Aerodynamics. — *Spectral analysis of an irregular function.* By J. M. BURGERS. (Mededeling No. 58*b* uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

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1. The object of the following lines is to give a few comments on the concept of the spectrum of an irregularly oscillating function of the time, which is treated by R. BETCHOV in the accompanying article [1]. BETCHOV has represented the function $v(t)$ considered by him, by means of a Fourier integral. In order to be able to do this, it was necessary to suppose that the function $v(t)$ existed only in an interval of finite duration, $-T < t < +T$, outside of which it was replaced by zero. Although this is sufficient for many purposes, there is something artificial in the procedure, as actually the function $v(t)$ may represent some natural phenomenon continually going on and exhibiting always the same statistical character (one may think *e.g.* of the oscillations of the atmospheric pressure). The Fourier integral, however, cannot be applied for an infinite interval of time in such a case, as the functions considered here do not satisfy the condition of having "limited total fluctuation" over an infinite interval.

It is possible to define the spectrum of such a function in a different way, by establishing a relation between the spectrum and the "coefficient of correlation" of the function $v(t)$. This has been shown by TAYLOR [2]. For the demonstration of his formula TAYLOR still refers to the Fourier integral of $v(t)$. We will show that a relation, very similar to that given by TAYLOR, can be deduced by investigating what is obtained when an electric signal, proportional to $v(t)$, is passed through a "filtering circuit". In this way we keep as closely as possible to the experimental method applied for spectral analysis, without making use of any supposition concerning the possibility of Fourier analysis of the function $v(t)$.

2. Many types of filtering circuits can be constructed, but in order to keep the mathematical deductions as simple as possible, the following example has been taken, in which the transformed signal $w(t)$ is connected with the signal $v(t)$ by the differential equation:

$$\frac{d^2 w}{dt^2} + 2 p \omega \frac{dw}{dt} + \omega^2 w = 2 \sqrt{p}\, \omega \frac{dv}{dt} \quad \ldots \ldots \quad (1)$$

Here ω and p are adjustable constants; p must be smaller than 1 and

[1]) R. BETCHOV, L'analyse spectrale de la turbulence, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 1063—1072 (1948).

[2]) G. I. TAYLOR, The spectrum of turbulence, Proc. Royal Society London, A **164**, 476 (1938).

actually will be taken very small. The factor before dv/dt on the right hand side is unimportant, as the signal coming out of the filtering circuit can be amplified; the factor $2\sqrt{p}\,\omega$ introduced in (1) has been chosen for convenience.

That the differential equation (1) defines a filter, is seen if v is replaced by e^{ist}. The value of w then becomes:

$$w = \frac{2\sqrt{p}}{2p + i(s/\omega - \omega/s)}\, e^{ist}. \quad \ldots \ldots \quad (2)$$

This gives a maximum for $s = \omega$; the sharpness of the maximum increases when p is taken smaller and smaller, its actual width being proportional to $p\,\omega$ [3]).

When $v(t)$ is an irregular function, always presenting the same statistical character, we can use the integral of (1):

$$w = 2\sqrt{\frac{p}{1-p^2}} \int_0^\infty d\tau\; v\left(t - \frac{\tau}{\omega}\right) e^{-p\tau} \cos\left(\tau\sqrt{1-p^2} + \varepsilon\right) \quad . \; . \quad (3)$$

(where ε is defined by $\sin\varepsilon = p$) for the calculation of mean values referring to w, from mean values referring to v.

The most important quantities in this respect are $\overline{w(t)\; w(t+\eta)}$ and $\overline{v(t)\; v(t+\eta)}$, where η is a fixed interval of time. In both cases the mean values are taken with respect to t, over a period of sufficient duration in order that mean values can be treated as invariant with respect to a displacement of this period forward or backward along the time-scale. We suppose that the second quantity is given, in the form:

$$\overline{v(t)\, v(t+\eta)} = A^2 R(\eta). \quad \ldots \ldots \quad (4)$$

where A^2 is the mean value of $v(t)^2$, while $R(\eta)$ represents the *coefficient of correlation* of the function $v(t)$.

Knowledge of the value of A^2 and of the function $R(\eta)$ will embody all that we assume to be given concerning the statistical character of the function $v(t)$. The function $R(\eta)$ is a measure for the rapidity of change with time of the function $v(t)$ on one hand, and for the degree of regularity or irregularity of $v(t)$ on the other hand. We have $R(0) = 1$, which at the same time is the maximum value $R(\eta)$ can attain. In all practical cases it is possible to find a value θ, such that:

$$R(\eta) = 0 \text{ for } \eta > \theta. \quad \ldots \ldots \quad (5a)$$

It is not to be excluded that within the range $0 < \eta < \theta$ the coefficient of correlation $R(\eta)$ may pass through negative values.

The function $R(\eta)$ is symmetrical, so that

$$R(\eta) = R(-\eta). \quad \ldots \ldots \quad (5b)$$

[3]) As dr. BETCHOV has told me, the differential equation for the filtering circuit described by him can be brought into a form similar to (1).

3. Starting from the integral (2) the following relation can be deduced:

$$\left.\begin{aligned}&\overline{w(t)\,w(t+\eta)} = \\ &= \frac{4p}{1-p^2}\int\limits_0^\infty d\tau_1 \int\limits_0^\infty d\tau_2\, \overline{v\left(t-\frac{\tau_1}{\omega}\right) v\left(t+\eta-\frac{\tau_2}{\omega}\right)}\, e^{-p(\tau_1+\tau_2)} \\ &\qquad\qquad \cos(\tau_1\sqrt{1-p^2}+\varepsilon)\cos(\tau_2\sqrt{1-p^2}+\varepsilon).\end{aligned}\right\} \quad (6)$$

This equation can be transformed by introducing $\sigma = \tau_2 + \tau_1$ and $\delta = \tau_2 - \tau_1$ as new variables. We will pass over the details and mention the result [4]):

$$\left.\begin{aligned}&\overline{w(t)\,w(t+\eta)} = \\ &= \frac{A^2}{\cos\varepsilon}\int\limits_0^\delta d\delta \left\{R\left(\eta+\frac{\delta}{\omega}\right)+R\left(\eta-\frac{\delta}{\omega}\right)\right\} e^{-p\delta}\cos(\delta\sqrt{1-p^2}+\varepsilon).\end{aligned}\right\} \quad (7)$$

The mean square of w is obtained from this formula if we take $\eta = 0$. Making use of ($5b$) and assuming p to be so small that p^2 can be neglected in comparison with 1, we find:

$$\overline{w^2} = 2A^2\int\limits_0^\infty d\delta\, R\left(\frac{\delta}{\omega}\right) e^{-p\delta}\cos(\delta+\varepsilon). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We introduce the functions:

$$I_1(\omega) = \omega\int\limits_0^\infty d\delta_1\, R(\delta_1)\, e^{-p\omega\delta_1}\cos\omega\,\delta_1 \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

$$I_2(\omega) = \omega\int\limits_0^\infty d\delta_1\, R(\delta_1)\, e^{-p\omega\delta_1}\sin\omega\,\delta_1 \quad . \quad . \quad . \quad . \quad . \quad (9b)$$

These functions are generalised Fourier transforms of the correlation coefficient $R(\eta)$. When p is so small that the quantity $p\,\omega\,\theta$ will be small compared with unity for an important range of values of ω, the exponential factor in ($9a$) and ($9b$) is of little importance and we obtain ordinary Fourier transforms.

Equation (8) now gives:

$$\overline{w^2} = 2A^2\,\{I_1(\omega) - p\,I_2(\omega)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence the mean amplitude of the signal coming out of the filtering circuit, for a given value of ω, is determined by the Fourier transform of the correlation function for that value of ω.

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[4]) The calculations necessary for the reduction of the right hand side of (6) are of similar nature as various calculations given by dr. TCHEN CHAN-MOU in: "Mean value and correlation problems connected with the motion of small particles" (thesis Delft, 1947; Meded. no. 51 Labor. v. Aero- en Hydrodynamica der T. H.; see pp. 88/89 and 99/100). — TCHEN has also given a definition of a function presenting always the same statistical character (l.c. p. 52).

4. In the second place we take $\eta > \theta$, which allows us to discard the term $R(\eta + \delta/\omega)$ in (7). We then find, again with $\cos \varepsilon \cong 1$:

$$\overline{w(t)\, w(t+\eta)} = 2A^2 e^{-p\omega\eta} \{ I_1^* \cos(\omega\eta + \varepsilon) + I_2^* \sin(\omega\eta + \varepsilon) \} \quad . \quad (11)$$

where

$$I_1^*(\omega) = \tfrac{1}{2}\,\omega \int_{-\infty}^{+\infty} d\delta_1\, R(\delta_1)\, e^{p\omega\delta_1} \cos \omega \delta_1 \quad . \quad . \quad . \quad . \quad (12a)$$

$$I_2^*(\omega) = \tfrac{1}{2}\,\omega \int_{-\infty}^{+\infty} d\delta_1\, R(\delta_1)\, e^{p\omega\delta_1} \sin \omega \delta_1 . \quad . \quad . \quad . \quad . \quad (12b)$$

In those cases where $p\,\omega\,\theta$ is small compared with unity, I_1^* will be practically equal to I_1, while I_2^* will be of the order of p.

From (11) it appears that the signal $w(t)$ transmitted by the filter exhibits a correlation extending over an interval of time which is no longer directly dependent on θ (the maximum duration for which correlation is found in the original function $v(t)$), and which actually can be much greater than θ. The presence of the harmonic functions of η in the right hand member of (11) proves that the transmitted signal is nearly a harmonic oscillation of frequency ω. The exponential factor $e^{-p\omega\eta}$, however, indicates that the correlation does not extend over an indefinite period, but becomes gradually less when η is increased. This is exactly what we must expect, when the filter does not give a pure harmonic oscillation, but a band of frequencies of breadth proportional to $p\,\omega$.

We can interprete this result by saying that the filter *produces* an approximately harmonic oscillation out of the irregular function $v(t)$, in such a way that the amplitude of this oscillation depends on the Fourier transform of $R(\eta)$. This is in accordance with TAYLOR's result, but we have obtained it in a form adapted to the experimental method for spectral analysis. In particular it was not necessary to assume that $v(t)$ itself was composed of harmonic components; nor did we require any knowledge of phase relations for such components, if they were present — this in contrast with the Fourier integral, where the phase of the transformed function is of great importance.

The result obtained can also be considered as an illustration of the controversy, once famous in the theory of optics, whether the colours of the spectrum can be said to be originally present in white light, or whether they are produced by the spectroscope.

(To be continued.)