Mathematics. — *On the densest packing of convex domains.* By LÁSZLÓ FEJES TÓTH. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Let C be a convex domain, $\mathfrak{p}$ and $\mathfrak{q}$ two linearly independent vectors, $C_{nm} \equiv C + n\mathfrak{p} + m\mathfrak{q}$ the domain arising from C by displacement by the vector $n\mathfrak{p} + m\mathfrak{q}$. The set of domains C_{nm}; $n, m = 0, \pm 1, \pm 2, \ldots$ is called a lattice of the domain C. The set of points $\mathfrak{x} = \lambda\mathfrak{p} + \mu\mathfrak{q}$, $0 < \lambda < 1$, $0 < \mu < 1$ is the basic parallelogram P of the lattice.

Consider the quotient $d = |C|/|P|$, denoting by the sign of the absolute value the area of the corresponding domain. d can be interpreted as the ratio of the sum of the areas $|C_{nm}|$ to the whole plane. We call d the density of the lattice.

The problem to find for a given C the maximal value $d(C)$ of the density of all separated lattices, i.e. of lattices in which no two domains overlap, was studied by LORD KELVIN [1]) and H. MINKOWSKI [2]). R. COURANT [3]) announced the conjecture that for any centrally symmetrical convex domain C,

$$d(C) \geqq \pi/\sqrt{12} \qquad (= 0 \cdot 9069\ldots),$$

with equality only if C is bounded by an ellipse.

Recently the problem of COURANT was studied by K. REINHARDT [4]) and K. MAHLER [5]). They proved the very surprising fact that COURANT's conjecture is false. Accordingly the constant $\pi/\sqrt{12}$ must be replaced by a smaller one. The extremal domain and the exact constant are still unknown and their determination seems to involve considerable difficulties.

Therefore it has some interest to show that one can determine by means of entirely elementary tools a lower bound of $d(C)$ which encloses the required value of the exact constant between narrow limits. We give in

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1) LORD KELVIN, Baltimore lectures on molecular dynamics, p. 618, London (1904).

2) H. MINKOWSKI, Dichteste gitterförmige Lagerung kongruenter Körper. Ges. Abh. 2, p. 3—42.

3) See W. BLASCHKE, Vorlesungen über Differentialgeometrie II. p. 65, Berlin (1923). where the conjecture is announced for any convex domain. Probably Courant tacitly supposed the symmetricity of the domains, since in the opposite case a triangle may yield a trivial counter-example. — It may be still mentioned that the problem to determine the lower bound of $d(C)$ for a certain special class of centrally symmetrical convex domains was already considered by MINKOWSKI.

4) K. REINHARDT, Ueber die dichteste gitterförmige Lagerung kongruenter Bereiche in der Ebene und eine besondere Art konvexer Kurven. Abh. Math. Sem. Hamb. Univ. **10**, 216—230 (1934).

5) K. MAHLER, On the area and the densest packing of convex domains. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 108—118 (1947).

this paper a lower bound which is only about 4.5 % less then $\pi/\sqrt{12}$. More precisely:

If any centrally symmetrical convex domain is given, there can be found a separated lattice of the domain having a density

$$(^*) \qquad d > \sqrt{3}/2 \qquad (= 0 \cdot 8660\ldots).$$

Together with the problem of the densest packing, we can also consider the dual problem of the smallest packing, i.e. to find to a given C the minimal density $D(C)$ of the lattices covering the whole plane. It is remarkable, that the dual problem of the problem of COURANT is solved by the following inequality [6])

$$D(C) \leqq \sqrt{12}\,\pi/9 \qquad (= 1 \cdot 2092\ldots),$$

which holds for any centrally symmetrical convex domain C. Equality holds only if C is bounded by an ellipse.

If we drop the postulate of the symmetry of C, then in both cases the triangles are proved to be the extremal domains [7]):

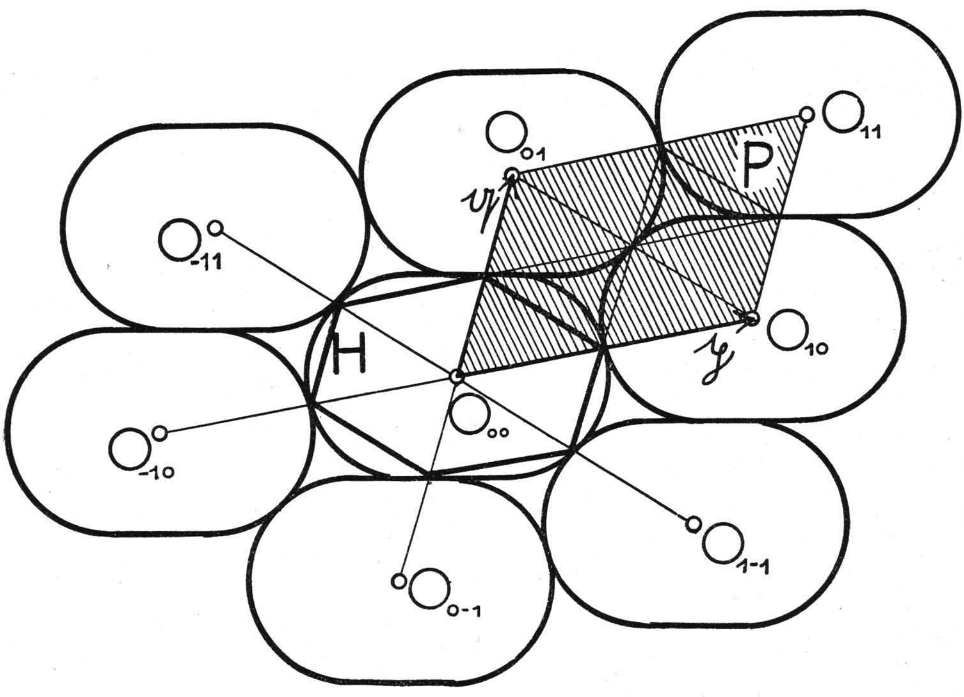

Fig. 1.

[6]) L. FEJES, Eine Bemerkung über die Bedeckung der Ebene durch Eibereiche mit Mittelpunkt. Acta Scientiarum Mathematicarum **11**, 93—95 (Szeged 1946).

[7]) These, as yet unpublished, results was find by I. FÁRY to my suggestion.

: The density of the densest separated lattice of any convex domain is $\geqq 2/3$.

The density of the smallest lattice of any convex domain covering the plane is $\leqq 3/2$.

In both cases equality holds only for triangles.

Let us now turn to the proof of (*).

Let $\mathfrak{p}$ be the least vector of a given direction such, that the centrally symmetrical domain $C \equiv C_{00}$ and the domain $C_{10} \equiv C + \mathfrak{p}$ arising from C by a displacement with $\mathfrak{p}$ have no common inner point. Choose $\mathfrak{q}$ so that $C_{01} \equiv C + \mathfrak{q}$ has common boundarypoints with C_{00} and C_{10} whithout having common inner points. The hexagon $O_{10}\, O_{01}\, O_{-11}\, O_{-10}\, O_{0-1}\, O_{1-1}$, where O_{nm} denote the centre of C_{nm}, is obviously affine regular. Consider the homothetic hexagon H inscribed in C, which arises from the above hexagon by a similitude with respect to O_{00} and the factor of similitude equal to 1/2. Decompose H in 6 triangles having the common vertex O_{00}. Since the basic parallelogram $P \equiv O_{00}\, O_{10}\, O_{11}\, O_{01}$ is composed of 8 equiareal triangles we have $|P|/|H| = 8/6 = 4/3$. Hence the density of the lattice $\{C_{nm}\}$ is given by

$$d = \frac{3}{4} \frac{|C|}{|H|}.$$

Thus, by $|C| \geqq |H|$, for any direction of $\mathfrak{p}$ a density $\geqq 3/4$ is already guaranted.

Let us now choose the direction of $\mathfrak{p}$ so that $|H|$ takes its minimal value

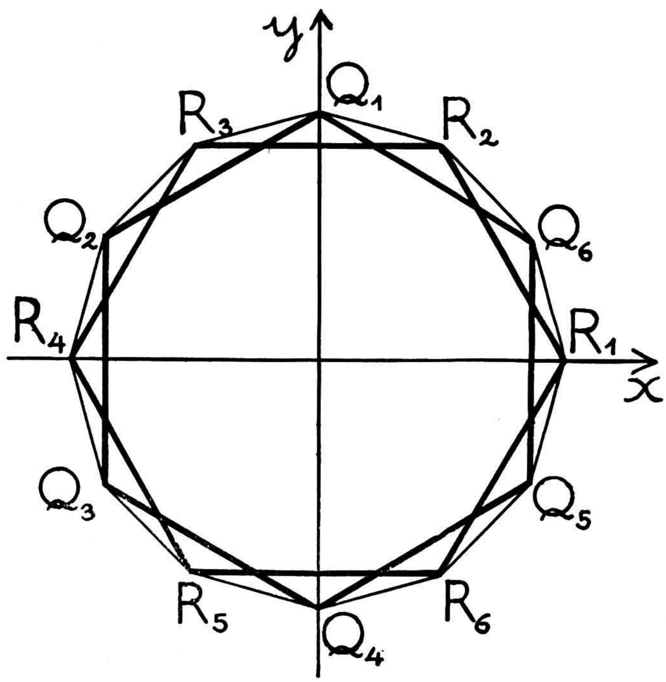

Fig. 2.

$|\bar{H}|$. We may suppose that $\bar{H} \equiv Q_1 Q_2 \dots Q_6$ is a regular hexagon with centre at the origine so that $Q_1 \equiv Q_1(0, 1)$. C contains however by supposition a second affine regular hexagon $H \equiv R_1 R_2 \dots R_6$ concentric to $\bar{H}$ so that $|H| \geqq |\bar{H}| = 3\sqrt{3}/2$ holds and that R_1 lies on the x axis: $R_1 \equiv R_1(\varrho, 0)$. Moreover the vertices of $\bar{H}$ lie outside H, and conversely.

Consider the least convex envelope E of $\bar{H}$ and H. Displace the line segments R_2R_3 and R_5R_6 of lengths ϱ in their own line in a position symmetric to the y axis. By this operation $|E|$ remains invariant, so that we may suppose that H is symmetric with respect to the y axis. If we put $R_2 \equiv R_2(\varrho/2, \eta)$, the condition $|H| \geqq 3\sqrt{3}/2$ becomes equivalent to $2\varrho\eta \geqq \sqrt{3}$.

We have now

$$|E| = |\bar{H}| + 2|R_1 Q_5 Q_6| + 4|R_2 Q_1 Q_6| = 3\sqrt{3}/2 + (\varrho - \sqrt{3}/2) + 2h,$$

where $h = \frac{1}{2}(\varrho/2 + \sqrt{3}\eta - \sqrt{3})$ denotes the distance of $R_2(\varrho/2, \eta)$ from the line Q_1Q_6 having the normal equation $\frac{1}{2}(x + \sqrt{3}y - \sqrt{3}) = 0$. Hence

$$|E| = 3\sqrt{3}/2 + \varrho - \sqrt{3}/2 + \varrho/2 + \sqrt{3}\eta - \sqrt{3} \geqq \tfrac{3}{2}(\varrho + 1/\varrho)$$

and by $\varrho + 1/\varrho \geqq 2$ we obtain $|E| \geqq 3$.

Equality holds only if $\varrho = 1$ and $|H| = |\bar{H}|$, i.e. if E is a regular 12-gon. But in this case $|C| > |E|$ holds, as otherwise $\bar{H}$ were not the least affine regular hexagon inscribed in C. Hence in any case we have $|C| > 3$ and thus

$$d(C) = \frac{3}{4}\frac{|C|}{|H|} > \frac{3}{4}\frac{3}{3\sqrt{3}/2} = \frac{\sqrt{3}}{2} \qquad \text{q. e. d.}$$

Remark during the proofreading. Later I have learnt the fact, that some time ago an inequality equivalent with (*) was already given by K. MAHLER, The theorem of MINKOWSKI–HLAWKA, Duke Math. Journal 13, 611—621 (1946); but the proof given above is a rather different one.