Mechanics. — *Non-linear relations between viscous stresses and instantaneous rate of deformation as a consequence of slow relaxation.* By J. M. BURGERS. (Mededeling No. 56 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

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1. In connection with preparatory work for the International Rheological Congress to be held in September 1948 at Scheveningen (Holland) I had to study some papers by WEISSENBERG referring to the stresses called forward by deformation in visco-elastic materials, where laminar flow is accompanied by elastic deformations [1]). WEISSENBERG's treatment is rather of an abstract character and the reader is puzzled by the problem how a continuously progressing deformation as is found in laminar flow and a permanent elastic deformation can be present together, in particular as attention is drawn to the circumstance that the principal directions of these deformations may be different. However, a tangible picture can be obtained if we start from the idea that flow is possible in consequence of a relaxation phenomenon and bear in mind that, owing to the finite rate at which the re-arrangement of molecular structure takes place, every element of volume of a flowing medium will bear in its molecular pattern reminiscences of its past. In consequence of this circumstance the actual pattern during flow deviates from the equilibrium pattern. There is thus a state of physical deformation, which must be clearly distinguished from the progressively increasing deformation of the boundary surface of an element of volume as it occurs in flow.

In the ordinary theory of viscosity based on this idea it is supposed that the state of physical deformation is proportional to the instantaneous rate of deformation, as will be the case when the re-arrangement of the molecules takes place sufficiently quickly. On the other hand in liquids or fluids where this is not the case, "memory" will extend further into the past and the physical deformation will be determined by less simple relations.

The idea of a "memory" in matter has been introduced by BOLTZMANN, and was afterwards taken up by VOLTERRA and by VON KARMAN [2]). It

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[1]) Compare in particular: K. WEISSENBERG, La mécanique des corps déformables, Arch. Sciences phys. et natur. (Genève) (5) **17**, p. 1—105, 1935. — Of more recent publications may be mentioned: K. WEISSENBERG, A continuum theory of rheological phenomena, Nature **159**, p. 310, March 1, 1947.

[2]) See: V. VOLTERRA, Drei Vorlesungen über neuere Fortschritte der mathematischen Physik, Archiv d. Mathematik u..Physik (3) **22**, p. 97—182, 1914, in particular p. 155—171.

TH. VON KÁRMÁN, Das Gedächtnis der Materie, Die Naturwissenschaften **4**, p. 489, 1916.

is present also in an important model proposed by PRANDTL for the explanation of hysteresis and relaxation [3]). Although VOLTERRA has given some attention to the general case, the examples mainly considered refer to systems with displacements in a single direction, so that the problem of non coinciding spatial directions does not occur. In the following lines it will be shown how such a problem can arise for the elementary case of laminar motion, even if we keep to an extremely simplified qualitative treatment.

2. A few words on the theory of hydrodynamic viscosity may precede the consideration of the example. In hydrodynamical theory the components of the viscous stresses in a flowing liquid of constant density [4]) are assumed to be proportional to the corresponding components of the instantaneous rate of deformation. If the velocity components are denoted by u_i ($i = 1, 2, 3$, corresponding to the three axes of a system of rectangular coordinates x_1, x_2, x_3), the components of the instantaneous rate of deformation are given by [5]):

$$D_{ik} = \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the components of the stress tensor have the values:

$$\tau_{ik} = \eta \, D_{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where η is the viscosity.

The theoretical explanation of this relationship starts from the idea that (*a*) every deformation of an element of volume of the liquid calls forward a change in the arrangement of the molecules, and that (*b*) when the element is left to itself a short but finite interval of time is needed before the normal statistical distribution of molecular distances and velocities has been restored. Whereas in the normal equilibrium state of an ordinary liquid the molecular field is statistically isotropic, there can be present consequently an anisotropy for a short period. The anisotropy of the arrangement gives rise to anisotropy of the field of inter-molecular forces, and as a result of this, to the appearance of stresses.

When deformations are changed or repeated periodically in intervals of time short compared with the time necessary for re-arrangement, the

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[3]) L. PRANDTL, Ein Gedankenmodell zur kinetischen Theorie der festen Körper, Zeitschr. f. angew. Mathem. u. Mechanik **8**, p. 85—106, 1928. Compare: First Report on Viscosity and Plasticity, Verhand. Kon. Nederl. Akademie v. Wetenschappen, (1) **15**, no. 3, p. 41—64 (1939).

[4]) To simplify incompressibility has been assumed so that the equation of continuity $\partial u_i / \partial x_i = 0$ (summation with respect to repeated indices) is satisfied by the components of the velocity. Also it is assumed that the stress components τ_{ik} refer to the deviatoric stresses only, so that $\tau_{ii} = 0$. No attention is given to the hydrostatic part of the stress (hydrostatic pressure).

[5]) A factor $\frac{1}{2}$ is sometimes inserted; in that case the factor η in eq. (2) must be replaced by 2η.

medium in general will show elastic behaviour, in first approximation with proportionality between stress and strain. On the other hand when the material is subjected to a process of progressive deformation, in particular when a stationary state of flow is present, the resultant effect of deformation and re-arrangement leads to the appearance of a statistically stationary anisotropic state of the molecular field, and thus to the appearance of a stationary system of stresses.

In the usual form of the theory the permanent state of physical deformation of the flowing medium, *i.e.* the state of anisotropy which is to be found in the pattern of molecular arrangement and of velocity distribution, is assumed to be equal to the product of the rate of deformation into a quantity having the dimensions of a time and called the "relaxation time" λ. When the stress is put equal to the product of the permanent physical deformation into the shear modulus G, the stress becomes equal to the rate of deformation multiplied by $G\lambda$. The product $G\lambda$ thefore represents the viscosity η, which can be calculated theoretically when it is possible to find G and λ [6]).

It is assumed in this mode of reasoning that the time of relaxation λ is very short compared with the time in which an appreciable geometrical deformation of the boundary surface of an element of volume takes place. The resulting physical deformation then will be slight, and as the equilibrium state itself is isotropic, the directions characterising the resulting anisotropy of the field will be the same as the directions characterising the tensor of the rate of deformation D_{ik}.

On the contrary when the relaxation time becomes large we must expect that the deviation of the molecular arrangement will bear reminiscences of a more remote past. The physical deformation will then no longer be simply proportional to the instantaneous rate of deformation, but will be determined by some integrated quantity. This brings the possibility that the tensor describing the physical deformation in general will not be parallel and proportional to the tensor of the instantaneous rate of deformation. At the same time the physical deformation may assume such a magnitude that in calculating the stresses account must be taken of stress-strain relationships for large deformations, which in general cannot be represented by linear formulae.

3. We now turn to the case of laminar flow. We assume the velocity u (parallel to the x-axis) to be given by $k\,y$; the component v being zero (we restrict ourselves to the x, y-plane). The tensor of the instantaneous rate of deformation is then given by:

$$D = \begin{Bmatrix} 0 & k \\ k & 0 \end{Bmatrix} \quad \ldots \ldots \ldots \ldots (3)$$

[6]) The modern development of this theory is due to M. BORN and H. GREEN, and a particularly clear exposition is given by GREEN in a paper to be read before the International Rheological Congress, which will be published in the Proceedings of the Congress.

Now consider a point which at the instant $\bar{t}$ has the coordinates $\bar{x}, \bar{y}$; at the instant $t = \bar{t} - \lambda$ the coordinates have been:

$$x = \bar{x} - k\lambda\bar{y} \quad ; \quad y = \bar{y},$$

from which:

$$\bar{x} = x + k\lambda y \quad ; \quad \bar{y} = y. \qquad \qquad (4)$$

The physical state of deformation to be found at the instant $\bar{t}$ may be dependent on the whole series of preceding states, so that it should properly be calculated by means of an integral extending over the time from $t = -\infty$ until $t = \bar{t}$ [7]). However, the features necessary to explain WEISSENBERG's formulae come out already if we assume that the physical state of deformation is described by equations (4) in which λ will be considered as some given, finite quantity. The point of importance is that in this case we have to do with a *physical deformation of finite magnitude* and that account must be taken of this circumstance in calculating the stresses. Various methods have been proposed for dealing with finite deformations and the stresses accompanying them; in the present case it is convenient to make use of the system of formulae developed by WEISSENBERG for that purpose [8]). The transformation from the coordinates x, y to the coordinates $\bar{x}, \bar{y}$ is described with the aid of the matrix equation:

$$\| \bar{x}, \bar{y} \| = \| x, y \| \cdot \begin{Vmatrix} 1 & 0 \\ k\lambda & 1 \end{Vmatrix}. \qquad \qquad (5)$$

Here $\| \bar{x}, \bar{y} \|$ and $\| x, y \|$ are matrices with a single row only, while the usual rule for matrix multiplication must be applied. We write ψ for the transformation matrix, $\tilde{\psi}$ for its transposed form, so that:

$$\psi = \begin{Vmatrix} 1 & 0 \\ k\lambda & 1 \end{Vmatrix} \quad ; \quad \tilde{\psi} = \begin{Vmatrix} 1 & k\lambda \\ 0 & 1 \end{Vmatrix}. \qquad \qquad (6)$$

The matrix ψ is unsymmetrical and combines in itself a rotation with a deformation. The latter can be described by means of either of two symmetrical matrices ϑ_a, ϑ_p, determined by the equations [9]):

$$\vartheta_a^2 = \psi\,\tilde{\psi} = \begin{Vmatrix} 1 & k\lambda \\ k\lambda & 1 + k^2\lambda^2 \end{Vmatrix} \qquad \qquad (7a)$$

$$\vartheta_p^2 = \tilde{\psi}\,\psi = \begin{Vmatrix} 1 + k^2\lambda^2 & k\lambda \\ k\lambda & 1 \end{Vmatrix}. \qquad \qquad (7b)$$

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[7]) See VOLTERRA, l.c. (footnote 2) above), p. 157 seq. — VOLTERRA, however, calculates the deformation at a given instant from the forces applied during the previous period, whereas in the text we are interested in the present state of physical deformation as it results from the geometrical deformation experienced by the material in the previous period.

[8]) K. WEISSENBERG, Arch. Sciences phys. et natur. (Genève) (5) **17**, p. 11 seq., 1935.

[9]) It is reminded that any power of a symmetrical matrix can be obtained as follows:

The matrices ϑ_a and ϑ_p differ in the order in which they must be combined with a matrix φ describing the rotation so as to obtain back the matrix ψ. This matrix φ (which is anti-symmetrical) is given by:

$$\varphi = \begin{Vmatrix} a & -\frac{1}{2}k\lambda a \\ \frac{1}{2}k\lambda a & a \end{Vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where a has been written for $(1+\frac{1}{4}k^2\lambda^2)^{-1/2}$, and we have the equations:

$$\psi = \vartheta_a \cdot \varphi = \varphi \cdot \vartheta_p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence ϑ_a represents a transformation which must precede the rotation, while ϑ_p is the transformation to be applied when the rotation is performed first.

Now the stress tensor is a function of ϑ_p [10]). For our purpose it is not so important which function is chosen, but the point to be observed is that the principal axes of the stress tensor are parallel to those of ϑ_p. It is not difficult to find the directions of the principal axes of ϑ_p and to calculate the corresponding eigenvalues. The first axis makes the angle Φ with the x-axis (counted anticlockwise from x to y) given by

$$tg\,\Phi = -\tfrac{1}{2}k\lambda + \sqrt{1+\tfrac{1}{4}k^2\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which is smaller than 1, so that $\Phi < 45^\circ$; the second axis is perpendicular to the first one. The eigenvalues are:

$$\tfrac{1}{2}k\lambda + \sqrt{1+\tfrac{1}{4}k^2\lambda^2} \quad ; \quad -\tfrac{1}{2}k\lambda + \sqrt{1+\tfrac{1}{4}k^2\lambda^2} \quad . \quad . \quad . \quad (11)$$

so that there is extension in the direction of the first axis and compression in that of the second axis.

Hence we see that when $k\lambda$ is treated as a finite quantity the principal direction of extension is *turned towards the x-axis*. We must expect the same for the principal tension stress. This is the result which is brought forward by WEISSENBERG; it can be interpreted by saying that a tension stress directed according to the x-axis is superposed on the usual system of shearing stresses τ_{yx} and τ_{xy}.

The principal directions of the given matrix are determined and its eigenvalues p_1, p_2, p_3 (for the threedimensional case) are calculated. The m-th power of the matrix then is a matrix with eigenvalues:

$$(p_1)^m \quad , \quad (p_2)^m \quad , \quad (p_3)^m ,$$

having the same orientation as the original matrix. Its expression with respect to any system of rectangular coordinates can be easily found in this way. By way of example we mention:

$$\vartheta_p = \begin{Vmatrix} (1+\frac{1}{2}k^2\lambda^2)\,a & \frac{1}{2}k\lambda a \\ \frac{1}{2}k\lambda a & a \end{Vmatrix},$$

where $a = (1+\frac{1}{4}k^2\lambda^2)^{-1/2}$.

[10]) Compare K. WEISSENBERG, l.c. p. 93, sub *b*). — It is confirmed by applying the equations given by R. S. RIVLIN, Large elastic deformations of isotropic materials, Philos. Trans. Roy. Soc. London A **240**, p. 459, 1948, eqs. (3.9) and (3.10).

4. It will be possible to extend the formulae of the preceding section in such a way that they embrace the general case of a homogeneous field of deformation. On the other hand, when the velocity components are non linear functions of the coordinates, other features will come into the picture. It is doubtful whether a phenomenological treatment of such a case would be worthwhile, as it is to be expected that the actual physical relations and a proper analysis of the relaxation phenomenon and of the range of the inter-molecular forces (which range may be considerably extended in certain directions when the molecules are very long) will play an important part. We therefore leave aside the case of non-homogeneous fields.

Résumé.

Le but de cette note est de montrer comment certaines formules de WEISSENBERG ayant trait aux relations entre tensions et déformations dans un milieu „plasto-élastique" peuvent obtenir une illustration si on se base sur la théorie de la relaxation pour expliquer la possibilité d'un mouvement illimité d'un tel corps, et si on suppose que le temps de relaxation soit assez grand pour que l'état physique du corps en mouvement diffère beaucoup de l'état normal.

Resumo.

La jena artikolo celas montri kiamaniere kelkaj formuloj de WEISSENBERG pri la rilatoj inter tensioj kaj aliformiĝoj en medio plastik-elasta povas esti ilustrataj, kiam oni bazas sin sur la teorio de la malstreĉiĝo por klarigi la eblon de nelimita movado de tia korpo, supozante ke la tempo de malstreĉiĝo daŭru sufiĉe longe por ke la stato fizika de la korpo moviĝanta diferencu multe de la stato normala.