Physics. - Superquantization. I. By H. J. Groenewold. (Koninklijk Nederlands Meteorologisch Instituut te De Bilt.) (Communicated by Prof. F. A. Vening Meinesz.)
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## 1. Introduction.

1.1 Quantization processes. We begin with shortly recalling the general situation in the current theories of quantization.
1.11 Historical situation. 1.111 Classical theory. In classical theory, which is the starting point, one has particles (e.g. electrons) and fields (e.g. the electromagnetic field). If necessary we shall distinguish classical concepts by a prefix $c$ and e.g. write c-particles and c-fields. c-particles and c-fields may interact with each other: c-particles generate c-fields; c-fields act upon c-particles. In the usual dualistic theory c-particles and $c$-fields are quantized separately in the following steps.
1.112 Particle quantization. The classical quantities belonging to cparticles are replaced by quantum operators. The latter act on wave functions, which represent the states of the c-particles. If there is a fixed number $n$ of c-particles, the wave functions depend on $n$ complete sets of coordinates and are called $n$-particle wave functions. The classical equations of motion are replaced by wave equations (SchröDINGER representation) or by operator equations (Heisenberg representation). The wave functions can be considered to describe quantum fields ( $q$-fields) related to the c-particles.
1.113 Field quantization. c-fields are quantized in a somewhat analogous way. The field quantities are replaced by field operators. The field equations are replaced by operator equations. The quantized c-fields show discrete properties, which can be interpreted from a particle point of view. They can be considered to describe quantum particles ( $q$-particles) related to the c-fields (e.g. photons related to the electromagnetic field). The field operators can be expressed in terms of creation and annihilation operators, which describe creation (emission) and annihilation (absorption) of $q$-particles.
1.114 Superquantization. The 1-particle wave functions of ordinarily quantized c-particles can in their turn be superquantized by a process, which is almost entirely analogous with the quantization of c-fields. The 1 -particle wave functions are replaced by wave operators, the wave equations by superquantized wave equations. Also this form of field quantization leads to the aspect of $q$-particles. The wave operators can again be expressed in terms of creation and annihilation operators, which describe creation and annihilation of $q$-particles.
1.12 Logical situation. 1.121 Particle quantization. Historically ordinary particle quantization makes the progressive step from the less correct classical theory towards the relatively more correct ordinary quantum theory. As soon as the latter has entirely been established, the quantization process has performed its pioneering duty. This historical aspect of the relation between classical and quantum theory has to be sharply distinguised from logical problems on the inner structure of ordinary quantum theory and its relations to observation. In this paper we shall not make such problems.
1.122 Superquantization. With regard to 1-particle wave functions the process of superquantization, which introduces the aspect of many $q$-particles, undoubtedly makes a progressive step. It is, however, perhaps not always clearly apprehended that superquantization does not lead one step beyond the complete ordinary quantum theory of c-particles with manyparticle wave functions. In fact superquantization can be entirely understood within the scope of the latter theory. This paper only contains news for those, who are still surprised by this point of view.

We shall assume the reader to be familiar with the usual theories of superquantization. As it has been said before, these theories introduce wave operators and creation and annihilation operators, which describe creation and annihilation of $q$-particles. In our line of reasoning we shall in the ordinary quantum theory with many-particle wave functions of c-particles introduce creation and annihilation operators, which act on the manyparticle wave functions and which describe creation and annihilation of c-particles, and wave operators. This introduction is not a matter of a new assumption, which leads beyond the ordinary quantum theory, it is merely a matter of definitions within this theory. Our aim will be to chose these definitions in such a way, that our creation and annihilation operators and wave operators become isomorphic to those of superquantization. In distinction with the latter process we shall denote our process as "superquantization".

If we succeed to establish the isomorphy, the two descriptions are entirely equivalent. The one does not go beyond the other. A further consequence of the isomorphy between the creation and annihilation operators of $c$-particles and those of $q$-particles is that there is no logical distinction between c-particles and $q$-particles. Therefore the two concepts can be identified and the prefix can be omitted.

In introducing the wave operators we shall not immediately make the right choice, which gives the required isomorphy with superquantization. We shall begin with a preliminary choice, first in statical, thereafter in dynamical representation. Later on comparison with the present theories of superquantization will show how the choice still has to be modified. Historically the introduction of wave operators and every step in our derivation is entirely guided by superquantization. On logical grounds,
however, the latter would deserve to be put between the humiliating quotation marks, which we have reserved for our process, just as well.
1.123 Field quantization. It is beyond doubt that historically field quantization is a progressive step. As soon as the quantization has been established we may with lack of historical reverence come to a logically more coherent picture by the following reinterpretation. Though originally we had no c-particles, the $q$-particles related to the c-field are identified with c-particles and the prefix is omitted. The creation and annihilation operators act on many-particle wave functions. The latter describe the $q$-field related to the particles. From this point of view the situation after field quantization is entirely similar to the situation after particle quantization. c-fields must then be considered as similar to 1 -particle wave functions (appropriately normalized), field quantization as similar to superquantization.
1.124 Particles and fields. It may be useful to state precisely what we now mean with particles and fields. Our fields are described by ordinary wave functions depending on complete sets of coordinates. Every complete set represents one particle (for indiscernible particles there is no individual one-to-one correspondence, only an assembly of $n$ sets corresponds to an assembly of $n$ particles).
1.2 Source particles and carrier particles. 1.21 Dualistic theory. In classical theory there are particles and fields (in the sense of 1.111, not of 1.124). Though they may interact with each other, their equations of motion are treated separately. Therefore the theory is dualistic. After quantization, which is also done separately, we get in both cases wave functions related to particles. That does not mean, however, that from our point of view quantum theory is unitary. For one obtains different kinds of particles. They may interact with each other, but their wave equations are still treated separately. Therefore the theory is still dualistic.
1.211 Interaction. In classical theory a field generated at a certain place and time by one particle may at another place and time act on another particle (e.g. an electromagnetic field generated by and acting on electrons). In this way by intervention of the field the particles indirectly interact with each other at different places and times. After quantization we get two kinds of particles. A particle of the first kind in a certain region of space-time emits a particle of the second kind, which in another region of space-time is absorbed by another particle of the first kind (e.g. a photon emitted and absorbed by electrons). In this way by intervention of the particles of the second kind the particles of the first kind interact with each other in different regions of space-time. Because of this behaviour we shall call particles of the first kind source particles and particles of the second kind carrier particles. If particles of a certain kind interact with particles of various other kinds, it may occur that in some interactions they behave as source particles and in other interactions as carrier particles (e.g.
charged mesons interacting with photons act as source particles, interacting with nucleons they act as carrier particles). Therefore the classification is not always unique.
1.212 Observability. The ordinary particle operators in general represent observables (particle observables). For some kinds of particles also the wave operators (if they are of commutator type) represent observables (field observables), for other kinds (if they are of anticommutator type) they do not. Whereas particles of both kinds can in classical theory be approximated by means of c-particles, only particles of the first kind can in classical theory be approximated by means of c-fields. The latter approximation will in general be used if they act as carrier particles, the former if they act as source particles.
1.22 Unitary theories. There are two ideals of unitary theories. They are opposite to each other, though a priori they need not exclude each other.
1.221 Unitary f-theory. Classical unitary field theory considers particles as singularities of the fields. The equations of motion and all other properties of the particles are entirely determined by those of the fields. Also the quantization of particles is determined by the quantization of fields. A complete theory has not been given. It might be difficult to fit such a theory in our scheme in which particle quantization and not field quantization is treated as the important step.
1.222 Unitary p-theory. Classical unitary particle theory starts with nothing else than particles, which have retarded and/or advanced interaction at a distance. The description of this interaction may be extremely complicated. Not before all equations of motion have been established, fields may be introduced in order to simplify the description. The equations of motion and all other properties of the fields are entirely determined by those of the particles and/or by the way in which the fields are introduced. Also the quantization of fields is entirely determined by the quantization of interacting particles. Or: quantum unitary p-theory starts with nothing else than source particles, which have retarded and/or advanced interaction at a distance. Not before all wave equations have been established, carrier particles may be introduced. The wave equations and all other properties of the carrier particles are entirely determined by those of the source particles and/or the way in which the carrier particles are introduced. Also here a complete theory has not been given. Such a theory would readily fit into our scheme and greatly simplify its principles.

In a unitary $p$-theory field observations could be entirely reduced to particle observations.
1.3 Plus and minus troubles. It is obvious that our whole picture badly suffers from oversimplification. Among all more or less concealed complications there is one group of difficulties, which during our derivations will become too apparent to be entirely ignored. They are about positive and negative states and positive and negative particles. We shall have to deal
with them in some extent, even though we shall not bother about the malignant divergencies with which they are connected.
1.31 Positive and negative states. 1.311 Energy states. In all cases we shall meet positive and negative energy states. There are two kinds of difficulties with negative states:
$d_{1}$ how to distinguish them properly from positive states;
$d_{2}$ how to get rid of them or to give them a proper physical meaning.
1.312 Density states. For some kinds of particles (those for which the wave equations contain second order time derivatives) one gets also saddled with negative density states. They entail the same two kinds of difficulties as negative energy states. The negative density states combine with the negative energy states in such a way, that it sometimes nearly looks as if the difficulties cancel.
1.32 Positive and negative particles. The creation or annihilation of pairs of opposite particles should be treated with due regard to the interaction between the two particles. If one tries to do so, one gets entangled with divergencies. These and other difficulties have not been solved, they have only been circumnavigated by means of an ingeneous trick. This trick (the hole theory) is to neglect the interaction and to put on a par creation or annihilation of a particle in a given energy state and annihilation or creation of an opposite particle in the opposite energy state. This trick has either openly or tacitly been performed in nearly all current theories. If we want to establish isomorphy with these theories, we are also bound to perform the trick. That truly makes a step beyond the original ordinary theory. It should be emphasized, however, that this step is entirely distinct from processes of quantization. And further it should not be forgotten that it is a tentative capriole rather than a firm step, which leads out of the difficulties.
1.4 Typical processes and queries. The following processes $P_{1}, P_{2}$ and $P_{3}$ and queries $Q_{1}, Q_{2}$ and $Q_{3}$ are loosely in parallel with each other and with the foregoing sections $1.1,1.2$ and 1.3.
1.41 Processes. Creation and annihilation operators serve to describe the elementary processes of creation and annihilation of particles. There are some typical compound processes, which can be described by pairs of such operators and which deserve our special attention:
$P_{1}$ transition of a particle from one state to another state;
$P_{2}$ birth and death of a carrier particle;
$P_{3}$ birth or death of a pair of opposite particles.
$P_{1}$ can be conceived so that a particle is annihilated in one state and a particle of the same kind is created in another state. In $P_{2}$ the carrier particle is first created somewhere and later annihilated elsewhere. The trick of the hole theory tries to reduce the pair creation or annihilation in $P_{3}$ to a transition of the type $P_{1}$ between a positive and a negative energy state.
1.42 Queries. Some typical problems in which we are interested are:
$Q_{1}$ does superquantization lead beyond ordinary quantum theory of particles?
$Q_{2}$ how are the relations between the properties of carrier particles and those of source particles; can in particular the former be derived from the latter?
$Q_{3}$ how shall negative states and pair processes be dealt with?
We shall be concerned with $Q_{1}$ and in the meantime leave $Q_{2}$ aside. We cannot do so with $Q_{3}$, which interferes so strongly with $Q_{1}$, that it has to be considered to some extent.
1.5 Present theories. There are two types of present theories based on superquantization ${ }^{1}$ ) with which we shall try to establish isomorphy. Of the one type ${ }^{2}$ ) Dirac's hole theory ${ }^{3}$ ) is typical for half-odd spin and Pauli-Weisskopf's theory ${ }^{4}$ ) for integer spin. The other type is Dirac's 1942 theory ${ }^{5}{ }^{6}{ }^{6}$ ). We shall find that the latter type is entirely equivalent with the ordinary quantum theory of particles. The theories of the first type can only be obtained after performing the trick of hole theory. From this point of view they are all hole theories. We shall not discuss the trick. That entirely belongs to the problems of $Q_{3}$.

It needs hardly to be repeated that in all this there is nothing new. This paper does not claim any originality.

## 2. "Superquantization".

In this section we introduce in the ordinary quantum theory of particles preliminary wave operators, which already strongly resemble those of the present theories.
2.1 Wave functions. Before doing so we have first to deal with the wave functions.
2.11 Notation. Our notation will appear rather cumbersome. In fact a careful detailed notation is essential for a rigorous discussion. The usual notations are not sufficient for this special purpose.
2.12 Wave equations. A system of $n$ particles can in ordinary quantum theory be described by a many-times theory, in which there is a separate time coordinate for each particle, and in some cases also by a single-time theory, in which there is only one time coordinate for the whole system. Let $\left(x_{k}\right)$ stand for the complete set of all individual variables of the $k$ th particle except for the individual time $t_{k}$ or the common time $t$. $\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi\right.$ or $\left(x_{1}, \ldots x_{n} ; t \mid \Psi\right.$ will stand for the wave function, $\left.\Psi^{\dagger} \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right)$ or $\left.\Psi^{\dagger} \mid t ; x_{n}, \ldots x_{1}\right)$ for its Hermitian adjoint. Before a $\Psi$ an operator in the $\left(x_{k}\right)$ and $t_{k}$ or $t$ will act to the right, after a $\Psi^{\dagger}$ it will act to the left (between a $\Psi^{\dagger}$ and a $\Psi$ it does in general not matter which way).

For a system of identical particles the wave equations have the form

$$
\begin{equation*}
\mathbf{K}\left\{x_{k} t_{k}\right\}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi=0 \quad(k=1, \ldots n)\right. \tag{1}
\end{equation*}
$$

in many-times theory, whereas in single-time theory there is a single wave equation

$$
\mathbf{K}\left\{x_{1}, \ldots x_{n} ; t\right\}\left(x_{1}, \ldots x_{n} ; t\right) \Psi=0 .
$$

The Hermitian operators $\mathbf{K}$ may depend on and act on the variables mentioned in curled brackets and also on variables of particles of different kinds with which those of the kind considered interact. $\mathbf{K}\left\{x_{1}, \ldots x_{n} ; t\right\}$ is symmetrical in the $\left(x_{k}\right)$.

Even in intrinsical relativistic invariant theories the explicit invariance may be more or less obscured by the particular part of the time coordinates. A striking case is a single-time theory derived from a relativistic invariant many-times theory.

Where in the following no special reference is made to expressions for single-time theory, they will (as far as they exist) be supposed to be similar to those of many-times theory with all $t_{k}(k=1, \ldots n)$ replaced by $t$.
2.13 Permutations. The permutations $\mathbf{P}$ of all sets $\left(x_{1} t_{1}\right), \ldots\left(x_{n} t_{n}\right)$ or $\left(x_{1}\right), \ldots\left(x_{n}\right)$ form the symmetrical group $S_{n}$. Every even or odd permutation $\mathbf{P}_{\text {even }}$ or $\mathbf{P}_{\text {odd }}$ can be regarded as the product of respectively an even or odd number of interchanges of pairs. In a 1 -dimensional representation of $S_{n}$ all pair interchanges must have the same representative, either $-1(F-D$ representation $)$ or +1 ( $B-E$ representation). The representative $\delta_{\mathbf{P}}$ of a permutation $\mathbf{P}$ is then given by

|  | $F-D$ | $B-E$ |
| :---: | :---: | :---: |
| $\mathbf{P}_{\text {even }}$ | +1 | +1 |
| $\mathbf{P}_{\text {odd }}$ | -1 | +1 |

2.14 Statistics. The equations (1) permute if we commute all $\mathbf{K}$ with a $\mathbf{P}$. In ( $1^{\prime}$ ) $\mathbf{K}$ commutes with all $\mathbf{P}$. Therefore all $\mathbf{P}$ are integrals of motion for (1) as well as for ( $1^{\prime}$ ).

Because identical particles are indiscernible, the wave functions $\Psi$ must provide a 1-dimensional representation of $S_{n}$

$$
\begin{equation*}
\mathbf{P} \Psi=\delta_{\mathbf{P}} \Psi \tag{3}
\end{equation*}
$$

If this is a $F-D$ representation, all $\Psi$ are antisymmetrical (Fermi-Dirac statistics), if it is a $B-E$ representation, all $\Psi$ are symmetrical (BoseEinstein statistics).

For a given representation the symmetry operator $\mathbf{S}_{n}$ will be defined by

$$
\begin{equation*}
\mathbf{S}_{n}=\frac{1}{n!} \sum_{\mathbf{P}} \delta_{\mathbf{P}} \mathbf{P} \tag{4}
\end{equation*}
$$

where the sum is over all $n!$ permutations. $\mathbf{S}_{n}$ is Hermitian and idempotent ( $\mathbf{S}_{n}^{2}=\mathbf{S}_{n}$ ), the eigenvalues are 1 and 0 . The eigenfunctions belonging to 1 are of the required symmetry (either antisymmetrical or symmetrical), those belonging to 0 must be rejected. $S_{n}$ is a projection operator selecting the (anti-) symmetrical component of $\Psi$. As soon as all $\Psi$ have the required symmetry, $\mathbf{S}_{n}$ can be replaced by the eigenvalue 1 .
2.15 Density operator. For particles of a given kind we have Hermitian density operators $\varrho\left\{x_{k} t_{k}\right\}$ (commuting for different $k$ ). In a relativistic theory they are the time components of the 4 -velocity operators. In some cases (e.g. if (1) is a second order differential equation in the time coordinates i.e. for integral spin) the density operators are indefinite. Then we have to distinguish between positive and negative density states. That is the first difficulty $d_{1}$. We try to form operators $\eta\left\{x_{k} t_{k}\right\}$ (also commuting for different $k$ ), which satisfy the conditions
$c_{1} \boldsymbol{\eta}$ is Hermitian $\boldsymbol{\eta}=\eta^{\dagger}$ and unitary $\eta^{2}=1$;
$c_{2} \quad \eta$ commutes with $@$;
$c_{3} \eta \varrho$ (which is Hermitian by $c_{1}$ and $c_{2}$ ) is positive definite;
$c_{4} \eta$ is uniquely determined and in a relativistic invariant theory it is invariant.

Because of $c_{1}$ the eigenvalues of $\eta$ are +1 and $-1 . c_{2}$ and $c_{3}$ can be met by taking for $\boldsymbol{\eta}$ the operator with the same eigenstates and the same sign of eigenvalues as $\varrho$. $c_{4}$ is liable to give trouble. In fact it may be too stringent. -If $\eta$ can be found, the eigenstates with positive eigenvalue will determine the positive density states, those with negative eigenvalue the negative density states. If $\rho$ is positive definite, then $\eta=1$ and all states are positive.
2.16 Inner product. The density operators determine the metric in the Hilbert space of wave functions. In case of an indefinite density operator this metric would also become indefinite. That is the second difficulty $d_{2}$. We pass it off in a rather primitive way. If $\eta$ has been found, we make use of $c_{3}$ and take as the inner product of $\Psi^{\dagger}$ and $\Psi^{\prime}$

$$
\left.\begin{array}{r}
\left.\int \ldots \int\left(d x_{1}\right) \ldots\left(d x_{n}\right) \Psi^{\dagger} \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right) \eta\left\{x_{1} t_{1}\right\} \varrho\left\{x_{1} t_{1}\right\} \ldots \eta\left\{x_{n} t_{n}\right\} \varrho\left\{x_{n} t_{n}\right\}  \tag{5}\\
\\
\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi^{\prime} .\right.
\end{array}\right\}
$$

Thus by intercalating the factors $\eta$ we have obtained a positive definite metric. $\int(d x)$ means integration over all continuous variables and summation over all discrete variables of the set $(x)$. If the wave functions are spinor quantities we get an inner spinor product, if they are 4-tensor quantities we get an inner 4-tensor product. The density operators are subjected to the condition that if $\Psi$ and $\Psi^{\prime}$ satisfy the wave equations (1) or ( $1^{\prime}$ ), the inner product (5) or ( $5^{\prime}$ ) has to be an integral of motion. In a relativistic theory more generally the 4-divergence of the 4 -velocity has to vanish.

It cannot be said that in 2.15 and 2.16 the difficulties $d_{1}$ and $d_{2}$ with negative density states have been solved in an elegant or even in a satisfactory way.
$2.17(x)$ and $(\mu)$ representation. We shall have to consider transformations between various representations.

Choose a complete system of orthonormal individual wave functions $(x t|\psi| \mu)$

$$
\left.\begin{array}{l}
\int(d x)\left(\mu\left|\psi^{\dagger}\right| x t\right) \eta\{x t\} \varrho\{x t\}\left(x t|\psi| \mu^{\prime}\right)=\delta_{\mu \mu^{\prime}} ;  \tag{6}\\
\int\left(d x^{\prime}\right)\left\{\sum_{(\mu)}(x t|\psi| \mu)\left(\mu\left|\psi^{\dagger}\right| x^{\prime} t\right)\right\} \eta\left\{x^{\prime} t\right\} \varrho\left\{x^{\prime} t\right\}\left(x^{\prime} t \mid \psi=(x t \mid \psi,\right. \\
\left.\left.\int(d x) \psi^{\dagger} \mid x t\right) \eta\{x t\} \varrho\{x t\}\left\{\sum_{(\mu)}(x t|\psi| \mu)\left(\mu\left|\psi^{\dagger}\right| x^{\prime} t\right)\right\}=\psi^{\dagger} \mid x^{\prime} t\right) .
\end{array}\right\} .
$$

$\Sigma$ means summation over all discrete parameters and integration over all ${ }_{(\mu)}$
continuous parameters of the set $(\mu)$. The first equation, which is concerned with the inner product of the wave functions, expresses the orthonormality. The second and third equation are Hermitian adjoint to each other. They are concerned with the outer product of the wave functions and express the completeness.

For the product representation we need the complete system of orthonormal (anti-)symmetrized products

$$
\begin{equation*}
\left(x_{1} t_{1}, \ldots x_{n} t_{n}|\Psi| \mu_{n}, \ldots \mu_{1}\right)=\left(\frac{n!}{\Pi z_{r}!}\right)^{1 / 2} \mathbf{S}_{n}\left(x_{1} t_{1}|\psi| \mu_{1}\right) \ldots\left(x_{n} t_{n}|\psi| \mu_{n}\right) \tag{7}
\end{equation*}
$$

$z_{v}$ is the number of $\left(\mu_{k}\right)$, which are equal to $(\nu)$, i.e. the occupation number of the state $(v)$. For Fermi-Dirac statistics all terms with $z_{v}>1$ cancel (Pauli's exclusion principle), so that always $z_{\nu}!=1$. In order that the wave functions (7) shall be complete in single-time theory, it is necessary that $\boldsymbol{\eta}\left\{x_{k} t\right\} \varrho\left\{x_{k} t\right\}$ commutes with $\left(x_{l} t|\psi| \mu\right)$ for $l \neq k$, so that it may not contain differentiation with respect to $t$. In those cases (e.g. if (1) is a second order differential equation in the time coordinates i.e. in case of integral spin) for which this condition cannot be satisfied a description in single-time theory is excluded.
(7) may serve as the kernel of the transformation between $(x)$ and $(\mu)$ representation. The two representations are not entirely similar, because they stand on a different footing with symmetry. In the following it is in general sufficient to remember that the wave functions can be linearly expressed in the products (7). A throughout performance of the transformation would, however, be rather illustrative.
2.2 Wave operators; statical representation. Now we shall introduce creation and annihilation operators and preliminary wave operators, first in a statical representation.
2.21 Creation and annihilation operators. We define the operators ( $\left.\mu\left|\mathbf{a}^{\dagger}\right| x t\right\}$ and $\{x t|\mathbf{a}| \mu$ ) by requiring that for arbitrary $n$ the operators $\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\} z_{\mu}^{-1 / 2}$ and $z_{\mu}^{-1 / 2}\{x t|\mathbf{a}| \mu)$ respectively take a factor $\left(x_{n} t_{n}|\psi| \mu\right)$ out of $\left(x_{1} t_{1}, \ldots x_{n} t_{n}|\Psi| \mu_{n}, \ldots \mu_{1}\right)$ or insert a factor $\left(x_{n} t_{n}|\psi| \mu\right)$ into $\left(x_{1} t_{1}, \ldots x_{n_{-1}} t_{n_{-1}}\left|\Psi^{\prime}\right| \mu_{n-1}, \ldots \mu_{1}\right)$ and further restore the (anti-) symmetry and normalization. At this stage the separation of the factor $z_{\mu}^{-1 / 2}$ is still rather artificial. It is obvious that this choice has already been made
with the purpose of later establishing the required isomorphy. $\mathbf{a}^{\dagger}$ and a are now given by

$$
\left.\begin{array}{l}
\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi=\right. \\
=n^{1 / 2} \int\left(d x_{n}\right)\left(\mu\left|\psi^{\dagger}\right| x_{n} t_{n}\right) \eta\left\{x_{n} t_{n}\right\} \varrho\left\{x_{n} t_{n}\right\} \mathbf{S}_{n}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi\right.  \tag{8}\\
\{x t|\mathbf{a}| \mu)\left(x_{1} t_{1}, \ldots x_{n-1} t_{n-1} \mid \Psi=n^{1 / 2} \mathbf{S}_{n}\left(x_{n} t_{n}|\psi| \mu\right)\left(x_{1} t_{1}, \ldots x_{n-1} t_{n-1} \mid \Psi\right.\right.
\end{array}\right) .
$$

Acting to the left they give

$$
\left.\begin{array}{l}
\left.\left.\Psi^{\dagger} \mid x_{n-1} t_{n-1}, \ldots x_{1} t_{1}\right)\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\}=\Psi^{\dagger} \mid x_{n-1} t_{n-1}, \ldots x_{1} t_{1}\right)\left(\mu\left|\psi^{\dagger}\right| x_{n} t_{n}\right) \mathbf{S}_{n} n^{1 / 2}, \\
\left.\Psi \dagger \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right)\{x t|\mathbf{a}| \mu)=  \tag{+}\\
\left.\left.\quad=\int d x_{n}\right) \Psi \dagger \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right) \mathbf{S}_{n} \eta\left\{x_{n} t_{n}\right\} \varrho\left\{x_{n} t_{n}\right\}\left(x_{n} t_{n}|\psi| \mu\right) n^{1 / 2} .
\end{array}\right\} .
$$

So they are Hermitian adjoint to each other. If acting to the right, $\{x t|\mathbf{a}| \mu)$ will be denoted as the creation operator, $\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\}$ as the annihilation operator belonging to the state $(\mu)$. If acting to the left, they reverse their roles. They satisfy the commutation relations

$$
\left.\begin{array}{l}
{\left[\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\},\left\{x t|\mathbf{a}| \mu^{\prime}\right)\right]^{ \pm}=\delta_{\mu \mu^{\prime}} \mathbf{S} ;}  \tag{9}\\
{\left[\{x t|\mathbf{a}| \mu),\left\{x t|\mathbf{a}| \mu^{\prime}\right)\right]^{ \pm}=0 \quad, \quad\left[\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\},\left(\mu^{\prime}\left|\mathbf{a}^{\dagger}\right| x t\right\}\right]^{ \pm}=0}
\end{array}\right\} .
$$

(upper sign for $F-D$ statistics, lower sign for $B-E$ statistics).
2.22 Transition operators. Incidentally it may be mentioned that the products $\left\{x t|\mathbf{a}| \mu^{\prime}\right)\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\}$ can serve as transition operators between the states $(\mu)$ and ( $\mu^{\prime}$ ) (process $P_{1}$ ). The products with $\mu=\mu^{\prime}$ can be considered as projection operators of the states $(\mu)$.
2.23 Wave operators. Now we define the preliminary statical wave operators by

$$
\left.\begin{array}{l}
(y s|\psi| x t\}=\sum_{(\mu)}(y s|\psi| \mu)\left(\mu\left|\mathbf{a}^{\dagger}\right| x t\right\}  \tag{10}\\
\left\{x t\left|\psi^{\dagger}\right| y s\right)=\sum_{(\mu)}\{x t|\mathbf{a}| \mu)\left(\mu\left|\psi^{\dagger}\right| y s\right) .
\end{array}\right\}
$$

They are Hermitian adjoint to each other. In many-times theory $s$ will always be kept equal to $t_{n}$, if the annihilation operator acts on a wave function with $n$ sets ( $x_{k} t_{k}$ ) (or the creation operator on a wave function with $n-1$ sets). In single-time theory $s$ will be kept equal to $t$. (10) can somehow be considered as a transformation between $(\mu)$ and ( $y$ ) representation. Meanwhile the operation is always on the ( $x t$ ).

As a consequence of (10) the wave operators satisfy the commutation relations
$\left.\begin{array}{l}{\left[(y s|\psi| x t\},\left\{x t\left|\psi^{\dagger}\right| y^{\prime} s\right)\right]^{ \pm}=\sum_{(\mu)}(y s|\psi| \mu)\left(\mu\left|\psi^{\dagger}\right| y^{\prime} s\right) S ;} \\ {\left[\left\{x t\left|\psi^{\dagger}\right| y s\right),\left\{x t\left|\psi^{\dagger}\right| y^{\prime} s\right)\right]^{ \pm}=0,\left[(y s|\psi| x t\},\left(y^{\prime} s|\psi| x t\right\}\right]^{ \pm}=0 .}\end{array}\right\}$.
The sum in the first relation has the property described in the second and third equation of (6).
2.24 Substitution operators. The wave operators, which operate on the $(x t)$, often occur combined with an operation in the $(y s)$ in such a way
that the total resulting operation has a simple meaning. We represent the combination by the substitution operators $\{y s|\boldsymbol{\psi}| x t\}$ and $\left\{x t\left|\boldsymbol{\psi}^{\dagger}\right| y s\right\}$ (mind the backets, which distinguish them from the wave operators and which indicate operation on the ( $x t$ ) as well as on the ( $y s)$ ). They are defined by

$$
\left.\begin{array}{l}
\{y s|\psi| x t\}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi=(y s|\psi| x t\}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi\right.\right. \\
\left\{x t\left|\dot{\psi}^{\dagger}\right| y s\right\} ' x_{1} t_{1}, \ldots x_{n-1} t_{n-1} ; y s \mid=  \tag{12}\\
\quad=\int(d y)\left\{x t\left|\psi^{\dagger}\right| y s\right) \eta\{y s\} \varrho\{y s\}\left(x_{1} t_{1}, \ldots x_{n-1} t_{n-1} ; y x \mid \Psi .\right.
\end{array}\right\} .
$$

If they act to the left we get

$$
\begin{gather*}
\left.\Psi^{\dagger} \mid y s ; x_{n-1} t_{n-1}, \ldots x_{1} t_{1}\right)\{y s|\psi| x t\}= \\
\left.=\int(d y) \Psi^{\dagger} \mid y s ; x_{n-1} t_{n-1}, \ldots x_{1} t_{1}\right) \eta\{y s\} \varrho\{y s\}(y s|\psi| x t\} \\
\left.=\begin{array}{l}
\left.\left.\Psi^{\dagger} \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right)\left\{x t\left|\Psi^{\dagger}\right| y s\right\}=\Psi^{\dagger} \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right)\left\{x t\left|\psi^{\dagger}\right| y s\right),
\end{array}\right\} .
\end{gather*}
$$

which are the Hermitian adjoint relations.
It may seem as if the operators introduced in 2.21, 2.23 and 2.24 have by little and little become more and more complicated. In fact, however, with this chain of definitions we have nearly gone round. The meaning of the substitution operators is clearly illustrated by the result of (12) and (12 ${ }^{\dagger}$ )

$$
\left.\begin{array}{rl}
\{y s|\psi| x t\}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi=\left(n^{1 / 2} \mathbf{S}_{n}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi\right)_{\left(x_{n} t_{n}\right) \rightarrow(y s)}\right.\right.  \tag{13}\\
\left\{x t\left|\psi^{\dagger}\right| y s\right\}\left(x_{1} t_{1}, \ldots x_{n-1} t_{n-1} ; y s \mid \Psi=\right. \\
=S_{n} n^{1 / 2}\left(\left(x_{1} t_{1}, \ldots x_{n-1} t_{n-1} ; y s \mid \Psi\right)_{(y s) \rightarrow\left(x_{n} t_{n}\right)}\right.
\end{array}\right\}
$$

and

$$
\left.\begin{array}{rl}
\left.\Psi \dagger \mid x_{1} t_{1}, \ldots x_{n-1} t_{n-1} ; y s\right) & \{y s|\psi| x t\}= \\
& \left.=\left(\Psi^{\dagger} \mid x_{1} t_{1}, \ldots x_{n-1} t_{n-1} ; y s\right)\right)_{(y s) \rightarrow\left(x_{n} t_{n}\right)} n^{1 / 2} \mathbf{S}_{n}  \tag{+}\\
\left.\Psi^{\dagger} \mid x_{1} t_{1}, \ldots x_{n} t_{n}\right)\left\{x t\left|\psi^{\dagger}\right| y s\right\}=\left(\Psi^{\dagger}\left(x_{1} t_{1}, \ldots x_{n} t_{n}\right) \mathbf{S}_{n} n^{1 / 2}\right)\left(x_{n} t_{n}\right) \rightarrow(y s)
\end{array}\right\} .(
$$

with $s=t_{n}$ in all substitutions. This shows that:
$S_{s}$ An annihilation substitution operator (anti-)symmetrizes the wave function in all sets $\left(x_{k} t_{k}\right)$, multiplies it by the square root of the number of these sets and replaces the last set by a new set ( $y s$ ) with the same value of time. A creation substitution operator replaces a set ( $y s$ ) by an additional set $\left(x_{k} t_{k}\right)$ with the same value of time, multiplies the wave function by the square root of the number of all sets $\left(x_{k} t_{k}\right)$ and (anti-) symmetrizes the result.
2.25 Particle operators. As particle operators we take the ordinary Hermitian operators acting on the $\left(x_{k} t_{k}\right)$ and symmetrical in these sets, i.e. commuting with all permutation operators $\mathbf{P}$. For a given number $n$ of particles such an operator can be written as a sum of homogeneous particle operators $\mathbf{R}^{(m)}$, each with a different value of $m$ ( $m \leq n$; in practice $m=0,1,2$ ), of the form
$\mathbf{R}^{(m)}\left\{x_{1} t_{1}, \ldots x_{n} t_{n}\right\}=\sum_{k} \quad \mathbf{R}\left\{x_{k_{1}} t_{k_{1}}, \ldots x_{k_{m}} t_{k_{m}}\right\} \quad\left(k_{1}, \ldots k_{m}\right.$ all different $)$.
$\mathbf{R}\left\{x_{1} t_{1}, \ldots x_{m} t_{m}\right\}$ needs not to be symmetrical, but it can be taken so and then it is uniquely determined by $\mathbf{R}^{(m)}$. If it operates on a function of the required symmetry type, $\mathbf{R}^{(m)}$ can be written

$$
\begin{equation*}
\mathbf{R}^{(m)}\left\{x_{1} t_{1}, \ldots x_{n} t_{n}\right\}=n \ldots(n-m+1) \mathbf{S}_{n} \mathbf{R}\left\{x_{n} t_{n}, \ldots x_{n-m+1} t_{n-m+1}\right\} \mathbf{S}_{n} \tag{15}
\end{equation*}
$$

According to $S_{s}$ this is equivalent with

$$
\begin{align*}
& \mathbf{R}^{(m)}\left\{x_{1} t_{1}, \ldots\right\}=\left\{x t\left|\psi^{\dagger}\right| y_{1} s_{1}\right\} \ldots  \tag{16}\\
& \left.\quad \ldots\left\{x t\left|\psi^{\dagger}\right| y_{m} s_{m}\right\} \mathbf{R}\left\{y_{1} s_{1}, \ldots y_{m} s_{m}\right\}\left\{y_{m} s_{m}|\psi| x t\right\} \ldots\left\{y_{1} s_{1}|\psi| x t\right\} .\right\}
\end{align*}
$$

This form is rather trivial. With the help of (12) it can be written

$$
\begin{align*}
& \mathbf{R}^{(m)}\left\{x_{1} t_{1}, \ldots\right\}= \\
& \quad=\int \ldots \int\left(d y_{1}\right) \ldots\left(d y_{m}\right)\left\{x t\left|\psi^{\dagger}\right| y_{1} s_{1}\right) \ldots\left\{x t\left|\psi^{\dagger}\right| y_{m} s_{m}\right) \eta\left\{y_{1} s_{1}\right\} \varrho\left\{y_{1} s_{1}\right\} \ldots  \tag{17}\\
& \quad \ldots \eta\left\{y_{m} s_{m}\right\} \varrho\left\{y_{m} s_{m}\right\} \mathbf{R}\left\{y_{1} s_{1}, \ldots y_{m} s_{m}\right\}\left(y_{m} s_{m}|\boldsymbol{\psi}| x t\right\} \ldots\left(y_{1} s_{1}|\psi| x t\right\},
\end{align*}
$$

if it acts to the right and with the help of (12 ${ }^{\dagger}$ )

$$
\left.\begin{array}{l}
\mathbf{R}^{(m)}\left\{x_{1} t_{1}, \ldots\right\}=  \tag{+}\\
\left.=\int \ldots \int\left(d y_{1}\right) \ldots\left(d y_{m}\right)\left\{x t\left|\boldsymbol{\psi}^{\dagger}\right| y_{1} s_{1}\right) \ldots\left\{x t\left|\psi^{\dagger}\right| y_{m} s_{m}\right) \mathbf{R}\left\{y_{1} s_{1}, \ldots y_{m} s_{m}\right\}\right\} \\
\quad \eta\left\{y_{m} s_{m}\right\} \varrho\left\{y_{m} s_{m}\right\} \ldots \eta\left\{y_{1} s_{1}\right\} \varrho\left\{y_{1} s_{1}\right\}\left(y_{m} s_{m}|\psi| x t\right\} \ldots\left(y_{1} s_{1}|\psi| x t\right\}
\end{array}\right\}
$$

if it acts to the left. In case of noncommutability of $\mathbf{R}$ with $\eta \varrho$ the distinction between (17) and $\left(17^{\dagger}\right)$ should well be observed. The expressions in the right hand members of (16) and (17), (17 ${ }^{\dagger}$ ) do not explicitly contain the number $n$ of particles.

In many-times theory the important homogeneous particle operators are of the type $\mathbf{R}^{(1)}\left\{x_{1} t_{1}, \ldots\right\}$, formed from the individual operators $\mathbf{R}\left\{x_{k} t_{k}\right\}$. 2.26 Observable particle quantities. All observable particle quantities can be built up from expressions of the type

$$
\begin{equation*}
\left.\int \ldots \int\left(d x_{1}\right) \ldots\left(d x_{n}\right) \Psi^{\dagger} \mid x_{n} t_{n}, \ldots x_{1} t_{1}\right) \mathbf{R}^{(m)}\left\{x_{1} t_{1}, \ldots x_{n} t_{n}\right\}\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi^{\prime}\right. \tag{18}
\end{equation*}
$$

$\mathbf{R}^{(m)}$ may contain $\varrho$ and $\eta$ and the determination of its correct form may be a thorny matter. But that belongs to problem $Q_{3}$ and for problem $Q_{1}$ we need not care about it. In (18) one can if one likes express $\mathbf{R}^{(m)}$ according to (16) or (17), (17 ${ }^{\dagger}$ ).
2.3 Wave operators; dynamical representation. Now we turn from the statical to the dynamical representation.
2.31 Dynamical representations. If we consider the time dependence of (18), the part of the motion of the system is entirely contained in the wave functions $\Psi$ (Schrödinger representation). This dynamical time dependence can be transfered to the operators $\mathbf{R}$ (Heisenberg representation). In the latter case the wave functions $\Psi$ are replaced by their "initial" values $\Psi^{\prime}$ and the operators $\mathbf{R}$ by operators $\mathbf{R}^{\prime}$, which contain, besides the explicit time dependence of $\mathbf{R}$, also the motion of the system.

Also in Heisenberg representation the particle operators can be expressed according to (16) or (17), (17 $)$, in which the $\mathbf{R}^{\prime}\{x t\}$ and $\mathbf{R}^{\prime}\{y s\}$ then appear with a dash. If one is willing to use these expressions throughout, it is also possible to transfer the dynamical time dependence to the substitution and wave operators $\boldsymbol{\psi}$. In that case we get substitution and
wave operators $\boldsymbol{\psi}^{\prime}$ containing the motion of the system and in (16) and (17), ( $17^{\dagger}$ ) the $\mathbf{R}^{\prime}\{x t\}$ appear with a dash, the $\mathbf{R}\{y s\}$ without.

The direct transfer of the dynamical time dependence from the $\Psi$ to the $\mathbf{R}$ can hardly be described in such general terms as we are using at the moment. But if we succeed in transfering it from the $\Psi$ to the $\psi$, the transfer from the $\boldsymbol{\psi}$ to the $\mathbf{R}$ can immediately be found from (16) or (17), ( $17^{\dagger}$ ) (with undashed $\mathbf{R}\{y s\}$ and dashed $\psi^{\prime}$ and $\mathbf{R}^{\prime}\{x t\}$ ).

The various dynamical representations are compared in outline in the following scheme. The explicit time dependence of the undashed operators $\mathbf{R}$ has been left out of account. The suffixes 0 are explained further on. The underlined quantities contain the motion of the particles. The initial conditions are always contained in $\Psi$.

| elementary | $e_{1}$ Schrödinger | $e_{2}$ Heisenberg |  |
| :---: | :---: | :---: | :---: |
| "superquantized" | $\underline{(x t \mid \Psi}$ | $\left(x t_{0} \mid \Psi\right.$ |  |
|  | $\mathbf{R}\{x\}$ | $\underline{\mathbf{R}^{\prime}\{x t\}}$ |  |
|  | $s_{1}$ | $s_{2}$ "Schrödinger" | $s_{3}$ "Heisenberg" |
|  | $\underline{(x t \mid \Psi}$ | $\left(x t_{0} \mid \Psi^{\prime}\right.$ | $\left(x t_{0} \mid \Psi^{\prime}\right.$ |
|  | $\overline{(y s\|\boldsymbol{\psi}\| x t\}}$ | $\underline{\left(y s\left\|\psi^{\prime}\right\| x t_{0}\right\}}$ | $\left(y s_{0}\|\boldsymbol{\psi}\| x t_{0}\right\}$ |
|  | $\mathbf{R}\{\underline{y}\}$ | $\mathbf{R}\{y\}$ | $\mathbf{R}^{\prime}\{y s\}$ |

2.32 "Superquantized wave equations". In transfering the dynamical time dependence from the $\Psi$ to the $\psi$ (representation $s_{2}$ ), we replace the wave functions $\left(x_{1} t_{1}, \ldots x_{n} t_{n} \mid \Psi\right.$ by their value $\left(x_{1} t_{10}, \ldots x_{n} t_{n_{0}} \mid \Psi^{\prime}\right.$ at a fixed set of initial times $t_{k_{0}}(k=1, \ldots n)$. The statical wave operators $(y s|\psi| x t\}$ and $\left\{x t\left|\boldsymbol{\psi}^{\dagger}\right| y s\right)$ are replaced by the dynamical wave operators ( $\left.y s\left|\boldsymbol{\psi}^{\prime}\right| x t_{0}\right\}$ and $\left\{x t_{0}\left|\psi^{\prime}\right| y s\right)$. The values of $s$ are not fixed to the $t_{0}$, but keep step with the $t$ in the same way as before. These substitutions should not alter the initial value and the time dependence of (18), in which (16) or (17), $\left(17^{\dagger}\right)$ has been inserted for $\mathbf{R}^{(m)}\{x t\}$. At $s=t_{0}$ the dashed operators have to be equal to the undashed ones at $t=t_{0}$. Further also at $s=t_{0}$ they have to satisfy (i1). Finally they have to satisfy the "superquantized wave equations"

$$
\left.\begin{array}{rl}
\mathbf{K}\{y s\}\left(y s\left|\psi^{\prime}\right| x t_{0}\right\} & =0 \\
\left\{x t_{0}\left|\psi^{\prime}\right| y s\right) \mathbf{K}\{y s\} & =0 \tag{19}
\end{array}\right\}
$$

in many-times theory or

$$
\left.\begin{array}{l}
\left(\mathbf{K}\left\{x_{1}, \ldots y ; t\right\}-\mathrm{K}\left\{x_{1}, \ldots ; t\right\}\right)\left(y s\left|\psi^{\prime}\right| x t_{0}\right\}=0  \tag{19'}\\
\left\{x t_{0}\left|\boldsymbol{\psi}^{\prime+}\right| y s\right)\left(\mathbf{K}\left\{x_{1}, \ldots, y ; t\right\}-\mathbf{K}\left\{x_{1}, \ldots ; t\right\}\right)=0
\end{array}\right\} .
$$

in single-time theory. (With the help of $\mathbf{K}^{(1)}\left\{x_{1} t_{1}, \ldots\right\}$ (19) can also be written similar to ( $19^{\prime}$ )).

The $\boldsymbol{\psi}$ were independent of the dynamics of the system. They are relatively simple universal operators. The $\psi^{\prime}$ contain (for given initial conditions) the entire motion of the system. They will be frightfully complicated.

