Mathematics. — *Modern operational calculus based on the two-sided Laplace integral.* I. By BALTH. VAN DER POL and H. BREMMER.

(Laboratorium voor Wetenschappelijk Onderzoek, N.V. Philips' Gloeilampenfabrieken, Eindhoven, Nederland.)

(Communicated at the meeting of September 28, 1948.)

1. *Introduction.*

The operational calculus, as often used by technicians, goes back to OLIVER HEAVISIDE, who introduced his heuristic methods with very great practical success. Although his approach is far from being mathematically rigorous, HEAVISIDE himself already drew attention to the fact that his methods and procedures could be derived from the Laplace transform [1]).

HEAVISIDE's operational methods were mainly meant as a tool for investigating linear electrical systems to which at the time $t = 0$ suddenly an electromotive force was applied, the system being originally at rest. Therefore the transform $f^*(p)$ of the time function $h(t)$, as used by HEAVISIDE and most of his followers, is the following:

$$f^*(p) = p \int_0^{\infty} e^{-pt}\, h(t)\, dt, \quad \ldots \ldots \quad (1)$$

with 0 as lower limit of integration.

All the work by BROMWICH, CARSON, VAN DER POL, NIESSEN, WAGNER, HUMBERT, MCLACHLAN and many others in this field is based on the *one-sided* Laplace transform. However, already before 1940 we worked out an operational calculus, well suited for practical applications, which is based *ab initio* on the *two-sided* Laplace transform

$$f(p) = p \int_{-\infty}^{\infty} e^{-pt}\, h(t)\, dt \quad \ldots \ldots \quad (2)$$

with $-\infty$ as lower limit of integration instead of 0. Henceforth, the integral relation (2) is shortly written as follows:

$$f(p) \doteqdot h(t), \qquad \alpha < \operatorname{Re} p < \beta,$$

where the strip of convergence of the integral (2), viz., $\alpha < \operatorname{Re} p < \beta$, has to be specified explicitly. When, moreover, an 'original' $h(x, y)$ of two variables x and y is transformed as

$$f(p, q) = pq \int_{-\infty}^{\infty} e^{-px} \int_{-\infty}^{\infty} e^{-qy}\, h(x, y)\, dx\, dy,$$

[1]) O. HEAVISIDE, Electromagnetic Theory, London, Benn Brothers, 1922. Vol. III, p. 236.

we write for short

$$f(p, q) \mathrel{\ddot{\overline{\underset{\cdot\cdot}{}}}} h(x, y), \qquad \text{etc.}^{2)}$$

Although the Laplace transform as such has long been known, there is certainly room for an operational calculus based on this Laplace transform (particularly the two-sided) because the operational (or symbolic) methods often lead in an extremely short way to a solution of complicated problems, once the rules and theorems of this calculus have been mastered. This is true not only for many technical problems, but also for large parts of the analysis, e.g., linear differential equations (both with constant and variable coefficients), difference equations, partial differential equations, integral equations, potential theory, number theory, etc. The situation here is analogous to that of the theory of linear equations or vector analysis where complicated calculations can often be reduced to simple procedures, owing to the introduction of determinants, matrices and concepts such as gradients, curl-vectors.

It is just this new symbolism of the two-sided Laplace transform which shows its great heuristic value, and many new results have been obtained during the eight years of its application [3]).

In practical applications of the operational calculus as expounded below, it is very seldom necessary to refer to the inversion integral corresponding to (2), viz.

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\frac{f(p)}{p}\,dp. \qquad (\alpha < c < \beta) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Moreover, an explicit use of the Laplace integral (2) as such is only rarely needed since the available rules usually enable us to find the solution of our problem right away. However, the Laplace transform being the rigorous mathematical basis, every step in the process of an operational

---

[2]) We originally introduced the symbol $\doteqdot$ for a one-sided Laplace transform. Some authors use with the same meaning the symbol $\supset$, which, however, might be confused with a similar symbol used in the theory of sets and which has already a different mathematical meaning. We shall therefore adhere to the definition of $\doteqdot$ as given above and which therefore represents the two-sided Laplace transform. The upper dot is always towards the original so that we can write either

$$f(p) \doteqdot h(t)$$

or

$$h(t) \doteqdot f(p),$$

both being short-hand notations for (2).

[3]) An extensive volume on the operational calculus based on the two-sided Laplace transform is now in course of publication at the Cambridge University Press. Several new results were already published during and after the war in the 'Wiskundige Opgaven', Groningen, Noordhoff; 1943, numbers 33, 34, 35, 36, 38; 1946, numbers 77, 120.

solution of a problem can fully be interpreted in terms of these transforms; thus a completely rigorous control of all the steps is always possible.

It is the purpose of the present paper to expound the general lines and also the advantages of this new form of the operational calculus and to point out the improvements and gains with respect to the older form based on the one-sided Laplace integral.

2. *The strip of convergence.*

In the consideration of one-sided Laplace integrals, the indication of the strip of convergence is usually omitted. This will cause no misunderstanding since, in that case, the strip is always a domain of the p-plane reaching to infinity at the right whereas the left boundary $\operatorname{Re} p = a$ is determined by the condition of convergence of the one-sided Laplace integral. In the case of two-sided Laplace integrals, however, there may exist several strips for which one and the same image function $f(p)$ corresponds to different originals $h(t)$. Consequently, the specification of the strip is absolutely necessary. In special cases this strip may cover the total p-plane, e.g., when the original is non-vanishing in a finite interval of t only. Less trivial examples showing strips coinciding with the total p-plane, are:

$$e^{-\alpha t^2} \doteqdot \sqrt{\frac{\pi}{\alpha}}\, p\, e^{\frac{p^2}{4\alpha}}, \qquad -\infty < \operatorname{Re} p < \infty,$$

$$\frac{1}{2}\left(\frac{d^2}{dt^2} - \frac{1}{4}\right)\left\{ e^{\frac{t}{2}}\, \theta_3(0, e^{2t}) \right\} \doteqdot p\, \xi(p + \tfrac{1}{2}), \qquad -\infty < \operatorname{Re} p < \infty,$$

where

$$\theta_3(0, x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},$$

and

$$\xi(p) = (p-1)\, \pi^{-\frac{p}{2}}\, \Pi\left(\frac{p}{2}\right) \zeta(p)$$

is, as usual, defined so that the functional equation for Riemann's ζ-function is equivalent to the observation that $\xi(p + 1/2)$ is an even function of p.

3. *The unit function.*

The discontinuous function $U(t)$ defined by

$$U(t) = \begin{cases} 1 & (t > 0) \\ \frac{1}{2} & (t = 0) \\ 0 & (t < 0) \end{cases}$$

is called the 'unit function'. It was already considered by CAUCHY, who named it 'coefficient limitateur' or 'restricteur' [4]). For our purpose, this

[4]) Enzyklopaedie Math. Wiss. II, 1, 2, p. 1324.

function is particularly important because in using it we can consider the one-sided operational calculus as a special case of the two-sided calculus. In fact, originals vanishing for $t < 0$ may be written as

$$h^*(t) = h(t)\, U(t),$$

while the corresponding two-sided Laplace transform

$$p \int_{-\infty}^{\infty} e^{-pt}\, h(t)\, U(t)\, dt = p \int_{0}^{\infty} e^{-pt}\, h(t)\, dt,$$

is automatically reduced to a one-sided transform. When using therefore the two-sided calculus, it is not allowed to omit the factor $U(t)$ in one-sided originals (which are therefore zero for $t < 0$). This is clear since a given original $h(t)$ may have an image without more, as well as another image after it has been replaced by zero for $t < 0$. An illustrative example is the following

$$\frac{1}{e^t + 1} \doteqdot -\frac{\pi p}{\sin(\pi p)}, \qquad -1 < \operatorname{Re} p < 0, \tag{4a}$$

$$\frac{U(t)}{e^t + 1} \doteqdot \frac{p}{2} \left\{ \psi\left(\frac{p}{2}\right) - \psi\left(\frac{p-1}{2}\right) \right\}, \qquad -1 < \operatorname{Re} p < \infty \tag{4b}$$

($\psi =$ logarithmic derivative of GAUSS's Π-function).

The unit function also plays a role in many other questions occurring in the operational calculus. In this respect we mention the notation

$$h(t)\; U(t-a)$$

for an arbitrary function $h(t)$ which is made to vanish for $t < a$ (compare the example of section 9).

4. *The shift rule.*

In the two-sided calculus most of the elementary 'rules' have a slightly simpler form than in the older one-sided calculus. E.g., in the differentiation rule (stating that a differentiation of the original corresponds to a multiplication by p of the image) the restriction $h(0) = 0$ of the one-sided calculus can be dropped (see section 7).

A simplification also occurs in the case of the 'shift rule'. In the one-sided calculus, which only concerns positive arguments of the original, this rule reads:

Given $$h(t) \doteqdot f(p),$$

then we have

(1) if $\lambda > 0$,

$$e^{\lambda p} f(p) \doteqdot h(t + \lambda);$$

(2) if $\lambda < 0$,

$$e^{\lambda p} f(p) \doteqdot \begin{cases} h(t+\lambda) & (t > -\lambda) \\ 0 & (0 < t < -\lambda). \end{cases} \tag{5a}$$

In the two-sided calculus, however, the distinction between positive and negative values of λ disappears. The final formulation there simply amounts to:

Given
$$h(t) \doteqdot f(p), \qquad \alpha < \operatorname{Re} p < \beta,$$
then we have
$$h(t+\lambda) \doteqdot e^{\lambda p} f(p), \qquad \alpha < \operatorname{Re} p < \beta. \tag{5b}$$

The one-sided rule (5a) is obtained as a special case of the two-sided rule (5b) by substituting in the latter $h(t)U(t)$ for $h(t)$.

We give here two applications of the general shift rule (5b) illustrating, moreover, the usefulness of the unit function.

(A) The construction of the image of 'step functions', i.e., of functions $h(t)$ that are constant between two consecutive integer values of t. As an example we consider the one-sided function that increases by unity at each of the points $t = \log n$ (n integer). This function may be represented by

$$\sum_{n=1}^{\infty} U(t - \log n) = \sum_{n=1}^{\infty} U(e^t - n) = \sum_{n=1}^{[e^t]} 1 = [e^t]$$

($[x]$ = greatest integer not greater than x).

Starting from the fundamental relation
$$U(t) \doteqdot 1, \qquad 0 < \operatorname{Re} p < \infty,$$
which follows at once from (2), the shift rule yields
$$U(t - \log n) \doteqdot e^{-p \log n} = \frac{1}{n^p}, \qquad 0 < \operatorname{Re} p < \infty,$$
so that we have
$$[e^t] = \sum_{n=1}^{\infty} U(t - \log n) \doteqdot \sum_{n=1}^{\infty} \frac{1}{n^p}.$$
The image here found is the Dirichlet series for the ζ-function; since this series only converges for $\operatorname{Re} p > 1$, we obtain at once the operational relation
$$[e^t] \doteqdot \zeta(p), \qquad 1 < \operatorname{Re} p < \infty.$$
This transform can be made the basis of a large part of modern arithmetic [5]).

(B) A function given originally in the interval $0 < t < 1$ can easily be continued periodically outside this interval. In these cases we can start from the relation
$$h(t)\{U(t) - U(t-1)\} \doteqdot f(p), \qquad -\infty < \operatorname{Re} p < \infty,$$

---

[5]) See BALTH. VAN DER POL, Application of the Operational or Symbolic Calculus to the Theory of Prime Numbers, Phil. Mag. **26**. 925 (1938).

expressing that the given function, which is zero for $t < 0$ and $t > 1$, will in general have some image $f(p)$. The periodic function in question is representable by

$$\begin{aligned} h(t-[t])\, U(t) = {} & h(t)\{U(t)-U(t-1)\} + \\ & + h(t-1)\{U(t-1)-U(t-2)\} + \\ & + h(t-2)\{U(t-2)-U(t-3)\} + \dots \end{aligned}$$

The shift rule at once leads to the corresponding image, viz.

$$h(t-[t])\, U(t) \doteqdot f(p) + e^{-p} f(p) + e^{-2p} f(p) + \dots$$

The summation of this geometric series is possible for all p having positive real part. Thus we arrive at the final result

$$h(t-[t])\, U(t) \doteqdot \frac{f(p)}{1-e^{-p}}, \qquad 0 < \operatorname{Re} p < \infty.$$

5. *The rule for the composition product.*

In the one-sided calculus this rule reads:

Given $$h_1(t) \doteqdot f_1(p) \quad ; \quad h_2(t) \doteqdot f_2(p),$$

then it follows that

$$\int_0^t h_1(\tau)\, h_2(t-\tau)\, d\tau \doteqdot \frac{1}{p} f_1(p)\, f_2(p). \qquad \dots \quad (6a)$$

The corresponding rule in the two-sided calculus is simpler insofar as the composition integral (sometimes the term 'convolution' is used) [6]) has constant limits of integration. In fact, the complete rule now becomes:

Given

$$\begin{aligned} h_1(t) \doteqdot f_1(p), \qquad & \alpha_1 < \operatorname{Re} p < \beta_1, \\ h_2(t) \doteqdot f_2(p), \qquad & \alpha_2 < \operatorname{Re} p < \beta_2, \end{aligned}$$

then it follows that

$$\left.\begin{aligned} \int_{-\infty}^{\infty} h_1(\tau)\, h_2(t-\tau)\, d\tau \doteqdot \frac{1}{p} f_1(p)\, f_2(p), \\ \max(\alpha_1, \alpha_2) < \operatorname{Re} p < \min(\beta_1, \beta_2). \end{aligned}\right\} \qquad \dots \quad (6b)$$

It has to be stressed that an image of a composition product exists only when the two initial strips of convergence overlap. The existence of the corresponding common strip is guaranteed in the case of one-sided originals

[6]) We prefer the term 'composition product' (as given by VOLTERRA) instead of 'convolution' because the fundamental principle can be extended to more (and even to an infinite number of) dimensions, in which case the 'folding' idea is lost.

(since each strip extends to infinity in the right part of the p-plane). In the two-sided calculus, however, the condition of overlapping is not self-evident; it is expressed analytically by

$$\max(\alpha_1, \alpha_2) < \operatorname{Re} p < \min(\beta_1, \beta_2).$$

In this connection we remark that, if a common strip is lacking, the rule (6*b*) is still applicable after a transformation, of one or both of the primary relations, with the aid of the 'attenuation rule'.

The latter states, in both the one-sided and the two-sided calculus:

Given $\qquad h(t) \doteqdot f(p), \qquad \alpha < \operatorname{Re} p < \beta,$

then it follows that

$$e^{-at} h(t) \doteqdot \frac{p}{(p+a)} f(p+a), \qquad \alpha - \operatorname{Re} a < \operatorname{Re} p < \beta - \operatorname{Re} a.$$

An example may illustrate the possibility of such an indirect application of the composition-product rule. We start from the relation

$$e^{-e^{-t}} \doteqdot \Pi(p), \qquad 0 < \operatorname{Re} p < \infty, \quad \ldots\ldots \quad (7)$$

which is easily verified by reducing its Laplace integral to Euler's second integral for the Π-function. According to the rule concerning the transformations of t into $-t$, we have also:

$$e^{-e^{t}} \doteqdot -\Pi(-p), \qquad -\infty < \operatorname{Re} p < 0. \quad \ldots\ldots \quad (8)$$

The strips of convergence of (7) and (8) are not overlapping but adjacent. In order to construct a composition product, we replace (7) by the following relation, obtained with the aid of the attenuation rule,

$$e^{-t} e^{-e^{-t}} \doteqdot \frac{p}{(p+1)} \Pi(p+1) = p\,\Pi(p), \qquad -1 < \operatorname{Re} p < \infty. \quad . \quad (9)$$

Now, the relations (8) and (9) have a common strip, viz. $-1 < \operatorname{Re} p < 0$. The composition-product rule can now be used and leads to

$$\int_{-\infty}^{\infty} e^{-\tau} e^{-e^{-\tau}} \cdot e^{-e^{t-\tau}} d\tau \doteqdot -\Pi(p)\,\Pi(-p), \qquad -1 < \operatorname{Re} p < 0. . \quad (10)$$

The substitution $e^{-\tau} = s$ transforms the integral into the original of the relation (4*a*), so that each of the two functions

$$-\Pi(p)\,\Pi(-p) \quad \text{and} \quad -\frac{\pi p}{\sin(\pi p)}$$

are found as image of $\dfrac{1}{e^t+1}$ (for $-1 < \operatorname{Re} p < 0$). By virtue of the uniqueness of the Laplace integral, we thus have demonstrated the relation

$$\Pi(p)\,\Pi(-p) = \frac{\pi p}{\sin(\pi p)}$$

for $-1 < \operatorname{Re} p < 0$. The validity of this formula for other values of p then follows from the principle of analytic continuation.

We conclude our considerations on the composition-product rule with two remarks:

(1) The one-sided form (6*a*) is obtained as a special case of the two-sided form (6*b*) by a substitution of $h_1(t)\,U(t)$ and $h_2(t)\,U(t)$ for $h_1(t)$ and $h_2(t)$ respectively, which substitution automatically introduces the limits of integration of (6*a*);

(2) By identifying $h_2(t) \doteqdot f_2(p)$ with the operational relations

$$U(t) \doteqdot 1, \qquad 0 < \operatorname{Re} p < \infty,$$
$$-U(-t) \doteqdot 1, \qquad -\infty < \operatorname{Re} p < 0,$$

respectively, we get the new *integration rule*:

Given $\qquad h(t) \doteqdot f(p), \qquad \alpha < \operatorname{Re} p < \beta,$

then it follows that

$$\int_{-\infty}^{t} h(\tau)\,d\tau \doteqdot \frac{1}{p} f(p), \qquad 0 < \operatorname{Re} p < \beta,$$

$$\int_{\infty}^{t} h(\tau)\,d\tau \doteqdot \frac{1}{p} f(p), \qquad \alpha < \operatorname{Re} p < 0.$$

6. *General advantages of the two-sided calculus.*

Some very striking advantages are:

(1) A simple formulation of the general operational rules (compare the two preceding sections).

(2) The possibility of treating functions whose two-sided image is simpler than their one-sided image or whose one-sided image even is lacking. In this respect we refer to the examples (4) and (7). Another typical two-sided original is

$$\frac{1}{e^{e^{-t}}-1} \doteqdot \Pi(p)\,\zeta(p), \qquad 1 < \operatorname{Re} p < \infty,$$

while many other relations of this kind are dealt with in section 10.

(3) The possibility of considering images having no original at all in the one-sided calculus. The increase of the number of available image functions follows, e.g., from the theorem that in the one-sided calculus there do not exist images $f(p)$ having equidistant zeros on a line parallel to the real p-axis. Such a restriction does not occur in the two-sided calculus. An example showing such a two-sided Laplace transform with an infinity of zeros on the real p-axis itself, is given by

$$2^{-\frac{p}{2}} \sin\left(\frac{\pi}{4} p\right) \Pi(p) \doteqdot e^{-e^{-t}} \sin(e^{-t}), \qquad -1 < \operatorname{Re} p < \infty.$$