Mathematics. — *A note on Mathieu functions.* By C. J. BOUWKAMP. (Communicated by Prof. BALTH. VAN DER POL.)

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To the best of my knowledge, there have been given four different proofs (see ref. 1–4) of the following theorem:

The differential equation of Mathieu (see ref. 5)

$$y'' + (a - 2q \cos 2z)\, y = 0. \qquad \text{(A)}$$

cannot have two linearly independent solutions periodic in z of period 2π unless $q = 0$, $a = n^2$, where n is an integer.

The main purpose of this note is to communicate of this theorem a fifth proof which may be of some interest on account of its simplicity. Henceforth, any solution of (A) that is periodic in z of period 2π will be termed Mathieu function. The parameters a and q are not restricted to real values; the case $q = 0$ is trivial.

As is well known (see ref. 5), there exist four different types of Mathieu function, each of wich possesses its characteristic Fourier-series expansion, *viz.*

(*i*) functions that are even in z, of (least) period π,

$$\mathrm{ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(q) \cos 2rz; \qquad (1)$$

(*ii*) functions that are even in z, of (least) period 2π,

$$\mathrm{ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(q) \cos (2r+1) z; \qquad (2)$$

(*iii*) functions that are odd in z, of (least) period 2π,

$$\mathrm{se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(q) \sin (2r+1) z; \qquad (3)$$

(*iv*) functions that are odd in z, of (least) period π,

$$\mathrm{se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(q) \sin (2r+2) z. \qquad (4)$$

The relation that is required between a and q, in order that Mathieu's differential equation shall admit of solutions of one of the types (1)–(4), is most simply expressed in terms of a transcendental equation involving an infinite continued fraction (see ref. 5).

These eigenvalue-equations are, in order,

$$0 = -\frac{a}{2} - \frac{q^2 \,|}{|\, 4-a} - \frac{q^2 \,|}{|\, 16-a} - \frac{q^2 \,|}{|\, 36-a} - \ldots, \quad . \quad . \quad . \quad \text{(I)}$$

$$0 = q + 1 - a - \frac{q^2 \,|}{|\, 9-a} - \frac{q^2 \,|}{|\, 25-a} - \frac{q^2 \,|}{|\, 49-a} - \ldots, \quad . \quad . \quad \text{(II)}$$

$$0 = -q + 1 - a - \frac{q^2 \,|}{|\, 9-a} - \frac{q^2 \,|}{|\, 25-a} - \frac{q^2 \,|}{|\, 49-a} - \ldots, \quad . \quad \text{(III)}$$

$$0 = 4 - a - \frac{q^2 \,|}{|\, 16-a} - \frac{q^2 \,|}{|\, 36-a} - \frac{q^2 \,|}{|\, 64-a} - \ldots, \quad . \quad . \quad \text{(IV)}$$

each of which has an infinite number of roots $a_\nu(q)$, where q is regarded as the independent variable.

All this has substantially been known since HEINE (ref. 6), well before the publication of INCE's paper referred to (ref. 1).

Let us now turn to the new proof in question. The theorem of INCE will have been proved when any pair out of equations (I)–(IV) is shown to have no roots in common ($q \neq 0$). Now, it is quite obvious that (I) and (IV) have no common root. Indeed, those values of a that satisfy (IV), for some fixed q, make the right-hand member of (I) infinite. Further, (II) and (III) cannot have any root in common because the right-hand side of (II) equals $2q \neq 0$ for those values of a that satisfy (III).

There does not seem to exist a direct proof that the remaining pairs of equations, *viz.* (I, II), (I, III), (II, IV), and (III, IV), behave in the same manner. However, the non-existence of common roots in these cases can be proved from the very start, that is, from the differential equation itself.

Obviously, any point z_0 of the finite plane of complex z is an ordinary point of Mathieu's equation. That is to say, the values of $y(z_0)$ and $y'(z_0)$ may be chosen at will; the relevant solution of Mathieu's equation then is unique, for any fixed a and q. This implies that the general solution of Mathieu's equation cannot vanish at some $z = z_0$. Neither can its derivative. In particular, the functions (1) and (2) cannot be coexistent solutions; if they were, the general solution would be an even function of z, thus having a vanishing derivative at $z = 0$. Similarly, the odd functions (3) and (4) cannot form a fundamental system of solutions. Further, the combinations (2,4) and (1,3) are impossible because then the general solution, respectively its derivative, would vanish at $z = \pi/2$. This completes the proof.

In concluding, I give some numerical information in connection with the eigenvalues of $\mathrm{ce}_0\,(z, q)$ and $\mathrm{ce}_2\,(z, q)$ for *purely imaginary* q ($= \pm is$, $s > 0$). As has been observed by MULHOLLAND and GOLDSTEIN (ref. 7), the function $a(q)$ may show branch points on the imaginary axis of q. The pair of these singularities nearest the origin, in case (I), is at $q = \pm i s_0$ where $s_0 \approx 1.468$ (Ref. 5 and 7). For this critical value of q the functions

$ce_0\ (z, q)$ and $ce_2\ (z, q)$ are no longer distinct; the relevant eigenvalue corresponds to a double root of (I). When $0 \leq s \leq s_0$ the eigenvalues of ce_0 and ce_2 are distinct and real; when $s > s_0$ they are conjugate-complex. It has been suggested by MC LACHLAN (ref. 5) that the critical eigenvalue $a_0 = a(i\, s_0)$ is equal to 2.

If this was true, s_0 would be the positive root of

$$1 = \frac{s^2 |}{| 2} + \frac{s^2 |}{| 14} + \frac{s^2 |}{| 34} + \frac{s^2 |}{| 62} + \ldots \qquad (5)$$

Since I do not understand why this should be so, and, moreover, since careful extrapolation of the numerical results of MULHOLLAND and GOLDSTEIN indicate that the critical eigenvalue slightly exceeds 2, I have taken the trouble to carry out some calculations in the critical range. It is thereby advisable to regard a as the independent variable instead of q. See table I.

TABLE I.

ce_0		ce_2	
a	s	a	s
2.0	1.467344...	2.20	1.466506...
2.05	1.468497...	2.15	1.468084...
2.075	1.468734...	2.10	1.468745...
2.080	1.468754...	2.095	1.468761...
2.085	1.468766...	2.090	1.468768...
$a_0 = 2.088...$		$s_0 = 1.468768...$	

Since $s = 1.46876852...$ when $a = 2.088$, we have, rounding off,

$$s_0 = 1.468769,$$

in which the error in the last decimal does not exceed half a unit. This should be compared with the root of (5), $\sigma = 1.467344....$

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