Mathematics. — *Non-homogeneous binary quadratic forms.* II. By P. VARNAVIDES. (Communicated by Prof. J. G. VAN DER CORPUT.)

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7. **Lemma 7.** *There exists a number* $n_0 = n_0(\delta)$ *depending only on* δ *such that, if* $a \geqq \tau/\theta$, *then there is exactly one integer* n *such that*

$$0 < n \leqslant n_0, \qquad (44)$$

$$\frac{1}{\xi_n} < a < \frac{1}{\xi_{n+2}}, \qquad (45)$$

and

$$\frac{1}{\eta_n} > \beta \geqslant \frac{1}{\eta_{n+1}}, \qquad (46)$$

where

$$\xi_n = \frac{1+\tau^{-n+1}}{\sqrt{2}\,\tau} \text{ and } \eta_n = \frac{1-\tau^{-n}}{\sqrt{2}} \qquad (47)$$

are integers of $k(\sqrt{2})$.

Proof. That ξ_n and η_n are integers of $k(\sqrt{2})$ follows from the congruence (21). We consider the sequence of numbers $(\eta_n)^{-1}$. As n increases from the initial value 1 and tends to infinity, this sequence decreases strictly from the initial value τ and tends to the limit $\sqrt{2}$. Thus, if n_0 is sufficiently large

$$\frac{1}{\eta_{n_0}} < \sqrt{2} + \tau^{-2}\delta.$$

Consequently, by Lemma 6,

$$\frac{1}{\eta_{n_0}} < \sqrt{2} + \tau^{-2}\delta < \beta < \sqrt{(2\tau)} < \tau = \frac{1}{\eta_1},$$

and there is just one integer $n \leq n_0$ for which (44) and (46) are satisfied. We have to prove that (45) is satisfied for this value of n.

Now using (17), (46) and (47)

$$a < \frac{2\tau}{\beta} \leqslant 2\tau\,\eta_{n+1} = \sqrt{2}\,\tau\,(1-\tau^{-n-1})$$

$$< \frac{\sqrt{2}\,\tau}{1+\tau^{-n-1}} = \frac{1}{\xi_{n+2}},$$

and so one of the inequalities (45) is satisfied. To prove the remaining inequality we use (15) with $\xi = 1$. If we neglect ε we obtain

$$\alpha \geqslant 1 + \frac{1}{\beta-1} = \frac{\beta}{\beta-1} > \frac{1}{1-\eta_n}$$
$$= \frac{\sqrt{2}\,\tau}{1+\tau^{-n+1}} = \frac{1}{\xi_n},$$

by using (46) and (47). When we do not neglect ε we obviously obtain an inequality of the form

$$\alpha \geqslant \frac{1}{\xi_n} - \nu_n\,\varepsilon, \qquad (48)$$

where ν_n is a positive number depending only on n. Using (15) with $\xi = \xi_n$, we have

$$|(\alpha\,\xi_n - 1)(\beta\,\xi_n' - 1)| \geqslant 1 - \varepsilon,$$

so that, by (37),

$$|\alpha\,\xi_n - 1|\,\{\sqrt{(2\tau)}\,|\xi_n'| + 1\} \geqslant 1 - \varepsilon. \qquad (49)$$

But, if α were less than or equal to $1/\xi_n$, the inequalities (48) and (49) would give a contradiction, provided ε was less than some positive number depending only on n_0. Hence $\alpha \geq 1/\xi_n$, and the lemma is proved.

Lemma 8. *If $\alpha \geq \tau/\theta$, the inequalities* (44), (45) *and* (46) *are satisfied for just one odd integer n.*

Proof. Suppose, if possible, that (44), (45) and (46) are satisfied for some even integer n. Then

$$\eta_n' = \frac{(-\tau)^n - 1}{\sqrt{2}} = \frac{\tau^n - 1}{\sqrt{2}},$$
$$\eta_{n+1}' = \frac{(-\tau)^{n+1} - 1}{\sqrt{2}} = -\frac{\tau^{n+1}+1}{\sqrt{2}}.$$

By (15) with $\xi = \eta_n'$ and with $\xi = \eta_{n+1}'$

$$(\eta_n'\,\alpha - 1)(1 - \eta_n\,\beta) \geqslant 1 - \varepsilon,$$
$$(|\eta_{n+1}'|\,\alpha + 1)(\eta_{n+1}\,\beta - 1) \geqslant 1 - \varepsilon,$$

the four factors on the left hand sides being positive, by (46) and our assumption that $\alpha \geq \tau/\theta$. Thus

$$1 - \eta_n\,\beta \geqslant \frac{1-\varepsilon}{\eta_n'\,\alpha - 1},$$

$$\eta_{n+1}\,\beta - 1 \geqslant \frac{1-\varepsilon}{|\eta_{n+1}'|\,\alpha + 1},$$

Eliminating β and using (45) we obtain

$$\left.\begin{aligned}
\frac{\eta_{n+1}-\eta_n}{1-\varepsilon} &\geqslant \frac{\eta_{n+1}}{\eta_n' a - 1} + \frac{\eta_n}{|\eta_{n+1}'| a + 1} \\
&\geqslant \frac{\eta_{n+1}\,\xi_{n+2}}{\eta_n' - \xi_{n+2}} + \frac{\eta_n\,\xi_{n+2}}{|\eta_{n+1}'| + \xi_{n+2}} \\
&= \frac{(1-\tau^{-n-1})(1+\tau^{-n-1})}{\sqrt{2}\,\{\tau(\tau^n-1)-(1+\tau^{-n-1})\}} + \frac{(1-\tau^{-n})(1+\tau^{-n-1})}{\sqrt{2}\,\{\tau(\tau^{n+1}+1)+(1+\tau^{-n-1})\}} \\
&= \frac{(1-\tau^{-2n-2})}{\sqrt{2}\,\{\tau^{n+1}-\tau-1-\tau^{-n-1}\}} + \frac{(1-\tau^{-n})}{\sqrt{2}\,\{\tau^{n+2}+1\}} \\
&> (^1/\sqrt{2})\,\tau^{-n-1}(1-\tau^{-2n-2})(1-\tau^{-n})^{-1} + (^1/\sqrt{2})\,\tau^{-n-2}(1-\tau^{-n})(1+\tau^{-n-2})^{-1} \\
&> (^1/\sqrt{2})\,\tau^{-n-1}(1-\tau^{-2n-2})(1+\tau^{-n}) + (^1/\sqrt{2})\,\tau^{-n-2}(1-\tau^{-n})(1-\tau^{-n-2}) \\
&= (^1/\sqrt{2})\,\tau^{-n-1}(1+\tau^{-1}+\tau^{-n}-\tau^{-n-1}-\tau^{-n-3}-\tau^{-2n-2}+\tau^{-2n-3}-\tau^{-3n-2}) \\
&= \tau^{-n-1} + (^1/\sqrt{2})\,\tau^{-2n-1}\,[1-\tau^{-1}-\tau^{-3}-\tau^{-n-3}(\tau-1)-\tau^{-2n-2}]
\end{aligned}\right\} \quad (50)$$

As n is an even positive integer the expression in the square brackets exceeds

$$1-\tau^{-1}-3\,\tau^{-3} = 23-16\sqrt{2},$$

which is positive. Thus provided ε is smaller than a certain number depending only on n_0,

$$\eta_{n+1}-\eta_n > \tau^{-n-1}$$

But

$$\eta_{n+1}-\eta_n = \frac{1-\tau^{-n-1}}{\sqrt{2}} - \frac{1-\tau^{-n}}{\sqrt{2}} = \tau^{-n-1}.$$

This contradiction proves that the unique integer n, for which (44), (45) and (46) are satisfied, is odd.

Lemma 9. *If $\alpha \geq \tau/\theta$, there is exactly one odd integer n such that the inequalities* (44) *and* (46) *are satisfied and*

$$\frac{1}{\xi_n} < \alpha < \frac{1}{\xi_{n+1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Proof. Suppose, if possible, that (44), (45) and (46) are satisfied for some odd integer n, but that

$$\frac{1}{\xi_{n+1}} \leqslant \alpha < \frac{1}{\xi_{n+2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

As n is odd

$$\xi_{n+1} = \frac{1+\tau^{-n}}{\sqrt{2}\,\tau}, \qquad \xi_{n+1}' = \frac{1-\tau^n}{\sqrt{2}}\,\tau$$

$$\eta_{n+1} = \frac{1-\tau^{-n-1}}{\sqrt{2}}, \quad \eta_{n+1}' = \frac{\tau^{n+1}-1}{\sqrt{2}}.$$

Using (15) with $\xi = \xi_{n+1}$ and with $\xi = \eta'_{n+1}$, we have

$$(\alpha \xi_{n+1} - 1)(\beta |\xi'_{n+1}| + 1) \geqslant 1 - \varepsilon, \quad \ldots\ldots \quad (53)$$

$$(\alpha \eta'_{n+1} - 1)(\beta \eta_{n+1} - 1) \geqslant 1 - \varepsilon, \quad \ldots\ldots \quad (54)$$

the factors on the left hand sides being positive by (19), (46) and (52).

We use (53) and (54), together with the inequality $\alpha\beta < 2\tau$ obtained from (17), to find lower bounds for α and β. In these calculations we neglect the effect of ε in the first instance. From (53)

$$(\xi_{n+1}\alpha - 1)\left(\frac{2\tau |\xi'_{n+1}|}{\alpha} + 1\right) \geqslant 1. \quad \ldots\ldots \quad (55)$$

It is convenient to write $P = \frac{1}{2}\xi_{n+1} |\xi'_{n+1}|$ so that

$$P = \tfrac{1}{2} |N(\xi_{n+1})| = \tfrac{1}{4} |N(1 + \tau^{-n})| = \tfrac{1}{4}(\tau^n - \tau^{-n}).$$

Writing $2P/\xi_{n+1}$ for $|\xi'_{n+1}|$ in (55),

$$4\tau P + \xi_{n+1}\alpha - \frac{4\tau P}{\xi_{n+1}\alpha} - 1 \geqslant 1.$$

It follows just as in DAVENPORT's work [3]) that

$$\xi_{n+1}\alpha > 1 + \tau^{-n-1} \quad \ldots\ldots\ldots \quad (56)$$

Now writing $Q = \frac{1}{2}\eta_{n+1}\eta'_{n+1}$ and using the inequality $\alpha\beta < 2\tau$ in (54) we have

$$4\tau Q - \eta_{n+1}\beta - \frac{4\tau Q}{\eta_{n+1}\beta} + 1 \geqslant 1.$$

Also

$$\beta\eta_{n+1} < \eta_{n+1}/\eta_n < \eta_{n+1}\eta'_{n+1} < 2\tau Q.$$

It follows just as in DAVENPORT's [4]) work that

$$\eta_{n+1}\beta > 1 + \frac{1}{4\tau Q}.$$

We have also

$$Q = \tfrac{1}{2} |N(\eta_{n+1})| = \tfrac{1}{4}\tau^{n+1}(1 - \tau^{-n-1})^2.$$

Hence

$$\eta_{n+1}\beta > 1 + \tau^{-n-2}. \quad \ldots\ldots\ldots \quad (57)$$

By (56) and (57)

$$\alpha\beta > (1 + \tau^{-n-1})(1 + \tau^{-n-2})/\xi_{n+1}\eta_{n+1}$$
$$= (1 + \sqrt{2}\,\tau^{-n-1} + \tau^{-2n-3})/\xi_{n+1}\eta_{n+1}.$$

But

$$\xi_{n+1}\eta_{n+1} = (1 + \tau^{-n})(1 - \tau^{-n-1})/2\tau$$
$$< (1 + \sqrt{2}\,\tau^{-n-1})/2\tau.$$

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[3]) D. IV (747—48).

[4]) D. IV (748).

Thus

$$\alpha\beta > 2\tau + 2\tau^{-2n-2}/(1+\sqrt{2}\tau^{-n-1}).$$

Provided ε is smaller than a certain number depending only on n_0, it is clear that, if we carried out these calculations without neglecting ε, we should still be able to conclude that

$$\alpha\beta > 2\tau,$$

contrary to (17). This contradiction proves that (51) is satisfied when n is the unique odd integer for which (44) and (46) are satisfied.

Lemma 10. *If n is odd*

$$|(a_n\xi - 1)(a_n'\xi' - 1)| = 1$$

for the following values of ξ,

$$1,\ \xi_n,\ \xi_{n+1},\ \eta_n',\ \eta_{n+1}'.$$

Proof. By (5)

$$\begin{aligned} a_n - 1 &= \frac{(\tau^{n+1}-1)(\tau-1)}{\tau^n+1} - 1 \\ &= \frac{\tau^n(\tau^2-\tau-1)-\tau}{\tau^n+1} \\ &= \tau\frac{\tau^n-1}{\tau^n+1}, \end{aligned}$$

which has norm 1 as n is odd. Similarly by (5) and (47) we have (since n is odd)

$$\begin{aligned} a_n\xi_n - 1 &= \tau^{-n}\frac{\tau^n-1}{\tau^n+1}, \\ a_n\xi_{n+1} - 1 &= \tau^{-n-1}, \\ a_n\eta_n' - 1 &= -\tau^{n+1}, \\ a_n\eta_{n+1}' - 1 &= \tau^{n+2}\frac{\tau^n-1}{\tau^n+1}, \end{aligned}$$

and these all have norm 1.

Lemma 11. *There exists a constant C depending only on δ such that, if $\alpha \geq \tau/\theta$, then there is a unique odd integer n, for which*

$$|\alpha - a_n| \leqslant C\varepsilon,\quad |\beta - a_n'| \leqslant C\varepsilon \quad \ldots\ldots \quad (58)$$

and (44), (46) and (51) are satisfied.

Proof. We take n to be the unique odd integer for which (44), (46) and (51) are satisfied. We write

$$\alpha = a_n + \mu,\quad \beta = a_n' + \nu.$$

By (15), for every integer ξ of $k(\sqrt{2})$

$$|\{(a_n+\mu)\xi-1\}\{(a'_n+\nu)\xi'-1\}|\geqq 1-\varepsilon$$

so that

$$\left|\left\{1+\frac{\xi}{a_n\xi-1}\mu\right\}\left\{1+\frac{\xi'}{a'_n\xi'-1}\nu\right\}\right|\geqq\frac{1-\varepsilon}{N(a_n\xi-1)}\cdot \quad . \quad . \quad (59)$$

Taking ξ equal to ξ_n, ξ_{n+1}, η'_n and η'_{n+1} and using Lemma 10 we obtain the four inequalities

$$|(1+\varrho_i\mu)(1+\sigma_i\nu)|\geqq 1-\varepsilon, \quad i=1,2,3,4$$

where

$$\left.\begin{aligned}\varrho_1 &= \frac{\xi_n}{a_n\xi_n-1}=\frac{1}{\sqrt{2}}\frac{(\tau^{n-1}+1)(1+\tau^{-n})}{1-\tau^{-n}}>0,\\ \sigma_1 &= \varrho'_1 = \frac{1}{\sqrt{2}}\frac{(1+\tau^{-n+1})(1-\tau^{-n})}{1+\tau^{-n}}>0,\end{aligned}\right\} \quad . \quad . \quad . \quad (60)$$

$$\left.\begin{aligned}\varrho_2 &= \frac{\xi_{n+1}}{a_n\xi_{n+1}-1}=-\frac{1}{\sqrt{2}}(\tau^n+1)<0,\\ \sigma_2 &= \varrho'_2 = \frac{1}{\sqrt{2}}(1-\tau^{-n})>0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$\left.\begin{aligned}\varrho_3 &= \frac{\eta'_n}{a_n\eta'_n-1}=\frac{1}{\sqrt{2}\,\tau}(1+\tau^{-n})>0,\\ \sigma_3 &= \varrho'_3 = -\frac{\tau}{\sqrt{2}}(\tau^n-1)<0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$\left.\begin{aligned}\varrho_4 &= \frac{\eta'_{n+1}}{a_n\eta'_{n+1}-1}=\frac{1}{\sqrt{2}\,\tau}\frac{(1-\tau^{-n-1})(1+\tau^{-n})}{1-\tau^{-n}}>0,\\ \sigma_4 &= \varrho'_4 = \frac{\tau^{n+2}}{\sqrt{2}}\frac{(1-\tau^{-n-1})(1-\tau^{-n})}{1+\tau^{-n}}>0.\end{aligned}\right\} \quad . \quad . \quad (63)$$

It is easy to verify that

$$\frac{1}{\eta'_n}<0<\frac{1}{\eta'_{n+1}}<\frac{1}{\xi_n}<a_n<\frac{1}{\xi_{n+1}}$$

and that

$$\frac{1}{\xi'_{n+1}}<0<\frac{1}{\xi'_n}<\frac{1}{\eta_{n+1}}<a'_n<\frac{1}{\eta_n}.$$

It follows from (46) and (51) that $\alpha\xi-1$ and $a_n\xi-1$ and $\beta\xi'-1$ and $a'_n\xi'-1$ have the same signs when ξ takes the values ξ_n, ξ_{n+1}, η'_n, η'_{n+1}. Consequently $(1+\varrho_i\mu)$ and $(1+\sigma_i\nu)$ are positive for $i=1,2,3,4$ and using the inequality of the arithmetic and geometric means

$$1+\tfrac{1}{2}(\varrho_i\mu+\sigma_i\nu)\geqq 1-\varepsilon,$$

i.e.

$$\varrho_i\mu+\sigma_i\nu\geqq-2\varepsilon \text{ for } i=1,2,3,4. \quad . \quad . \quad . \quad . \quad (64)$$

Eliminating first ν and then μ from the second and third of the inequalities (64),

$$(\varrho_2 |\sigma_3| + \varrho_3 \sigma_2) \mu \geqslant -2\varepsilon (\sigma_2 + |\sigma_3|),$$

$$(\sigma_2 \varrho_3 + \sigma_3 |\varrho_2|) \nu \geqslant -2\varepsilon (|\varrho_2| + \varrho_3).$$

But

$$\begin{aligned} \varrho_2 |\sigma_3| + \varrho_3 \sigma_2 &= \sigma_2 \varrho_3 + \sigma_3 |\varrho_2| \\ &= \tfrac{1}{2} \tau^{-1} (1 - \tau^{-n})(1 + \tau^{-n}) - \tfrac{1}{2} \tau (\tau^n + 1)(\tau^n - 1) \\ &= -\tfrac{1}{2} (\tau^{2n+1} - \tau^{-1})(1 - \tau^{-2n}) < 0. \end{aligned}$$

Thus the above inequalities imply that $\mu \leq c_n \varepsilon$, and $\nu \leq c'_n \varepsilon$ where c_n and c'_n depend only on n. Substituting these bounds for μ and ν in the first or the fourth of the inequalities (64), we obtain $\mu \geq -c''_n \varepsilon$ and $\nu \geq -c'''_n \varepsilon$. Since n is positive and less than n_0 there is a constant C depending only on δ such that

$$|\mu| \leqslant C\varepsilon, \quad |\nu| \leqslant C\varepsilon.$$

This proves the lemma.

Lemma 12. *If $\alpha \geq \tau/\theta$, then for some odd positive integer n,*

$$\alpha = \alpha_n, \quad \beta = \alpha'_n.$$

Proof. Let n be the integer of Lemma 11. We define numbers X_r and Y_r of $k(\sqrt{2})$ by the equations

$$\alpha_n X_r - 1 = \tau^{-r(n+1)+1} \frac{\tau^n - 1}{\tau^n + 1},$$

$$\alpha_n Y_r - 1 = -\tau^{-r(n+1)}.$$

It is clear that

$$N(\alpha_n X_r - 1) = 1,$$

$$N(\alpha_n Y_r - 1) = 1;$$

we have to prove that X_r and Y_r are integers of $k(\sqrt{2})$. Solving the equations for X_r and Y_r, and using (5)

$$\begin{aligned} X_r &= \frac{\tau^n + 1}{\sqrt{2}(\tau^{n+1} - 1)} \left\{ \tau^{-r(n+1)+1} \frac{\tau^n - 1}{\tau^n + 1} + 1 \right\} \\ &= \frac{\tau^{-r(n+1)+1}(\tau^n - 1) + (\tau^n + 1)}{(\tau - 1)(\tau^{n+1} - 1)}, \end{aligned}$$

and

$$Y_r = \frac{\tau^n + 1}{\sqrt{2}} \tau^{-r(n+1)} \frac{\tau^{r(n+1)} - 1}{\tau^{n+1} - 1}.$$

It is now clear that Y_r is an integer of $k(\sqrt{2})$. Also, since

$$\begin{aligned}
&\tau^{-r(n+1)+1}(\tau^n-1)+(\tau^n+1)\\
&\qquad\equiv \tau(\tau^n-1)+(\tau^n+1) \quad (\mathrm{mod}\,(\tau \cdot 1)(\tau^{n+1}-1))\\
&\qquad=(\tau+1)\tau^n-(\tau-1)\\
&\qquad=\tau(\tau-1)\tau^n-(\tau-1)\\
&\qquad\equiv(\tau-1)-(\tau-1) \qquad (\mathrm{mod}\,(\tau-1)(\tau^{n+1}-1))\\
&\qquad=0,
\end{aligned}$$

X_r is an integer.

We write $\alpha = a_n + \mu$ and $\beta = a'_n + \nu$. By Lemma 11 we must have

$$|\mu| \leqslant C\varepsilon \text{ and } |\nu| \leqslant C\varepsilon.$$

Applying (59) with $\xi = X_r$ and with $\xi = Y_r$, we obtain

$$|(1+R_r\mu)(1+R'_r\nu)| \geqslant 1-\varepsilon, \quad \ldots \ldots \quad (65)$$

$$|(1+S_r\mu)(1+S'_r\nu)| \geqslant 1-\varepsilon, \quad \ldots \ldots \quad (66)$$

where

$$\left.\begin{aligned}
R_r &= \frac{X_r}{a_n X_r - 1} = \frac{1}{a_n}\left\{1+\tau^{r(n+1)-1}\frac{\tau^n+1}{\tau^n-1}\right\},\\
R'_r &= \frac{1}{a'_n}\left\{1+\tau^{-r(n+1)+1}\frac{\tau^n-1}{\tau^n+1}\right\},
\end{aligned}\right\} \quad \ldots \quad (67)$$

$$\left.\begin{aligned}
S_r &= \frac{Y_r}{a_n Y_r - 1} = -\frac{1}{a_n}\{\tau^{r(n+1)}-1\}\\
S'_r &= \frac{1}{a'_n}\{1-\tau^{-r(n+1)}\}.
\end{aligned}\right\} \quad \ldots \ldots \quad (68)$$

The numbers R_r, $-R'_r$, S_r and $-S'_r$ vary with r in a way which is essentially similar to the variation the numbers R_r, R'_r, S_r and S'_r occurring in DAVENPORT's work [5]). It follows just as in DAVENPORT's investigation that $\mu = \nu = 0$.

8. **Lemma 13.** *If a, b are of the form* (4), *then*

$$|(\xi-a)(\xi'-b)| \geqslant \frac{1}{a_n a'_n}, \quad \ldots \ldots \quad (69)$$

for all integers ξ of $k(\sqrt{2})$; equality occurring for an infinite number of integral values of ξ.

Proof. It clearly suffices to consider the case when

$$a = \frac{1}{a_n}, \quad b = \frac{1}{a'_n}$$

[5]) D. IV (911—913).

where n is an odd positive integer. In this case we have to prove that

$$|N(a_n \xi - 1)| \geqslant 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

for all integers ξ of $k(\sqrt{2})$; and that equality occurs for an infinite number of integral values of ξ. We have seen in Lemma 12 that (70) is satisfied with equality when $\xi = X_r$ and when $\xi = Y_r$ and that X_r and Y_r are integers of $k(\sqrt{2})$ for all rational integers r.

When $n = 1$ we have $a_n = a'_n = 2$ so that $a_n\xi - 1 = 2\xi - 1$ is a non-zero integer of $k(\sqrt{2})$ whenever ξ is an integer of $k(\sqrt{2})$. Thus (70) is in this case satisfied for all integers ξ of $k(\sqrt{2})$.

When $n = 3$ we have

$$a_3 = \frac{4\sqrt{2}}{2\sqrt{2}-1}$$

$$a'_3 = \frac{4\sqrt{2}}{2\sqrt{2}+1}$$

so that, if $\xi = x + \sqrt{2}y$,

$$a_3 \xi - 1 = \frac{4\sqrt{2}(x + \sqrt{2}y) - 2\sqrt{2} + 1}{2\sqrt{2} - 1}$$

$$= \frac{2\sqrt{2}(2x - 1) + (8y + 1)}{2\sqrt{2} - 1}.$$

Thus

$$|N(a_3 \xi - 1)| = \tfrac{1}{7}\,|8(2x-1)^2 - (8y+1)^2|.$$

Now, for any integers x and y

$$8(2x - 1)^2 - (8y + 1)^2 \equiv > 7 \pmod{16}$$

Thus

$$|N(a_3 \xi - 1)| \geqslant 1$$

for all integers ξ of $k(\sqrt{2})$.

When $n \geq 5$ we use a different type of argument. We suppose that, for some integer η of $k(\sqrt{2})$,

$$|N(a_n \eta - 1)| < 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and we eventually arrive at a contradiction. Consider the substitution

$$a_n \eta - 1 = \tau^{n+1}(a_n \zeta - 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

We have

$$\eta = \tau^{n+1}\zeta - \frac{\tau^{n+1} - 1}{a_n} = \tau^{n+1}\zeta - \frac{\tau^n + 1}{\sqrt{2}},$$

$$\zeta = \tau^{-n-1}\eta - \frac{\tau^{-n-1} - 1}{a_n} = \tau^{-n-1}\eta + \tau^{-n-1}\frac{\tau^n + 1}{\sqrt{2}}.$$

Thus using (21) the transformation (72) transforms integers ζ into integers η and vice versa. Also

$$N(\alpha_n \eta - 1) = N(\alpha_n \zeta - 1).$$

It is clear that by repeated application of the transformation (72) or of its inverse we may obtain an integer η of $k(\sqrt{2})$ satisfying (71) and such that

$$\tau^{-n-1} \leqslant \left| \frac{\alpha_n \eta - 1}{\alpha_n' \eta' - 1} \right| < \tau^{n+1} \quad \ldots \ldots \quad (73)$$

It follows from (71) and (73) that

$$|\alpha_n \eta - 1| < \tau^{\frac{1}{2}(n+1)}, \quad |\alpha_n' \eta' - 1| < \tau^{\frac{1}{2}(n+1)}, \quad \ldots \ldots \quad (74)$$

further one of these two numbers must be less than 1. We consider two cases separately, but first we note that

$$\left. \begin{aligned} \tau\sqrt{2} > \alpha_n \geqslant \alpha_5 = (98 + 28\sqrt{2})/41 = 3{\cdot}356\ldots \\ \sqrt{2} < \alpha_n' \leqslant \alpha_5' = (98 - 28\sqrt{2})/41 = 1{\cdot}424\ldots \end{aligned} \right\} \quad \ldots \quad (75)$$

Case 1. Suppose that

$$|\alpha_n \eta - 1| < 1 \text{ and } |\alpha_n' \eta' - 1| < \tau^{\frac{1}{2}(n+1)}. \quad \ldots \ldots \quad (76)$$

The first inequality implies that

$$0 < \eta < \frac{2}{\alpha_n} < \tfrac{2}{3}, \quad \ldots \ldots \ldots \quad (77)$$

by (75).

Suppose, if possible, that $|\eta'| < \tau$. Then by (77),

$$0 < |\eta \eta'| < \frac{2\tau}{3} < 2,$$

so that η is a unit of $k(\sqrt{2})$. But there is no unit η of $k(\sqrt{2})$ with $|\eta| < \frac{2}{3}$ and $|\eta'| < \tau$. This contradiction proves that $|\eta'| \geq \tau$.

Multiplying the inequality

$$|\theta \tau \eta - 1| \leqslant |\alpha_n \eta - 1| + (\theta \tau - \alpha_n)\eta$$

by $|\alpha'_n \eta' - 1|$ and using (71) and (76), we deduce that

$$|(\theta \tau \eta - 1)(\alpha_n' \eta' - 1)| < 1 + (\theta \tau - \alpha_n)\eta \tau^{\frac{1}{2}(n+1)}.$$

Now, since $\theta\tau\eta - 1$ is a non-zero integer of $k(\sqrt{2})$,

$$|N(\theta \tau \eta - 1)| \geqslant 1.$$

Hence

$$\left| \frac{(\alpha_n' \eta' - 1)}{(\theta' \tau' \eta' - 1)} \right| < 1 + (\theta \tau - \alpha_n)\eta \tau^{\frac{1}{2}(n+1)}. \quad \ldots \ldots \quad (78)$$

Now the expression on the left hand side of (78) is either

$$\frac{a_n'|\eta'|-1}{\theta\tau^{-1}|\eta'|-1} \text{ or } \frac{a_n'|\eta'|+1}{\theta\tau^{-1}|\eta'|+1}$$

according as $\eta' > 0$ or $\eta' < 0$. The latter is the smaller, since $a'_n > \theta > \theta/\tau$. Hence, by (78)

$$\frac{a_n'|\eta'|+1}{\theta\tau^{-1}|\eta'|+1} < 1 + (\theta\tau - a_n)\,\eta\,\tau^{\frac{1}{2}(n+1)}.$$

As

$$\theta\tau - a_n = \sqrt{2}\,\frac{\tau+1}{\tau^n+1} = \frac{2\tau}{\tau^n+1} < 2\tau^{-n+1},$$

this implies that

$$\frac{a_n'|\eta'|+1}{\theta\tau^{-1}|\eta'|+1} < 1 + 2\eta\,\tau^{\frac{1}{2}(3-n)}. \quad \ldots\ldots \quad (79)$$

Using the fact that $|\eta'| \geq \tau$, (75), (79) and (77) we obtain

$$\frac{\sqrt{2}\tau+1}{\sqrt{2}+1} \leqslant \frac{\sqrt{2}|\eta'|+1}{\theta\tau^{-1}|\eta'|+1} < \frac{a_n'|\eta'|+1}{\theta\tau^{-1}|\eta'|+1} < 1 + 2\eta\,\tau^{\frac{1}{2}(3-n)} < 1 + \tfrac{4}{5}\tau^{-1},$$

since $n \geq 5$. This is a contradiction since the left hand side has the value 1.828 ... while the right hand side has the value 1.552 ....

Case 2. Suppose that

$$|a_n\eta - 1| < \tau^{\frac{1}{2}(n+1)} \text{ and } |a_n'\eta' - 1| < 1. \quad \ldots \quad (80)$$

The second inequality implies that

$$0 < \eta' < \frac{2}{a_n'} < \sqrt{2} \quad \ldots\ldots \quad (81)$$

by (75).

Suppose, if possible, that $|\eta| < \tau$. Then

$$0 < |\eta\eta'| < \tau\sqrt{2} < 4,$$

and so, as 3 and -3 are not norms in $k(\sqrt{2})$, we must have either $|\eta\eta'| = 1$ or $|\eta\eta'| = 2$. If $|\eta\eta'| = 1$, then η is a unit and as $0 < \eta' < \sqrt{2}$, $|\eta| < \tau$ we must have $\eta = 1$. This is impossible by Lemma 10 and the fact that η satisfies (71). If $|\eta\eta'| = 2$, then necessarily $\eta = \sqrt{2}\zeta$, $\eta' = -\sqrt{2}\zeta'$, where ζ is a unit of $k(\sqrt{2})$ satisfying $|\zeta| < \tau$, $|\zeta'| < 1$. But there is no such unit of $k(\sqrt{2})$. These contradictions prove that $|\eta| \geq \tau$.

Just as in case 1, using (71) and (80)

$$|(a_n\eta-1)(\theta\eta'-1)| \leqslant |(a_n\eta-1)(a_n'\eta'-1)| + |(a_n\eta-1)(a_n'-\theta)\eta'|$$
$$< 1 + (a_n'-\theta)\,\eta'\,\tau^{\frac{1}{2}(n+1)}.$$

Since $\theta\eta' - 1$ is a non-zero integer of $k(\sqrt{2})$ this implies that

$$\left|\frac{a_n\eta-1}{\theta\eta+1}\right| < 1 + (a_n'-\theta)\,\eta'\,\tau^{\frac{1}{2}(n+1)}.$$

The expression on the left hand side is either

$$\frac{a_n |\eta| + 1}{\theta |\eta| - 1} \quad \text{or} \quad \frac{a_n |\eta| - 1}{\theta |\eta| + 1},$$

according as $\eta \leq -\tau$ or $\eta \geq \tau$. Thus, in any case,

$$\frac{a_n |\eta| - 1}{\theta |\eta| + 1} < 1 + (a'_n - \theta)\, \eta'\, \tau^{\frac{1}{2}(n+1)}. \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

Now

$$a'_n - \theta = \sqrt{2}\, \frac{\tau^{n+1} - 1}{\tau(\tau^n - 1)} - \sqrt{2} = \frac{2}{\tau(\tau^n - 1)} = 2\, \tau^{-n-1} (1 - \tau^{-n})^{-1}. . \qquad (83)$$

Using the fact that $|\eta| \geq \tau$, (75), (82), (83) and (81) we obtain

$$\begin{aligned} \frac{3\tau - 1}{\sqrt{2}\,\tau + 1} \leqslant \frac{3|\eta| - 1}{2|\eta| + 1} < \frac{a_n |\eta| - 1}{\theta |\eta| + 1} & \\ < 1 + (a'_n - \theta)\, \eta'\, \tau^{\frac{1}{2}(n+1)} & \\ < 1 + 2\sqrt{2}\, \tau^{-\frac{1}{2}(n+1)} (1 - \tau^{-n})^{-1} & \\ < 1 + 2\sqrt{2}\, \tau^{-3} (1 - \tau^{-5})^{-1}, & \end{aligned}$$

as $n \geq 5$. This a contradiction since the left hand side has the value $\sqrt{2} = 1.414\ldots$ while the right hand side has the value $1.169\ldots$. This completes the proof of the lemma.

University College, London.