Mathematics. — *A note to a paper by* C. J. BOUWKAMP. By W. T. TUTTE. (Communicated by Prof. J. G. VAN DER CORPUT.)

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In a paper published recently in these Proceedings (On the dissection of rectangles into squares, III (Vol. 50, pp. 72—78)), C. J. BOUWKAMP called attention to certain pairs of "conformal squared rectangles". We use here the terminology employed by Dr BOUWKAMP. Following him we also denote by A the paper of R. L. BROOKS, C. A. B. SMITH, A. H. STONE and W. T. TUTTE entitled "The dissection of rectangles into squares" (Duke Mathematical Journal, Vol. 7 (1940), pp. 312—340).

In particular Dr BOUWKAMP pointed out three pairs of conformal rectangles in which the two members of each pair are of different orders. In his notation they are (*i*) IX, 130, *c* and XII, 585, *f*; (*ii*) X, 224, *a* and XIII, 1008, *b*; and (*iii*) X, 224, *b* and XIII, 1008, *e*. He noted that, for pairs (*ii*) and (*iii*), "upon transformation on the same size, four common elements are found in each case" [1]).

There is an interesting electrical interpretation of these results.

If P_r, P_s, P_t and P_u are vertices of an electrical network N, then we denote by $[rs, tu]$ the fall of potential from P_t to P_u given by KIRCHHOFF'S Laws in the case in which a current equal to $C(N)$, the complexity [2]) of N, enters the network at P_r and leaves at P_s. This is in accordance with the notation of A.

Let X be the set of integers $(1, 2, 3)$ and Y the set $(4, 5, 6)$. Let (p, q) and (r, s) be each an ordered pair of distinct integers. We suppose that for each of these pairs the two integers belong either both to X or else both to Y. If all four of p, q, r and s belong to the same set X or Y we write

$$e_{pq,\,rs} = 1,$$

but if p and q belong to one set X or Y, and r and s to the other, we write

$$e_{pq,\,rs} = -1.$$

Consider the two electrical networks N_1 and N_2 shown in figs 1 and 2 respectively. It may readily be verified, either by evaluating determinants or by counting trees [3]) that the complexities of N_1 and N_2, $C(N_1)$ and $C(N_2)$, are 6 and 27 respectively.

Taking the conductance of each wire to be 1, we consider the distri-

[1]) In pair (*i*), three common elements are found (not two as stated by Dr. BOUWKAMP).

[2]) A, p. 315; C. J. BOUWKAMP, Paper II (These Proceedings, Vol. 50, pp. 58—71), p. 59.

[3]) A, pp. 317—318.

butions of currents in the two networks when, in each case, a current I enters at P_1 and leaves at P_2. It is at once evident on inspection of the

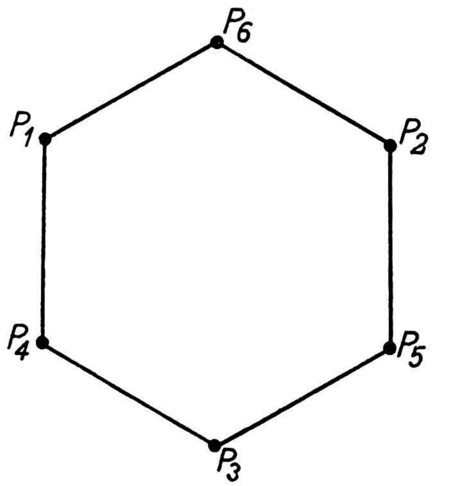

Fig. 1. Network N_1 of complexity 6.

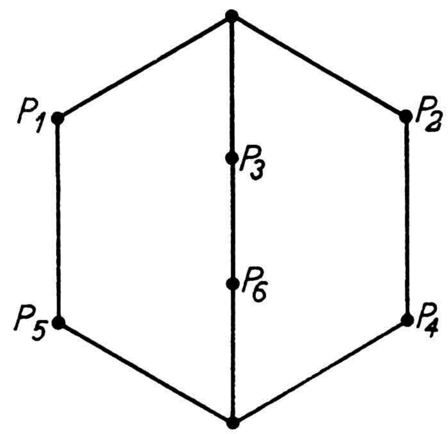

Fig. 2. Network N_2 of complexity 27.

diagrams that if u and v are two integers, either both in X or else both in Y, then the ratio

$$\frac{\text{Fall of potential from } P_u \text{ to } P_v \text{ in } N_1}{\text{Fall of potential from } P_u \text{ to } P_v \text{ in } N_2} = e_{12,uv}.$$

We transform to "full flows" [4]) by multiplying the currents in N_1 and N_2 by $C(N_1)/I$ and $C(N_2)/I$ respectively. The above equation then becomes

$$\frac{[12, uv]_1}{[12, uv]_2} = \frac{C(N_1)}{C(N_2)} e_{12,uv} = \tfrac{2}{9} e_{12,uv}.$$

(Quantities referring to N_1 and N_2 are distinguished by suffices 1 and 2 respectively.)

By using the symmetry of the networks we can generalize this at once to

$$\frac{[pq, rs]_1}{[pq, rs]_2} = \tfrac{2}{9} e_{pq,rs}, \quad \ldots \ldots \quad (1)$$

where, for each pair (p, q) or (r, s) the two members belong either both to X or both to Y.

It is a remarkable fact that equation (1) remains valid when the networks N_1 and N_2 are modified [5]) by any sequence of operations of the following kind; we choose two integers u and v belonging to the same set X or Y, and join P_u and P_v by a wire of arbitrarily chosen conductance c in each network (c having the same value in each network). Moreover

[4]) A, § (2.13); C. J. BOUWKAMP, Paper II, p. 62.

[5]) The modified networks will be denoted by N'_1 and N'_2.

the complexities $C(N'_1)$ and $C(N'_2)$ remain in the ratio $2:9$. There is no difficulty in proving this by induction, using the formulae of section (2.3) of A.

If therefore, after we have added the new wires, we can derive a squared rectangle from each network by taking P_1 and P_2 as poles, then the two rectangles will be conformal, and corresponding full sides will be in the ratio $2:9$ [6]). Further, those elements in the two rectangles which are represented by corresponding added wires will also be in the ratio $2:9$. Such elements thus become equal when the two conformal rectangles are made equal in size.

If the added wires join the pairs (P_5P_6), (P_4P_5) and (P_2P_3) we obtain the pair of conformal perfect rectangles IX, 130, *c* and XII, 585, *f*. On transforming these to the same size we find just three pairs of equal squares, corresponding to these three added wires.

If the added wires join the pairs (P_1P_3), (P_2P_3), $P_4P_5)$ and (P_4P_6) we obtain the pair X, 224, *a* and XIII, 1008, *b*. If the added wires join the pairs (P_2P_3), (P_4P_5), (P_5P_6) and (P_4P_6) we obtain the pair X, 224, *b* and XIII, 1008, *e*. The four common elements noted in each case by Dr BOUWKAMP when the two rectangles are made equal in size thus all correspond to "added" wires.

In each of these pairs, the full sides of the rectangle of lower order have a common factor 2, and the full sides of the other rectangle have a common factor 9 (because of the ratio 2/9 in formula (1)). We could therefore predict from the reduction theorems of A (pp. 325—326) that the reductions of the two rectangles divide by 2 and 3 respectively, without actually working out the values of the elements.

Similar conformal pairs, which are however compound [7]), can be obtained by replacing the "added wires" by polar networks of squared rectangles, which can be regarded as resistances whose value is (in general) different from unity.

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[6]) In each rectangle the full sides are $C(N'_i)$ and $[12, 12]_i$ $(i = 1, 2)$. (By A, § (2.13)).

[7]) A, § (5.21); C. J. BOUWKAMP, Paper I (These Proceedings, Vol. 48, pp. 1176—1188), p. 1177.