which equation (2) will hold.

We will provisionally assume that for large values of t an asymptotic solution of this type is possible, although the conclusion cannot be accepted as final, in consequence of the peculiar conditions of overlapping which are found in the neighbourhood of A.

Development of (13) and (14) for large values of ϑ gives:

$$\left.\begin{aligned} t=\tau &= \frac{\vartheta+\eta}{2g}+\frac{\eta\,(a^2-\eta^2)}{g\,\vartheta^2}-\frac{(a^2-5\eta^2)\,(a^2-\eta^2)}{2\,g\,\vartheta^3}+\dots \\ \zeta &= \frac{\vartheta^2+4\vartheta\eta+\eta^2+2a^2}{12g}+\frac{2\eta\,(a^2-\eta^2)}{3g\vartheta}-\frac{(3a^2-23\eta^2)\,(a^2-\eta^2)}{12g\vartheta^2} \end{aligned}\right\} \quad (24)$$

from which:

$$z=-\frac{(\vartheta-\eta)^2-4a^2}{24g}+\frac{\eta\,(a^2-\eta^2)}{6g\vartheta}-\dots \quad (25)$$

These formulae show that for large values of ϑ (and thus for large values of t) there is a simple and unambiguous correspondence between the region of the η, ϑ-plane limited by the lines O^*C, AB, and the region of the z, t-plane limited by the parabolae OC and A^*B.

For large values of ϑ eq. (20) can be approximated by:

$$(\vartheta-\eta)^5 \cong S a^5 + 10\,\vartheta^2\,(\eta a^2-\eta^3),$$

from which:

$$\vartheta \cong C+\eta+2\eta\,(a^2-\eta^2)/C^2 \quad (26)$$

where $C=S^{1/5}a$. It follows that the curve in the η, ϑ-plane representing the motion of an element of volume of the gas (for shortness we shall call such a curve an "S-curve") starts from $\vartheta \cong C-a$; $\eta=-a$ and arrives at $\vartheta \cong C+a$; $\eta=+a$. From (24) and (25) we deduce that the corresponding actual motion takes place in the interval of time from $t=C/2g-a/g$ until $t=C/2g+a/g$, and that the change of level is small. There is a slight (and slow) up and down motion, in which the value of z first rises over an amount of approximately $a^4/4gC^2$ and then almost comes back to the original level, the resulting increase of z (as deduced from a closer approximation) amounting to $\frac{16}{15}\,a^5/gC^3$ only. It can also be deduced from (21) that for this motion $u \cong -\eta(a^2-\eta^2)/\vartheta^2$, so that u changes sign approximately at $\eta=0$. Hence the motion of the gas, started by the initial wave which travels along the path OC, is brought to rest again by a final wave travelling along the path A^*B.

6. *Appearance of shock waves.* — It has been mentioned that in establishing a correspondence between the η, ϑ-plane and the z, t-plane a certain overlapping is encountered. This overlapping becomes evident in various ways. In discussing it we conveniently make use of the ζ, τ-plane instead of the z, t-plane, as this relieves us from the necessity of calculating values of $\frac{1}{2}gt^2$. As the z, t-plane can be obtained from the ζ, τ-plane by a shift of every horizontal line over the appropriate distance $\frac{1}{2}gt^2$ to the left,

there is a one to one correspondence between the points of the relevant regions of the two planes, and intersections either of characteristics or of paths appearing in one plane will have their exact counterpart in the other plane.

We start with a short discussion of the course of the characteristics. Referring to eqs. (15*a*), (15*b*), (16) and (17) of section 4 and to the conclusions derived from them, it will be seen that characteristics of the set $\eta = constant$ intersect with the curve *AEDA* in the η, ϑ-plane if $\eta > \frac{1}{2}\sqrt{2} \cdot a$. At the point of intersection $\partial\tau/\partial\vartheta$ and $\partial\zeta/\partial\vartheta$ simultaneously become zero and change sign. This means that in the ζ, τ-plane the characteristic *turns back upon itself* and thus presents a cusp [5]). At the intersection with the upper branch of the curve *AEDA* a second cusp appears. — The characteristics of the set $\vartheta = constant$ intersect with the curve *ADF* if $\vartheta < 3a$; in the ζ, τ-plane these characteristics then likewise present a cusp, in consequence of the simultaneous vanishing of $\partial\tau/\partial\eta$ and $\partial\zeta/\partial\eta$.

More important is a discussion of the curves representing the motion of an individual element of volume (a layer) of the gas. The corresponding "*S*-curves" of the η, ϑ-plane are given by (20). The *S*-curves start from points of the line O^*C; if the starting point is given by $\vartheta = \vartheta_0$; $\eta = -a$, the value of S is found to be $S = (\vartheta_0/a + 1)^5 - 16$, so that S increases regularly with ϑ_0. The smallest possible value of S is obtained with $\vartheta_0 = a$, giving $S_{min} = 16$; the corresponding *S*-curve is the line O^*A in the η, ϑ-plane.

If we take a value of S slightly above S_{min} the corresponding *S*-curve will intersect with the lower branch of the curve *AEDA*. The point of intersection gradually moves from A to the left, until it reaches the point E when S has obtained the value $\frac{23}{10}\sqrt{2} = 16{,}26345$. For $S > 16{,}26345$ there is no intersection with the curve *AEDA*, although there can be intersection with the upper part of the curve *ADF* of fig. 1.

For any *S*-curve with $16 < S < 16{,}26345$ it follows from (17) and (19) that $d\vartheta/d\eta$ becomes infinite at the point of intersection with the curve *AE*. At the same point $\partial\tau/\partial\vartheta$ and $\partial\zeta/\partial\vartheta$ take the value zero and change sign. Hence when we proceed along the *S*-curve the corresponding path in the ζ, τ-plane turns back upon itself, *i.e.* it presents a cusp (both distance and time going back). Such behaviour in reality is impossible. It is to be observed that the velocity of the gas, which in the ζ, τ-plane is determined by $w = d\zeta/d\tau$, does *not* become zero at the cusp.

The vanishing of $\partial\tau/\partial\vartheta$ and $\partial\zeta/\partial\vartheta$ means that at the corresponding point of the ζ, τ-plane the quantity ϑ has an infinite gradient, *i.e.* ϑ presents an abrupt change. It appears from formulae (7) that this entails an abrupt change in w and in c (there is no abrupt change in η). Hence we have the situation characteristic for a *shock wave*.

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[5]) Compare J. HADAMARD, Leçons sur la propagation des ondes, p. 187.

The appearance of shock waves makes the problem considerably more difficult, as a shock wave introduces a non-isentropic change of state of the gas, so that the state can no longer be described by means of a single variable. This takes away the basis of the analysis given in sections **2** — **4**, and the η, ϑ-plane will lose part of its applicability. However, as the gas in its original state of rest was in isentropic equilibrium, a part of the diagram, situated along the boundary O^*C, will retain its applicability, even for indefinitely increasing values of ϑ and of t. Hence the shock waves will delimit certain domains in the η, ϑ-plane and in the ζ, τ-plane (or in the z, t-plane), within which the state of the gas no longer can be described by means of the value of c alone; these domains are embedded in regions to which the original analysis still applies.

In a continuation of this paper we shall consider the initial stage of the first shock wave, which appears to start from the point E.

(To be continued.)

Résumé.

Dans cette communication on considère l'influence de la pesanteur sur l'expansion d'un gas parfait, en supposant qu'il y a mouvement seulement dans la direction verticale. Dans le cas où $k = c_p/c_v = 5/3$, les équations peuvent être intégrées complètement par la méthode de RIEMANN; la fonction auxiliaire v de RIEMANN s'exprime en termes finies; la position z et le temps t caractérisant le mouvement d'un élément de volume sont exprimés par des fonctions rationelles de la vitesse même de l'élément et de la vitesse du son dans cet élément.

Quand on cherche à décrire le mouvement réel d'un élément de volume, on trouve des irrégularités dans certaines parties du champ, qui démontrent l'apparition des ondes de choc.

Resumo.

En ĉi tiu artikolo oni konsideras la influon de la pezo al ekspansio de perfekta gaso, supozante ke la gaso moviĝas en la direkto vertikala. Kiam $k = c_p/c_v = 5/3$ ekvacioj povas esti integrata per la metodo de RIEMANN; la helpfunkcio v de RIEMANN esprimiĝas en termoj finitaj; la pozicio z kaj la tempo t rilataj al moviĝo de elemento de volumeno esprimiĝas per funkcioj racionalaj de la rapido de la elemento mem kaj de la rapido de la sono en la elemento.

Kiam oni provas priskribi la realan moviĝon de elemento oni trovas neregulaĵoj en certaj partoj de la kampo, kiuj pruvas la aperon de ondoj de skuo.