Mathematics. — *On the Theory of Deduction, Part I. Derivation and its Generalizations.* By K. R. POPPER. (Communicated by Prof. L. E. J. BROUWER.)

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In sections I and II of this paper, new primitive rules for derivational logic will be formulated and proofs will be given of the principal rules of derivational logic proposed in three earlier papers [1]), P_1, P_2, and P_3, to which the present paper is a sequel.

In section III, a few concepts of the general theory of derivation will be introduced. Some of these will be used in the subsequent sections in which certain problems concerning the definitions of classical and intuitionist negation will be discussed.

I

The notation

$$\text{"}a_1, \ldots, a_n/b\text{"}$$

will be used, as in the earlier papers mentioned, to express the assertion: "The conclusion b is derivable from the premises $a_1, a_2, \ldots, a_n$." We shall call this the "/-notation". In addition to this notation, we shall use in the present paper another notation to express the same assertion. This new notation, viz.:

$$\text{"}D\,(b, a_1, \ldots, a_n)\text{"}$$

will be called the "*D*-notation".

We shall here use the *D*-notation as our fundamental notation; that is to say, we shall take

$$\text{"}D\,(a_1, \ldots, a_n)\text{"}$$

as our fundamental undefined concept, and we shall assume that the /-notation has been introduced with the help of the definition [2]):

$$(D/) \qquad a_2, \ldots, a_n/a_1 \longleftrightarrow D\,(a_1, \ldots, a_n)$$

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[1]) The following three papers by the author will be referred to: P_1 = *New Foundations for Logic* (MIND, vol. LVI, 1947, No. 223, pp. 193—235); see also the "Additions and Corrections" to this paper, forthcoming in MIND, vol. LVII, No. 225, 1948, and note 1 to P_3. — P_2 = *Logic without Assumptions* (Aristot. Soc. Proceedings, 1947, pp. 251—292). — P_3 = *Functional Logic without Axioms or Primitive Rules of Inference* (Kon. Ned. Akademie van Wetenschappen, Proceedings of the section of Sciences, vol. L, 1947, pp. 1214—1224). In its general approach, terminology, and symbolism, the present paper is based on these earlier papers.

[2]) As in P_1 and P_3, "$\longleftrightarrow$" abbreviates the metalinguistic use of "if and only if". Similarly, "$\rightarrow$"; "&"; and "V" abbreviate "if-then"; "and"; and "or-or both", respectively.

The use of the D-notation has the following advantages. First, it gives rise to a generalisation by suggesting that

$$\text{“}D(a_1, \ldots, a_n)\text{”}$$

may be meaningful for $n = 1$; and we shall indeed find (in section II) that

$$\text{“}D(a)\text{”}$$

turns out to be equivalent to "$\vdash \mathbf{a}$", i.e., to "a is demonstrable" (as defined in the earlier papers). Secondly, the D-notation assimilates our metalinguistic symbolism still more closely to that of HILBERT and BERNAYS. It makes it more obvious that "$D(a_1, \ldots, a_n)$" is an n-termed predicate of our metalanguage; also that it is our only [3]) undefined predicate, since the class of statements (including statement functions) can be defined as the domain of the relation $D(a_1, \ldots, a_n)$. Ultimately, the D-notation throws some light upon the function of the primitive rules which characterize the relation $D(a_1, \ldots, a_n)$, or upon the function of those secondary rules which take the place of the primitive rules if we employ, as in P_3, the method of laying down Basic Definitions. For in the D-notation, it becomes clear that the main function of these rules is to relate n-termed deducibility to 2-termed and $n + r$-termed deducibility (including, more especially, $n + 1$-termed deducibility, in the case of BI. 1–2; cp. also D 3.4).

In P_1, two methods of laying down primitive rules were distinguished, called "Basis I" and "Basis II", respectively. In the present section, a set forming a Basis I will be discussed, and in section II, a rule which (together with two definitions) suffices for Basis II.

The set of primitive rules for Basis I consists of the following two rules [4]), BI. 1 and BI. 2. (It is assumed, for all rules written in the D-notation, that $1 \leq n$ and $1 \leq r$.)

(BI. 1) $$D(a_1, a_2) \longleftrightarrow D(a_1, a_2, a_2).$$

(BI. 2) $$D(a_1, \ldots, a_n) \longleftrightarrow (a_{n+1}) \ldots (a_{n+r})(D(a_{n+r}, a_1, \ldots, a_n) \rightarrow D(a_{n+r}, \ldots, a_2)).$$

We shall first sketch the derivation of the most important rules belonging to the earlier sets.

Putting $n = 2$ and $r = 1$, we obtain from BI. 2:

(1. 1) $$D(a, b) \longleftrightarrow (c)(D(c, a, b) \rightarrow D(c, b)),$$

and from this, by substitution and BI. 1:

(1. 2) $$D(a, a);$$

(1. 3) $$D(a, a, a) \qquad \text{(by 1. 2; BI. 1).}$$

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3) Our undefined symbol "$a\binom{x}{y}$" (cp. P_1, P_2, P_3) is not, of course, a predicate, but a function whose values are statements (including statement functions) and whose arguments are a statement and two individual variables. But this symbol will not be used in the present paper (except in footnote 11, below).

4) In the present paper (in contradistinction to the earlier ones) I use the quantifiers "(a)" and "(Ea)" as abbreviations of the metalinguistic phrases "for every a" and "there exists at least one a such that".

Furthermore, we obtain from BI. 2 alone

$$(1.4) \qquad D(a_1, \ldots, a_n) \rightarrow (D(a_{n+r}, a_1, \ldots, a_n) \rightarrow D(a_{n+r}, \ldots, a_2)).$$

Except for the proof of

$$(1.5) \qquad D(a) \longleftrightarrow (b)\, D(a, b)$$

which we shall give in section II, in order to establish the equivalence of "$D(a)$" and "$\vdash a$" (as defined in the earlier papers), we shall give the following derivations in the /-notation, and we shall base them exclusively upon 1.2′, 1.3′ and 1.4′, i.e., upon 1.2, 1.3 and 1.4 expressed in the /-notation.

$$(1.2') \qquad a/a$$

$$(1.3') \qquad a, a/a$$

$$(1.4') \qquad a_1, \ldots, a_n/b \rightarrow (b, a_1, \ldots, a_n/c \rightarrow a_{n+r}, \ldots, a_1/c).$$

Our contention is that 1.2′ to 1.4′ are equivalent to the Basis I as characterized in the earlier papers.

We obtain, for $n = 1$ and $n + r = m$:

$$(1.41) \qquad a_m, \ldots, a_1/a_1, \qquad (1.2'; 1.3'; 1.4')$$

and therefore

$$(1.411) \qquad a_1, \ldots, a_n/a_n; \qquad (1.41)$$

$$(1.412) \qquad b, a_1, \ldots, a_n/a_n; \qquad (1.411)$$

$$(1.42) \qquad a_{n+r}, \ldots, a_1/a_n, \qquad (1.411; 1.412; 1.4')$$

which can also be written

$$(1.42') \qquad a_1, \ldots, a_n/a_i \quad (1 \leqslant i \leqslant n),$$

and which yields, more especially,

$$(1.421) \qquad b, a_1, \ldots, a_n/b.$$

We further obtain

$$(1.43) \qquad a_1, \ldots, a_n/b \rightarrow a_{n+r}, \ldots, a_1/b; \qquad (1.421; 1.4')$$

$$(1.44') \qquad a_1, \ldots, a_n/b \rightarrow a_n, \ldots, a_1/b; \qquad (1.43)$$

$$(1.45') \qquad a_1, \ldots, a_n/b \rightarrow a_1, \ldots, a_{n+r}/b. \qquad (1.43; 1.44')$$

From 1.4′ and 1.44′ we also obtain:

$$(1.46') \qquad a_1, \ldots, a_n/b \rightarrow (b, a_1, \ldots, a_n/c \rightarrow a_1, \ldots, a_n/c).$$

Rule 1.46′ may be called the "principle of the redundant first premise"; it can be formulated in words: "If the first premise of an inference is derivable from the remaining premises, then its omission does not invalidate the inference." For clearly, 1.46′ is the same as

$$(1.46'') \qquad a_2, \ldots, a_n/a_1 \rightarrow (a_1, \ldots, a_n/b \rightarrow a_2, \ldots, a_n/b),$$

or in our D-notation:

$$(1.46''') \quad D(a_1, \ldots, a_n) \rightarrow (D(b, a_1, \ldots, a_n) \rightarrow D(b, a_2, \ldots, a_n)).$$

Now 1.2′, 1.44′, 1.45′ form, together with the principle I have called [5]) "generalized transitivity principle", that is to say, with 1.47

$$(1.47)\quad (a_1, \ldots, a_n/b_1 \,\&\, \ldots \,\&\, a_1, \ldots, a_n/b_m) \rightarrow (b_1, \ldots, b_m/c \rightarrow a_1, \ldots, a_n/c)$$

a set of rules which constitutes a form of Basis I.

Our contention is that, in the presence of 1.44′ and 1.45′, the generalized transitivity principle 1.47 can be derived from 1.46′.

The proof of our contention will be given in the /-notation.

We first obtain, from 1.44′ and 1.45′

$$(1.48)\qquad b_1, \ldots, b_m/c \rightarrow a_1, \ldots, a_n, b_1, \ldots, b_m/c.$$

In the presence of this principle, we only need 1.45′ for our proof; for we need only to prove

$$(1.49)\quad (a_1, \ldots, a_n/b_1 \,\&\, \ldots \,\&\, a_1, \ldots, a_n/b_m) \rightarrow (a_1, \ldots, a_n, b_1, \ldots, b_m/c \rightarrow a_1, \ldots, a_n/c)$$

in order to obtain 1.47, since, clearly, 1.47 can be obtained by 1.48 from 1.49. We now sketch the proof of 1.49 which makes use of 1.45′ only (apart, of course, of 1.46′).

We observe that 1.46′ may be written:

$$(1.491)\quad \begin{aligned} a_1, \ldots, a_n, b_1, \ldots, b_{m-1}/b_m \rightarrow (a_1, \ldots, a_n, b_1, \ldots, b_m/c \rightarrow \\ \rightarrow a_1, \ldots, a_n, b_1, \ldots, b_{m-1}/c). \end{aligned}$$

Applying to this 1.45′, we obtain

$$(1.492)\quad a_1, \ldots, a_n/b_m \rightarrow (a_1, \ldots, a_n, b_1, \ldots, b_m/c \rightarrow a_1, \ldots, a_n, b_1, \ldots, b_{m-1}/c).$$

From this we obtain, by substitution,

$$(1.493)\quad a_1, \ldots, a_n/b_{m-1} \rightarrow (a_1, \ldots, a_n, b_1, \ldots, b_{m-1}/c \rightarrow a_1, \ldots, a_n, b_1, \ldots, b_{m-2}/c).$$

Combining 1.412 and 1.413, we get:

$$(1.494)\quad \begin{aligned} (a_1, \ldots, a_n/b_m \,\&\, a_1, \ldots, a_n/b_{m-1}) \rightarrow (a_1, \ldots, a_n, b_1, \ldots, b_m/c \rightarrow \\ \rightarrow a_1, \ldots, a_n, b_1, \ldots, b_{m-2}/c). \end{aligned}$$

We can continue this procedure for $m - 2$ steps, i.e., until the premises b_i are exhausted. The result is 1.49.

It may be remarked that a Basic Definition of "$b \wedge c$" (analogous and equivalent to DB 2′ in P_3) may be obtained by making use of 1.2′; 1.3′; and 1.4′; this Definition is DB 2^{I}:

$$(\mathrm{DB}\,2^{\mathrm{I}})\quad \begin{aligned} a//b \wedge c \longleftrightarrow (a_1) \ldots (a_{n+r})((a/a_n \longleftrightarrow b, c/a_n) \,\&\, (a_1, \ldots, a_n/b \rightarrow \\ \rightarrow (b, a_1, \ldots, a_n/c \rightarrow a_{n+r}, \ldots, a_1/c)) \,\&\, b/b \,\&\, b, b/b). \end{aligned}$$

This definition may be simplified by replacing "$b/b \,\&\, b, b/b$" by "$a_1, \ldots, a_n/a_1$"; or else, by "$a_1, \ldots, a_m/a_1$" together with the restriction "$(1 \leq m \leq 2)$"; but even in the latter case, the simplified definition is still a little stronger than necessary, that is to say, stronger than DB 2^{I}.

We have derived 1.2′; 1.44′; 1.45′ and 1.47 from our BI. 1 and BI. 2;

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[5]) Cp. P_1, pp. 199 f., rule 2.5 g.

the derivation of BI. 1 and BI. 2 from the rules mentioned, and therefore the equivalence of the two sets of rules for Basis I, is trivial [6]).

II

We now proceed to Basis II. This is not equivalent to Basis I, but to Basis I combined with the two definitions

(D B 1) $$a//b \longleftrightarrow (c)\,(c/a \longleftrightarrow c/b)$$

and

(D I $\wedge$) $$a//b \wedge c \longleftrightarrow (d)\,(a/d \longleftrightarrow b, c/d).$$

(The second of these definitions is incorporated in DB 2^{I}; cp. the end of the foregoing section.)

The simplest form of Basis II known to me consists of the *one primitive rule* BII (built in analogy to BI. 2), together with the two definitions, DBI and DII $\wedge$.

(B II) $$D(a_1, \dots, a_n) \longleftrightarrow (a_{n+1}) \dots (a_{n+r})\,(D(a_{n+r}, a_1) \rightarrow D(a_{n+r}, \dots, a_2)).$$

The difference between BII and BI. 2 consists in writing "$D(a_{n+r}, a_1)$" instead of "$D(a_{n+r}, a_1, \dots, a_n)$"; and although this difference makes it possible to derive BI.1 from BII, it makes it impossible to derive from BII the generalized transitivity principle without at least assuming, in place of DI $\wedge$, the more complicated definition

(D II $\wedge$) $$a//b\wedge c \longleftrightarrow (a_1) \dots (a_n)\,(a_1, \dots, a_n/a \longleftrightarrow (a_1, \dots, a_n/b \;\&\; a_1, \dots, a_n/c)).$$

If this definition is incorporated, together with BII, into a Basic

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[6]) The proof that we can obtain 4.7 (called in P_1 "generalized transitivity principle") from 4.6′ solves the problem of constucting a *complete* basis for the *general* theory of derivation — such as Basis I, as opposed to Basis II — without having to use 4.7 as primitive. The objection against 4.7 is that it makes use of an unspecified number of conjunctive components in its antecedent; this may be considered as introducing a new metalinguistic concept — something like an infinite product. The problem of avoiding 4.7 was discussed, but not solved, in P_1. The lack of a solution led me there to construct Basis II, the need for which, as it were, has now disappeared. But Basis II turend out to be of interest in itself; cp. especially the derivation of 2.2 from B II, below, and the remark (in the paragraph before the last, of section II) on the definition-like character of B II.

The solution mentioned makes it possible to construct Basic Definitions like those of P_3, but with Basis I as underlying basis. (P_3 uses Basis II for this purpose.) Such Basic Definitions — cp. DB 2^{I} — may define conjunction, but any other compound serves just as well, since Basis I, as opposed to Basis II, is neutral with regard to the various compounds. — It may be remarked that we can by various methods reduce the number of Basic Definitions form three to two, viz., to DB 1 plus a Basic Definition which, apart from incorporating the definition of some compound C and the rules of the basis chosen, also includes the six rules pertaining to substitution (i.e. P_3, rules 5.71 to 5.76). If Basis I is chosen, C may be one of the quantifiers, which would make this procedure more "natural".

Definition, then we obtain the following Basic Definition DB 2^{II}, which is equivalent to DB 2^{I}:

(DB 2^{II}) $$a//b \wedge c \longleftrightarrow (a_1) \ldots (a_{n+r})((a_1, \ldots, a_n/a \longleftrightarrow (a_1, \ldots, a_n/b \mathbin{\&} a_1, \ldots, a_n/c)) \mathbin{\&} (a_1, \ldots, a_n/b \rightarrow (b/c \rightarrow a_{n+r}, \ldots, a_1/c)) \mathbin{\&} b/b).$$

(This is equivalent to DB 2′, given in P_3).

I shall confine myself to showing that we can obtain, from BII, the rules 1.42′; 1.44′; 1.45′; and, besides, 2.46′, i.e.:

(2. 46′) $$a_1, \ldots, a_n/b \rightarrow (b/c \rightarrow a_1, \ldots, a_n/c)$$

which is, in Basis II, the transitivity principle that corresponds to the (stronger) principle 1.46′ of Basis I.

The main interest of BII is that, by putting $n = 2$ and $r = 1$, we obtain

(2. 1) $$D(a, b) \longleftrightarrow (c)(D(c, a) \rightarrow D(c, b))$$

which, without any further help, leads to

(2. 2) $$D(a, a);$$

for "$D(c, a) \rightarrow D(c, a)$" must be true, whatever the meaning of "$D(a_1, a_2)$" may be.

We obtain from BII immediately

(2. 4) $$D(a_1, \ldots, a_n) \rightarrow (D(a_{n+r}, a_1) \rightarrow D(a_{n+r}, \ldots, a_2)).$$

We thus have, in the /-notation:

(2. 2′) $$a/a$$

(2. 4′) $$a_1, \ldots, a_n/b \rightarrow (b/c \rightarrow a_{n+r}, \ldots, a_1/c).$$

Putting here $n = 1$ and $n + r = m$, we obtain:

(2. 41) $$a_m, \ldots, a_1/a_1, \qquad (2.\,2'; 2.\,4')$$

and from this

(2. 3′) $$a, a/a.$$

We further obtain:

(2. 43) $$a_1, \ldots, a_n/b \rightarrow a_{n+r}, \ldots, a_1/b; \qquad (2.\,2'; 2.\,4')$$

(2. 44′) $$a_1, \ldots a_n/b \rightarrow a_n, \ldots, a_1/b; \qquad (2.\,43)$$

(2. 45′) $$a_1, \ldots, a_n/b \rightarrow a_1, \ldots, a_{n+r}/b; \qquad (2.\,43; 2.\,44')$$

(2. 46′) $$a_1, \ldots, a_n/b \rightarrow (b/c \rightarrow a_1, \ldots, a_n/c). \qquad (2.\,4'; 2.\,44')$$

The principle

(2. 42) $$a_1, \ldots, a_{n+r}/a_n,$$

or

(2. 42′) $$a_1, \ldots, a_n/a_i \qquad (1 \leqslant i \leqslant n),$$

is obtained from 2.41 together with 2.45′.

Since 2.42′; 2.44′; 2.45′ are identical with the rules 1.42′; 1.44′; 1.45′, we have derived all the rules we wanted.

We now proceed to show that

$$(2.5) \qquad D(a) \longleftrightarrow (b)\, D(a,b),$$

which establishes the equivalence of "$D(a)$" and "$\vdash a$".

Putting in BII $n = 1$ and $r = 2$, we obtain

$$(2.51) \qquad D(a) \longleftrightarrow (b)(c)(D(c,a) \rightarrow D(c,b)),$$

and further

$$(2.52) \qquad D(a) \rightarrow (D(a,a) \rightarrow D(a,b)). \qquad \text{(by 2.51)}$$

From this and 2.2 we obtain

$$(2.53) \qquad D(a) \rightarrow D(a,b)$$

and thus

$$(2.54) \qquad D(a) \rightarrow (b)\, D(a,b).$$

In order to obtain from 2.51 the converse of 2.54, we need only

$$(2.55) \qquad (b)\, D(a,b) \rightarrow (b)(c)(D(c,a) \rightarrow D(c,b)),$$

which is an immediate consequence of the transitivity principle in its simplest form, i.e. of

$$(2.551) \qquad D(a,b) \rightarrow (D(c,a) \rightarrow D(c,b)).$$

This concludes the proof of 2.5. In order to show that 1.5, which is the same as 2.5, can be derived from BI. 1 and BI. 2, we first consider that we can obtain 2.51 from BI. 2 in the same way as from BII. Since we have also 2.2, i.e., 1.2, we obtain 2.45; and since we also have 2.55, which is obtainable from 1.47 by $n = 1$ and $m = 1$, we obtain 2.5, i.e. 1.5.

Concerning BII, it may be remarked that, in view of the method of deriving 2.2, it very closely resembles DB 1 (cp. P_3). It might therefore, perhaps, be described as a "quasi-definition"; for it defines, as it were, the n-termed relation $D(a_1, \ldots, a_n)$ for $1 \leq n$, in terms of the two-termed relation $D(a_1, a_2)$ and the $n + r$-termed relation $D(a_1, \ldots, a_{n+r})$, for $1 \leq r$.

Concerning our two Basic Definitions DB 2^{I} and DB 2^{II}, it may be noted that they can be made homogeneous (in the sense of P_3) simply by incorporating BI. 1 and BI. 2, or BII, respectively, as they stand. In this case, a second set of quantifiers appears within the right hand side of the definitions, which makes them more complicated; but we achieve, besides homogeneity, the derivability of 1.5 and 2.5. If, on the other hand, we wish to avoid this set of quantifiers within the right hand side, then 1.5 and 2.5 cannot be derived, and we have to revert to our earlier method of defining "$\vdash a$". But with this, the main advantage of the D-notation disappears. (Thus it seems that there is not much point in formulating DB 2^{I} and DB 2^{II} in the D-notation.)

III

We can distinguish between the *general* and the *special* theories of derivation (and proof). The special theories are the theories of the compounds, quantifiers, and modalities; closely connected with them are the theories obtained by adding existential postulates to our bases. (Such postulates may demand, for example, the existence of provable or refutable statements, or perhaps the existence of a conditional statement c to every pair of statements a and b, that is to say, of a weakest statement which, together with a, yields b). The general theories are erected on either BI. 1 and BI. 2, or on BII, without, however, introducing compound argument variables for $D(a_1, \ldots, a_n)$, or existential postulates. But such problems as the equivalence of existential postulates fall within the scope of the general theory.

In the present section, we shall first sketch that part of the general theory which is concerned with the complementarity or exhaustiveness (or disjunctness) and with the contradictoriness or exclusiveness (or conjunctness) of n statements. We shall write

$$\text{"} \vdash (a_1, \ldots, a_n) \text{"}$$

for "the statements $a_1, \ldots, a_n$, taken together, are complementary or exhaustive" and

$$\text{"} \urcorner (a_1, \ldots, a_n) \text{"}$$

for "the staments $a_1, \ldots, a_n$, taken together, are contradictory or exclusive". The definitions are:

(D 3. 1) $\qquad \vdash (a_1, \ldots, a_n) \longleftrightarrow (b)\,(c)\,((a_1/c \mathbin{\&} \ldots \mathbin{\&} a_n/c) \rightarrow b/c).$

(D 3. 2) $\qquad \urcorner (a_1, \ldots, a_n) \longleftrightarrow (b)\,(c)\,((b/a_1 \mathbin{\&} \ldots \mathbin{\&} b/a_n) \rightarrow b/c).$

An alternative and equivalent way of defining "$\urcorner$" is this:

(D 3. 2′) $\qquad \urcorner (a_1, \ldots a_n) \longleftrightarrow (b)\,(a_1, \ldots, a_n/b).$

Introducing the convention that the brackets after "$\vdash$" and "$\urcorner$" may be omitted, we obtain, for $n = 1$,

(3. 1) $\qquad \vdash a \longleftrightarrow (b)\,(b/a)$; $\qquad$ (2.2′; 2.46′; D 3. 1)

(3. 2) $\qquad \urcorner\, a \longleftrightarrow (b)\,(a/b)$ $\qquad$ (2.2′; 2.46′, D 3. 2)

This shows that, for $n = 1$, the two concepts coincide with demonstrability and refutability respectively, as defined in my earlier papers; furthermore, that for $n = 1$, $D(a_1, \ldots, a_n)$ and $\vdash (a_1, \ldots, a_n)$ coincide.

The two concepts may be generalized or relativized by introducing the following definition. (We assume $0 \leq n$; $0 \leq m$; $1 \leq n + m$.)

(D 3. 3) $\qquad (a_1, \ldots, a_n) \vdash (b_1, \ldots, b_m) \longleftrightarrow (c)\,(d)\,((b_1/d \mathbin{\&} \ldots \mathbin{\&} b_m/d) \rightarrow$
$\qquad\qquad \rightarrow ((c/a_1 \mathbin{\&} \ldots \mathbin{\&} c/a_n) \rightarrow c/d)).$

An alternative formulation can be obtained as before (cp. D3. 2′):

(D 3. 3′) $\qquad (a_1, \ldots, a_n) \vdash (b_1, \ldots, b_m) \longleftrightarrow (c)\,((b_1/c \mathbin{\&} \ldots \mathbin{\&} b_m/c) \rightarrow a_1, \ldots, a_n/c).$

We again introduce the convention that brackets can be omitted, before and after "$\vdash$".

The concept defined by D3. 3 may be called "relative demonstrability" (or "relative refutability"), and "$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$" may be read: "The statements $b_1, \ldots b_m$ are complementary relative to the demonstrability (or self-complimentarity) of all the statements $a_1, \ldots, a_n$." For it is clear, from D3. 3 and D3. 3', that every one of the a_i which stand in front of "$\vdash$" may be omitted if it is demonstrable (or self-complementary), without affecting the force of the whole expression. This shows that for $n = 0$ relative demonstrability degenerates into complementarity (or demonstrability) as defined by D3. 1.

A similar consideration shows that every b_i which stands after "$\vdash$" may be omitted if it is refutable or self-contradictory. If all the b_i are so omitted, we obtain, for $m = 0$, an expression which is equivalent to D3. 2. That is to say, we obtain:

(3. 3) $$a_1, \ldots, a_n \vdash \longleftrightarrow \neg\, a_1, \ldots, a_n.$$

Thus "$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$" may be read, alternatively: "The statements $a_1, \ldots, a_n$ are contradictory, relative to the refutability (or self-contradictoriness) of all the statements $b_1, \ldots, b_m$."

For certain purposes — especially if we wish to emphasize the duality or symmetry between "$\vdash$" and "$\neg$" — the use of "$(\ldots) \vdash$" turns out to be preferable to that of "$\neg(\ldots)$".

It should be noted that, for $m = 1$, relative demonstrability degenerates, as it were, into derivability; that is to say, we have

(3. 4) $$a_1, \ldots, a_n \vdash b \longleftrightarrow a_1, \ldots, a_n/b.$$

If, however, $m > 1$, then relative demonstrability means something else. (It is thus a further generalisation of "/".) Its meaning can be intuitively explained by remarking that, whenever disjunction is available,

(3. 5) $$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \longleftrightarrow a_1, \ldots, a_n/b_1 \vee b_2 \vee \ldots \vee b_m,$$

that is to say, the intuitive meaning (even if disjunction is not available) is the same as that of the derivability of the disjunction of the statements standing to the right of "$\vdash$" from the statements standing to the left of "$\vdash$". Relative demonstrability is thus about the same as GENTZEN's "Sequences" or CARNAP's "Involution" [7].

[7]) Cp. G. GENTZEN, *Untersuchungen über das logische Schliessen I & II* (Mathematische Zeitschrift vol. 39, 1935); and RUDOLF CARNAP, *Formalization of Logic*, 1943, esp. § 32, pp. 151 ff. — My remark that relative demonstrability is "*about*" the same as these earlier concepts alludes to the following differences. (1) It is not quite clear whether GENTZEN's concept is, like ours, a metalinguistic *predicate* asserting some kind of inference (it looks as if his horizontal stroke rather than his sequences were meant in this way) or a *name of an object-linguistic notation*. (2) CARNAP's concept on the other hand (which is not open to any objection of this kind) may be described as a *generalization* of "$\ldots \vdash \ldots$", in so far as *classes* of statements are admitted as arguments, besides statements. (3) Attention may be drawn, furthermore, to the fact that GENTZEN identifies the difference between classical and intuitionist logic with the difference (I am using my own terminology) between permitting m to be greater than *1*, and taking *1* as the upper limit of m. This does not agree with our results, and seems to be due only to GENTZEN's choice of his primitive rules for negation. We operate freely with $m \geq 2$, even within intuitionist logic.

Once we have adopted D3. 3 or D3. 3′, we obtain D3. 1 and 3. 1 as theorems, as well as theorems corresponding to D3. 2 and 3. 2 (with "(...) $\vdash$" instead of "$\urcorner$ (...)"); and we can, if we wish, even dispense with the further use of "/", in view of 3. 5.

In the next section, we shall make use of "$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$" more especially for $0 \leq n \leq 2$ and $0 \leq m \leq 2$. Implicit use will be made of the obvious theorems:

(3. 61) $$a \vdash a, b$$

(3. 62) $$b \vdash a, b$$

(3. 63) $$a \vdash b, c \rightarrow (c \vdash d \rightarrow (b \vdash d \rightarrow a \vdash d)).$$

(In the expression in brackets, "$\vdash$" may, of course, be here replaced by "/".)

The rules 6. 31 to 6. 33 show that we may characterize "$a \vee b$" by a characterizing rule which is precisely analogous to and dual of the simplest characterizing rule [8]) for "$a \wedge b$":

(3. 71) $$a \wedge b \vdash c \longleftrightarrow a, b \vdash c$$

(3. 72) $$a \vdash b \vee c \longleftrightarrow a \vdash b, c.$$

Similarly, we can characterize a new compound, called "anti-conditional" and denoted by "$a \not> b$", in precise analogy to (and as the dual of) $a > b$:

(3. 81) $$a \vdash b > c \longleftrightarrow a, b \vdash c$$

(3. 82) $$a \not> b \vdash c \longleftrightarrow a \vdash b, c.$$

"$a \not> b$" is, intuitively (and in the presence of classical negation), a name for the negation of $a > b$, or for the conjunction of a and the negation of b. But in the absence of classical negation, its meaning is less familiar. (Cp. 5. 32 and 5.42, and note 15, below.)

These characterizing rules can all be transformed into explicit definitions; for example, 3. 71 into DI $\wedge$ (quoted at the beginning of section II).

All the rules and definitions given apply equally to closed and open statements. But as long as we confine ourselves to closed statements, "... $\vdash$ ..." may, intuitively, be interpreted as asserting that at least one of the statements on the right of "$\vdash$" is true, provided all statements on the left are true. It should be noted, on the other hand, that two or more open statements — e.g. an open statement and its classical negation — may be complementary (or relative complementary) in our sense even though their A-closures, i.e. the results of universal quantification, are not complementary (or relative complementary). This is due to fact that A-closures and E-closures (i.e. results of existential quantification), are duals of each other, like conjunction and disjunction. For it follows from our definitions that every valid $\vdash$-formula remains valid if any of its statements to the left of "$\vdash$" are replaced by their A-closures, and any of its statements to the

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[8]) Cp. P_1 p. 215, rule 4.1; P_3, 5.2. — See also DI $\wedge$ (in section II, above).

right by their E-closures; and that, if there are no open statements to the left or to the right respectively, the disjunction or conjunction respectively of the remaining open statements may be replaced by either its A-closure or its E-closure.

So far we have considered generalizations of the idea of derivability; but in view of 3.4, which may be taken as a definition of derivability in terms of relative demonstrability, and in view of D3.3, which defines the latter with the help of only *two-termed derivability* (or inference from *one* premise), it is clearly possible to use two-termed derivability, "$D(a, b)$", characterized by (ordinary) reflexitivity and transitivity, as our *sole undefined predicate*. That is, we may introduce "$D(a, b)$" by the *one* rule 2.1, or, for example, by the following rule (constituting a "basis III"):

(BIII) $$D(a, b) \longleftrightarrow (c)(D(b, c) \rightarrow D(a, c)).$$

The explicit definition of "$D(a_1, \ldots, a_n)$" in terms of "$D(a, b)$" becomes, considering D3.3 and 3.4:

(D 3.4) $$D(a_1, \ldots, a_n) \longleftrightarrow (b)((D(a_n, b) \,\&\ldots\&\, D(a_2, b)) \rightarrow D(a_1, b)).$$

But this formula turns out to yield BIII. It therefore, surprisingly enough, suffices (without BIII) for a Basis I.