Mathematics. — *Uitbreiding van enige identiteiten*. II. By J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of May 29, 1948.)

2. Voorts gaan we uit van de betrekkingen (41) en (41′), voorkomende in I, nl.

$$S'_{\nu,2k+1}(x) = \nu^{2k}\frac{x}{2}I_{\nu-1}(x) + xS'_{\nu+1,0}(x) + x\sum_{p_1=0}^{k-1}\binom{2k}{2p_1+2}S'_{\nu+1,2p_1+2}(x). \quad (9)$$

en

$$S'_{\nu,2k+2}(x) = -(\nu-2)^{2k+1}\frac{x}{2}I_{\nu-1}(x) + x\sum_{p_1=0}^{k}\binom{2k+1}{2p_1+1}S'_{\nu-1,2p_1+1}(x), \quad . \quad (10)$$

waarin $S'_{\nu,k}(x) = \sum_{n=0}^{\infty}(\nu+2n)^k I_{\nu+2n}(x)$ is.

We kunnen hieraan nog toevoegen:

$$\left.\begin{aligned} S'_{\nu,2k+1}(x) &= -(\nu-2)^{2k}\frac{x}{2}I_{\nu-1}(x) + xS'_{\nu-1,0}(x) + \\ &\quad + x\sum_{p_1=0}^{k-1}\binom{2k}{2p_1+2}S'_{\nu-1,2p_1+2}(x) \end{aligned}\right\} . \quad (11)$$

en

$$S'_{\nu,2k+2}(x) = \nu^{2k+1}\frac{x}{2}I_{\nu-1}(x) + x\sum_{p_1=0}^{k}\binom{2k+1}{2p_1+1}S'_{\nu+1,2p_1+1}(x), \quad . \quad (12)$$

welke formules op overeenkomstige wijze zijn afgeleid als is aangegeven in I § 5 bij de afleiding van soortgelijke formules voor $S_{\nu,2k+2}(x)$.

Door nu met elkaar te combineren de betrekkingen (9) en (12), (11) en (10), (9) en (10), (11) en (12), op de wijze als in het voorgaande is uitgevoerd, vindt men de volgende algemene formules (ν willekeurig):

$$\left.\begin{aligned} S'_{\nu,2k+1}(x) &= \frac{x}{2}\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-1}\cdots \\ &\cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+1}{2p_{2r}+1}\{(\nu+2r)^{2p_{2r}}I_{\nu+2r-1}(x) + 2S'_{\nu+2r+1,0}(x)\} + \\ &+ 2\left(\frac{x}{2}\right)^2\sum_{r=0}^{k}x^{2r}I_{\nu+2r}(x)\sum_{p_1=0}^{k-\nu-1}\binom{2k}{2p_1+2r+2}\sum_{p_2=0}^{p_1}\binom{2p_1+2r+1}{2p_2+2r+1}\cdots \\ &\cdots\sum_{p_{2r+1}=0}^{p_{2r}}\binom{2p_{2r}+2}{2p_{2r+1}+2}(\nu+2r+1)^{2p_{2r+1}+1} = \\ &= \frac{x}{2}\sum_{r=0}^{k}x^{2r}\sum_{p_1=0}^{k-r}\binom{2k}{2p_1+2r}\sum_{p_2=0}^{p_1}\binom{2p_1+2r-1}{2p_2+2r-1}\cdots \\ &\cdots\sum_{p_{2r}=0}^{p_{2r-1}}\binom{2p_{2r-1}+1}{2p_{2r}+1}\{-(\nu-2r-2)^{2p_{2r}}I_{\nu-2r-1}(x) + 2S'_{\nu-2r-1,0}(x)\} - \end{aligned}\right\} . \quad (13)$$

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$$
\begin{aligned}
&-2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{k-1} x^{2r} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-2r-3)^{2p_{2r+1}+1} = \\
&= \frac{x}{2} \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ \nu^{2p_{2r}} I_{\nu-1}(x) + 2\, S'_{\nu+1,0}(x) \right\} - \\
&-2\left(\frac{x}{2}\right)^2 I_\nu(x) \sum_{r=0}^{k-1} x^{2r} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1} = \\
&= \frac{x}{2} \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ -(\nu-2)^{2p_{2r}} I_{\nu-1}(x) + 2\, S'_{\nu-1,0}(x) \right\} + \\
&+2\left(\frac{x}{2}\right)^2 I_{\nu-2}(x) \sum_{r=0}^{k-1} x^{2r} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1}.
\end{aligned}
\qquad (13)
$$

Hierin is, zo $r=0$ is, voor $\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} (\nu+r)^{2p_{2r}}$ resp. $(\nu-2r-2)^{2p_{2r}}$ resp. $\nu^{2p_{2r}}$ resp. $(\nu-2)^{2p_{2r}}$ te nemen ν^{2k} resp. $(\nu-2)^{2k}$.

Op gelijke wijze komt men door combinatie van de betrekkingen (12) en (9), (10) en (11), (12) en (11), (10) en (9) tot de volgende algemene formules (ν willekeurig):

$$
\begin{aligned}
S'_{\nu,2k+2}(x) &= 2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ (\nu+2r+1)^{2p_{2r+1}} I_{\nu+2r}(x) + 2S'_{\nu+2r+2,0}(x) \right\} + \\
&+ \frac{x}{2} \sum_{r=0}^{k} x^{2r} I_{\nu+2r-1}(x) \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1} =
\end{aligned}
\qquad (14)
$$

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$$
\left.
\begin{aligned}
&= 2\left(\frac{x}{2}\right)^2 \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\quad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{p_{2r}+1}{2p_{2r+1}+1} \{-(\nu-2r-3)^{2p_{2r+1}} I_{\nu-2r-2}(x) + 2S'_{\nu-2r-2,0}(x)\} - \\
&- \frac{x}{2} \sum_{r=0}^{k-1} x^{2r} I_{\nu-2r-1}(x) \sum_{p_1=0}^{k-r-1}{}' \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu-2r-2)^{2p_{2r}+1} = \\
&= \frac{x}{2} \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\quad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} [\nu^{2p_{2r}+1} I_{\nu-1}(x) + x \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \\
&\qquad\qquad \{-(\nu-1)^{2p_{2r+1}} I_\nu(x) + 2S'_{\nu,0}(x)\}] = \\
&= \frac{x}{2} \sum_{r=0}^{k} x^{2r} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\
&\quad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} [-(\nu-2)^{2p_{2r}+1} I_{\nu-1}(x) + x \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \\
&\qquad\qquad \{(\nu-1)^{2p_{2r+1}} I_{\nu-2}(x) + 2S'_{\nu,0}(x)\}]
\end{aligned}
\right\} \quad . \quad (14)
$$

Hierin is, zo $r=0$ is, voor

$$\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1}$$

resp. $(\nu-2r-2)^{2p_{2r}+1}$ resp. $\nu^{2p_{2r}+1}$ resp. $(\nu-2)^{2p_{2r}+1}$ te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$.

In verband met de identiteit, geldig voor alle ν, waarvan de juistheid o.a. door volledige inductie kan worden aangetoond:

$$\sum_{p=0}^{s} \frac{(-1)^p}{(s-p)!\,\Gamma(s+p+\nu+1)} = \frac{1}{(2s+\nu)\,s!\,\Gamma(s+\nu)},$$

kan men voor $S'_{\nu,0}(x)$ schrijven:

$$S'_{\nu,0}(x) = \sum_{n=0}^{\infty} I_{\nu+2n}(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+2n+\nu}}{m!\,\Gamma(m+2n+\nu+1)} =$$

$$= \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s} \frac{(-1)^p}{(s-p)!\,\Gamma(s+p+\nu+1)} =$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(2s+\nu)\,s!\,\Gamma(s+\nu)}.$$

Door nu in (13) te substitueren:

$$S'_{\nu,2k+1}(x) = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+1}}{(s-p)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{(\nu+2r)^2 p_{2r} I_{\nu+2r-1}(x) + 2S'_{\nu+2r+1,0}(x)\} =$$

$$= \sum_{s=2r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r)!\,\Gamma(s+\nu)} \left\{(\nu+2r)^2 p_{2r} - \frac{2(s-2r)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu+2r}(x) = -\sum_{s=2r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r-1)!\,\Gamma(s+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{-(\nu-2r-2)^2 p_{2r} I_{\nu-2r-1}(x) + 2S'_{\nu-2r-1,0}(x)\} =$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu)} \left\{-(\nu-2r-2)^2 p_{2r} + \frac{2(s-2r+\nu-1)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu-2r-2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu-1)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{\nu^2 p_{2r} I_{\nu-1}(x) + 2S'_{\nu+1,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)} \left\{\nu^2 p_{2r} - \frac{2(s-r)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu}(x) = -(-1)^r \sum_{s=r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r-1)!\,\Gamma(s-r+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+1} \{-(\nu-2)^2 p_{2r} I_{\nu-1}(x) + 2S'_{\nu-1,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)} \left\{-(\nu-2)^2 p_{2r} + \frac{2(s-r+\nu-1)}{2s-2r+\nu-1}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+2} I_{\nu-2}(x) = (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu-1)},$$

vindt men na gelijkstelling der coëfficienten van $\left(\frac{x}{2}\right)^{2s+\nu}$ in beide leden de algemene indentiteiten (ν willekeurig):

$$\sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+1}}{(s-p)!\,\Gamma(s+p+\nu+1)} =$$

$$\left.\begin{aligned}
&= \frac{1}{\Gamma(s+\nu)} \sum_{r=0}^{k} \frac{2^{2r}}{(s-2r)!} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ (\nu+2r)^{2p_{2r}} - \frac{2(s-2r)}{2s-2r+\nu-1} \right\} - \\
&- \frac{1}{\Gamma(s+\nu)} \sum_{r=3}^{k-1} \frac{2^{2r+1}}{(s-2r-1)!} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu+2r+1)^{2p_{2r+1}+1} =
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
&= \frac{1}{s!} \sum_{r=0}^{k} \frac{2^{2r}}{\Gamma(s-2r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ -(\nu-2r-2)^{2p_{2r}} + \frac{2(s-2r+\nu-1)}{2s-2r+\nu-1} \right\} - \\
&- \frac{1}{s!} \sum_{\nu=0}^{k-1} \frac{2^{2r+1}}{\Gamma(s-2r+\nu-1)} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-2r-3)^{2p_{2r+1}+1} =
\end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned}
&= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ \nu^{2p_{2r}} - \frac{2(s-r)}{2s-2r+\nu-1} \right\} + \\
&+ \sum_{r=0}^{k-1} \frac{(-1)^r\, 2^{2r+1}}{(s-r-1)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}+1} =
\end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned}
&= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\
&\qquad \cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} \left\{ -(\nu-2)^{2p_{2r}} + \frac{2(s-r+\nu-1)}{2s-2r+\nu-1} \right\} + \\
&+ \sum_{r=0}^{k-1} \frac{(-1)^r\, 2^{2r+1}}{(s-r)!\,\Gamma(s-r+\nu-1)} \sum_{p_1=0}^{k-\nu-1} \binom{2k}{2p_1+2r+2} \sum_{p_2=0}^{p_1} \binom{2p_1+2r+1}{2p_2+2r+1} \cdots \\
&\qquad \cdots \sum_{p_{2r+1}}^{p_{2r}} \binom{2p_{2r}+2}{2p_{2r+1}+2} (\nu-1)^{2p_{2r+1}}.
\end{aligned}\right\} \quad (18)$$

Hierin is, zo $r=0$ is, voor $\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+1}{2p_{2r}+1} (\nu+2r)^{2p_{2r}}$ resp. $(\nu-2r-2)^{2p_{2r}}$ resp. $\nu^{2p_{2r}}$ resp. $(\nu-2)^{2p_{2r}}$ te nemen ν^{2k} resp. $(\nu-2)^{2k}$.

Door in (14) te substitueren:

$$S'_{\nu,2k+2}(x) = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+\nu} \sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{(\nu+2r+1)^{2p_{2r+1}} I_{\nu+2r}(x) + 2S'_{\nu+2r+2,0}(x)\} =$$

$$= \sum_{s=2r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r-1)!\,\Gamma(s+\nu)} \left\{-(\nu+2r+1)^{2p_{2r+1}} + \frac{2(s-2r-1)}{2s-2r+\nu-2}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+1} I_{r+2r-1}(x) = \sum_{s=2r}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-2r)!\,\Gamma(s+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{-(\nu-2r-3)^{2p_{2r+1}} I_{\nu-2r-2}(x) + 2S'_{\nu-2r-2,0}(x)\} =$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu-1)} \left\{-(\nu-2r-3)^{2p_{2r+1}} + \frac{2(s-2r+\nu-2)}{2s-2r+\nu-2}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+1} I_{\nu-2r-1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{s!\,\Gamma(s-2r+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{-(\nu-1)^{2p_{2r+1}} I_\nu(x) + 2S'_{\nu,0}(x)\} =$$

$$= (-1)^r \sum_{s=r+1}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r-1)!\,\Gamma(s-r+\nu)} \left\{(\nu-1)^{2p_{2r+1}} - \frac{2(s-r+\nu-1)}{2s-2r+\nu-2}\right\},$$

$$\left(\frac{x}{2}\right)^{2r+1} I_{\nu-1}(x) = (-1)^r \sum_{s=r}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu)},$$

$$\left(\frac{x}{2}\right)^{2r+2} \{(\nu-1)^{2p_{2r+1}} I_{\nu-2}(x) + 2S'_{\nu,0}(x)\} =$$

$$= (-1)^r \sum_{s=r}^{\infty} \frac{(-1^{\,s} \left(\frac{x}{2}\right)^{2s+\nu}}{(s-r)!\,\Gamma(s-r+\nu-1)} \left\{(\nu-1)^{2p_{2r+1}} - \frac{2(s-r)}{2s-2r+\nu-2}\right\},$$

vindt men na gelijkstelling der coëfficienten van $\left(\frac{x}{2}\right)^{2s+\nu}$ in beide leden de volgende algemene identiteiten (ν willekeurig):

$$\sum_{p=0}^{s} \frac{(-1)^p (2p+\nu)^{2k+2}}{(s-p)!\,\Gamma(s+p+\nu+1)} =$$

$$\left.\begin{aligned} &= \frac{1}{\Gamma(s+\nu)} \sum_{r=0}^{k} \frac{2^{2r+1}}{(s-2r-1)!} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\ &\cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ -(\nu+2r+1)^{2p_{2r+1}} + \frac{2(s-2r-1)}{2s-2r+\nu-2} \right\} + \\ &+ \frac{1}{\Gamma(s+\nu)} \sum_{r=0}^{k} \frac{2^{2r}}{(s-2r)!} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\ &\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1} = \end{aligned}\right\} . \quad (19)$$

$$\left.\begin{aligned} &= \frac{1}{s!} \sum_{r=0}^{k} \frac{2^{2r+1}}{\Gamma(s-2r+\nu-1)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\ &\cdots \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ -(\nu-2r-3)^{2p_{2r+1}} + \frac{2(s-2r+\nu-2)}{2s-2r+\nu-2} \right\} - \\ &- \frac{1}{s!} \sum_{r=0}^{k} \frac{2^{2r}}{\Gamma(s-2r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r} \sum_{p_2=0}^{p_1} \binom{2p_1+2r-1}{2p_2+2r-1} \cdots \\ &\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu-2r-2)^{2p_{2r}+1} = \end{aligned}\right\} . \quad (20)$$

$$\left.\begin{aligned} &= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\ &\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} \left[\nu^{2p_{2r}+1} + 2(s-r) \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \left\{ (\nu-1)^{2p_{2r+1}} - \frac{2(s-r+\nu-1)}{2s-2r+\nu-2} \right\} \right] = \end{aligned}\right\} . \quad (21$$

$$\left.\begin{aligned} &= \sum_{r=0}^{k} \frac{(-1)^r\, 2^{2r}}{(s-r)!\,\Gamma(s-r+\nu)} \sum_{p_1=0}^{k-r} \binom{2k+1}{2p_1+2r+1} \sum_{p_2=0}^{p_1} \binom{2p_1+2r}{2p_2+2r} \cdots \\ &\cdots \sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} \left[-(\nu-2)^{2p_{2r}+1} + 2(s-r+\nu-1) \sum_{p_{2r+1}=0}^{p_{2r}} \binom{2p_{2r}+1}{2p_{2r+1}+1} \right. \\ &\qquad \left. \left\{ (\nu-1)^{2p_{2r+1}} - \frac{2(s-r)}{2s-2r+r-2} \right\} \right] . \end{aligned}\right\} . \quad (22)$$

Hierin is, zo $r=0$ is, voor $\sum_{p_{2r}=0}^{p_{2r-1}} \binom{2p_{2r-1}+2}{2p_{2r}+2} (\nu+2r)^{2p_{2r}+1}$ resp. $(\nu-2r-2)^{2p_{2r}+1}$ resp. $\nu^{2p_{2r}+1}$ resp. $(\nu-2)^{2p_{2r}+1}$ te nemen ν^{2k+1} resp. $(\nu-2)^{2k+1}$.

Daar in (17) voor $r=s$ geldt: $\nu^{2p_{2r}}-\frac{2\,(s-r)}{2s-2r+\nu-1}=\nu^{2p_{2s}}$, terwijl deze uitdrukking $=0$ is voor $r\neq s$ en $\nu=1$, evenzo in (18) voor $r=s$ geldt: $-(\nu-2)^{2p_{2r}}+\frac{2\,(s-r+\nu-1)}{2\,s-2r+\nu-1}=2-(\nu-2)^{2p_{2r}}$, terwijl deze uitdrukking $=0$ is voor $r\neq s$ en $\nu=1$, volgt uit (17) zowel als (18) voor $\nu=1$:

$$\left.\begin{aligned}\sum_{p=0}^{s}\frac{(-1)^p(2p+1)^{2k+1}}{(s-p)!\,(s+p+1)!}=(-1)^s\,2^{2s}\sum_{p_1=0}^{k-s}\binom{2k}{2p_1+2s}\sum_{p_2=0}^{p_1}\binom{2p_1+2s-1}{2p_2+2s-1}\cdots\\ \cdots\sum_{p_{2s}=0}^{p_{2s-1}}\binom{2p_{2s-1}+1}{2p_{2s}+1}\end{aligned}\right\}. \quad (23)$$

Zo geldt in (22) voor $r=s$: $(\nu-1)^{2p_{2r+1}}-\frac{2\,(s-r)}{2s-2r+\nu-2}=(\nu-1)^{2p_{2s+1}}$, terwijl deze uitdrukking $=0$ is voor $r\neq s$ en $\nu=2$. Derhalve volgt uit (22) voor $\nu=2$:

$$\left.\begin{aligned}\sum_{p=0}^{s}\frac{(-1)^p(2p+2)^{2k+2}}{(s-p)!\,(s+p+2)!}=(-1)^s\,2^{2s+1}\sum_{p_1=0}^{k-s}\binom{2k+1}{2p_1+2s+1}\sum_{p_2=0}^{p_1}\binom{2p_1+2s}{2p_2+2s}\cdots\\ \cdots\sum_{p_{2s+1}=0}^{p_{2s}}\binom{2p_{2s}+1}{2p_{2s+1}+1}\end{aligned}\right\}. \quad (24)$$

Deze beide identiteiten stemmen overeen met die, welke vermeld zijn in I onder (47) en (48).

Substitueren wij in (15) t/m (18), evenzo in (19) t/m (20) $k=0$, dan vinden we (ν willekeurig):

$$\sum_{p=0}^{s}\frac{(-1)^p(2p+\nu)}{(s-p)!\,\Gamma(s+p+\nu+1)}=\frac{\nu-1}{(2s+\nu+2)\,s!\,\Gamma(s+\nu)} \quad . \;.\;. \quad (25)$$

en

$$\sum_{p=0}^{s}\frac{(-1)^p(2p+\nu)^2}{(s-p)!\,\Gamma(s+p+\nu+1)}=\frac{\nu(\nu-2)}{(2s+\nu-2)\,s!\,\Gamma(s+\nu)} \quad . \;.\;. \quad (26)$$

Onderstellen we $k>0$, dan volgt uit (18) voor $\nu=2$:

$$\left.\begin{aligned}\sum_{p=0}^{s}\frac{(-1)^p(2p+2)^{2k+1}}{(s-p)!\,(s+p+2)!}=\frac{2}{(2s+1)\,(s!)^2}+\\ +2\sum_{r=0}^{k-1}\frac{(-1)^r\,2^{2r}}{\{(s-r)!\}^2}\sum_{p_1=0}^{k-r-1}\binom{2k}{2p_1+2r+2}\sum_{p_2=0}^{p_1}\binom{2p_1+2r+1}{2p_2+2r+2}\cdots\\ \cdots\sum_{p_{2r+1}=0}^{p_{2r}}\binom{2p_{2r}+2}{2p_{2r+1}+2}\left[1+2(s-r)\left\{2p_{2r+1}+1-\frac{2\,(s-r)}{2s-2r-1}\sum_{p_{2r+2}=0}^{p_{2r+1}}\binom{2p_{2r+1}+1}{2p_{2r+2}+1}\right\}\right]\end{aligned}\right\}. \quad (27)$$

Evenzo volgt, zo $k>0$ is, uit (21) en (22) voor $\nu=1$:

$$\left.\begin{aligned}&\sum_{p=0}^{s}\frac{(-1)^p\,(2p+1)^{2k-2}}{(s-p)!\,(s+p+1)!}=-\frac{1+4sk+4s^2\sum\limits_{p_1=0}^{k-1}\binom{2k+1}{2p_1+3}}{(2s-1)\,(s!)^2}-\\&-4\sum_{r=0}^{k-1}\frac{(-1)^r\,2^{2r}}{\{(s-r-1)!\}^2}\sum_{p_1=0}^{k-r-1}\binom{2k+1}{2p_1+2r+3}\sum_{p_2=0}^{p_1}\binom{2p_1+2r+2}{2p_2+2r+2}\cdots\\&\cdots\sum_{p_{2r+2}}^{p_{2r+1}}\binom{2p_{2r+1}+2}{2p_{2r+2}+2}\left[1+2(s-r-1)\left\{2p_{2r+2}+1-\frac{2(s-r-1)}{2s-2r-3}\sum_{p_{2r+3}=0}^{p_{2r+2}}\binom{2p_{2r+2}+1}{2p_{2r+3}+1}\right\}\right]\end{aligned}\right\}. \quad (28)$$

In zekere zin zijn (27) en (28) tegenhangers van (23) en (24).