Physics. — *"Superquantization."* II. By H. J. GROENEWOLD. (Koninklijk Nederlands Meteorologisch Instituut te De Bilt.) (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of September 25, 1948.)

2.33 *Dynamical wave operators.* We can write down an explicit expression for the preliminary dynamical wave operators if we have a complete system of orthonormal solutions (satisfying (6)) of

$$\mathbf{K}\{xt\}(xt|\psi=0 \tag{20}$$

in many-times theory or of

$$(\mathbf{K}\{x_1,\dots x_{n-1},x;t\}-\mathbf{K}\{x_1,\dots x_{n-1};t\})(xt|\psi_n=0 \tag{20'}$$

in single-time theory. In single-time theory the solutions $(xt\,|\,\psi_n\,|\,\mu)$ will in general depend on the variables $(x_1,\dots x_{n-1})$. In many cases of single-time theory and in all interesting cases of many-times theory the operator $\mathbf{K}$ and therefore also the solutions $(xt\,|\,\psi'_n\,|\,\mu)$ (the suffix n has to be dropped in many-times theory) contain creation and annihilation operators of particles of another kind (in particular of carrier particles) with which those of the considered kind interact. All this makes them extremely complicated.

The dynamical wave operators are then given by

$$\left.\begin{aligned}(ys\,|\,\psi'\,|\,xt_0\}(x_1t_{10},\dots x_nt_{n_0}|\,\Psi'=\sum_{(\mu)}(ys\,|\,\psi_n\,|\,\mu)\int(dx_n)(\mu\,|\,\psi'_n\,|\,xt_0)\,n^{1/2}\mathbf{S}_n\\ \eta\{x_nt_{n_0}\}\varrho\{x_nt_{n_0}\}(x_1t_{10},\dots x_nt_{n_0}|\,\Psi',\\ \{xt_0\,|\,\psi'^{\dagger}\,|\,ys)(x_1t_{10},\dots x_{n-1}t_{n-10};ys\,|\,\Psi'=\sum_{(\mu)}\mathbf{S}_n\,n^{1/2}(x_nt_{n_0}|\,\psi'_n\,|\,\mu)(\mu\,|\,\psi_n^{\dagger}\,|\,ys)\\ (x_1t_{10},\dots x_{n-1}t_{n-10};ys\,|\,\Psi',\end{aligned}\right\} \tag{21}$$

if operating to the right and by the HERMITIAN adjoint relations (21†), if operating to the left and with $s=t_n$ everywhere. The ψ' appear as a generalization of the ψ. If we replace the t_{k0} by t_k, we get the undashed ψ again.

The dynamical wave operators satisfy the commutation relations

$$[(ys\,|\,\psi'\,|\,xt_0\},\{xt_0\,|\,\psi'^{\dagger}\,|\,y's')]^{\pm}=\sum_{(\mu)}(ys\,|\,\psi_n\,|\,\mu)(\mu\,|\,\psi_n^{\dagger}\,|\,y's')\mathbf{S}_n,\quad\text{etc.,}\tag{22}$$

similar to (11), only s and s' may now have different values. The sum satisfies the wave equations similar to (20) or (20') with $\mathbf{K}\{ys\}$ (at the left) as well as with $\mathbf{K}\{y's'\}$ (at the right). For $s=s'$ it has the properties described in (6).

2.34 *Dynamical substitution operators.* If we form the dynamical substitution operators similar to (12), (12†), their meaning can still readily be demonstrated.

We shall say that with respect to the kth set of variables a wave function is up to date if the time coordinate has the value t_k and that it is at the

beginning if the value of the time coordinate is t_{k0}. In the representation s_1 the Ψ are up to date in all sets (xt) and (ys). In s_3 the Ψ' are at the beginning in all sets (xt_0) and (ys_0). In s_2, with which we are dealing for the moment, the Ψ' are at the beginning in the sets (xt_0) and up to date in the sets (ys).

If we compare the dynamical substitution operators with the statical ones, particularly observing the time dependence, we see that:

S_d The dynamical substitution operators have the same properties as the statical substitution operators as summed up in S_s, but in addition a dynamical annihilation substitution operator brings the wave function up to date in the set (ys), which replaces the last set $(x_k t_{k0})$, and a dynamical creation substitution operator puts the wave function back to the beginning in the new set $(x_k t_{k0})$, which replaces the set (ys).

2.35 *Dynamical particle operators.* If in (16) or (17), (17†) the ψ are replaced by ψ', we find the dynamical homogeneous particle operators $\mathbf{R}'^{(m)}\{x_1 t_1, \ldots\}$, which contain the entire motion of the system. Operators of the type $\mathbf{R}'\{x_1 t_1, \ldots; y_1 s_1, \ldots\}$ will be dynamical in the sets (xt), not in the sets (ys).

If in many-times theory $\mathbf{R}^{(1)}\{x_1 t_1, \ldots\}$ is formed from the individual operators $\mathbf{R}\{x_k t_k\}$, we can define the dynamical individual operators $\mathbf{R}'\{xt\}$ by $\mathbf{R}'^{(1)}\{x_1 t_1, \ldots; xt\} - \mathbf{R}'^{(1)}\{x_1 t_1, \ldots\}$.

The dynamical $\mathbf{K}'\{xt\}$ in many-times theory or $\mathbf{K}'\{x_1, \ldots; t\}$ in single-time theory vanish according to the "superquantized wave equation" (19) or (19′).

If after having formed the dynamical particle operators one forgets everything about wave operators, one is left with the elementary HEISENBERG representation e_2.

3. Special cases.

Before facing the general formalism developed so far with present theories, we first derive the explicit expression for the right hand member of (22) (in which we omit $\mathbf{S}_n$) in some special cases of typical kinds of particles. Successively we consider particles of spin ½, 0 and 1.

3.1 *Spin ½.* The 1-particle wave functions are spinors. In many-times theory the operators $\mathbf{K}\{xt\}$ in (1) read

$$\mathbf{K}\{xt\} = \left(\frac{\hbar}{i}\frac{\partial}{\partial t} - e\varphi(x)\right) + \vec{\alpha}\left(\frac{\hbar c}{i}\frac{\vec{\partial}}{\partial x} + e\vec{a}(x)\right) + \beta m c^2. \quad . \quad (23)$$

The 4-velocity operator is $(1, \vec{\alpha})$, so the density operator $\varrho = 1$. This is positive definite, therefore $\eta = 1$.

For free particles (zero external field $(\varphi, \vec{a})$ or zero charge e) a complete

system of orthonormal solutions (satisfying (6)) of (20) is given by

$$(xt|\psi|\xi_{\pm}r) = b|\xi_{\pm}r)\, e^{-\frac{i}{\hbar c}(\pm(\xi^2+m^2c^4)^{1/2}ct - \vec{\xi}\vec{x})}/(hc)^{3/2} \quad . \quad . \quad . \quad (24)$$

with two spinors $b|\vec{\xi}_{\pm}r)$ $(r = 1, 2)$, for which

$$\left.\begin{aligned} &(r\vec{\xi}_{\pm}|\, b\cdot b|\vec{\xi}_{\pm}s) = \delta_{rs}, \\ &\sum_r b|\vec{\xi}_{\pm}r)\,(r\vec{\xi}_{\pm}|\, b = (\pm(\xi^2+m^2c^4)^{1/2} + \vec{\alpha}\vec{\xi} + \beta mc^2)/\pm 2(\xi^2+m^2c^4)^{1/2}. \end{aligned}\right\} \quad (25)$$

This gives for the right hand member of (22)

$$\left.\begin{aligned} &\sum_r \sum_{\pm} \int (d\vec{\xi})\,(ys|\psi|\vec{\xi}_{\pm}r)(r\vec{\xi}_{\pm}|\psi^{\dagger}|y's') = \\ &\qquad = \left(-\frac{\hbar}{i}\frac{\partial}{\partial s} + \vec{\alpha}\frac{\hbar c}{i}\frac{\vec{\partial}}{\partial y} + \beta mc^2\right) D_a(y-y', s-s'). \end{aligned}\right\} \quad (26)$$

3.2 *D-fuctions.* The functions D_a above in 3.1 and D_s below in 3.3 and 3.4 are given by

$$\left.\begin{aligned} D_{\substack{a\\s}}(x,t) &= \int \frac{(d\vec{\xi})}{(hc)^3} \frac{e^{-\frac{i}{\hbar}(\xi^2+m^2c^4)^{1/2}t} \mp e^{\frac{i}{\hbar}(\xi^2+m^2c^4)^{1/2}t}}{2(\xi^2+m^2c^4)^{1/2}} e^{\frac{i}{\hbar c}\vec{\xi}\vec{x}} = \\ &= \frac{1}{4\pi\hbar c x}\frac{\partial}{\partial x} F_{\substack{a\\s_0}}\left(\frac{mc}{\hbar}|c^2t^2-x^2|^{1/2}\right) = \\ &= \frac{m}{4\pi\hbar^2} F_{\substack{a\\s_1}}\left(\frac{mc}{\hbar}|c^2t^2-x^2|^{1/2}\right)\Big/|c^2t^2-x^2|^{1/2}. \end{aligned}\right\} \quad (27)$$

Inside and outside the lightcone F_a and F_s stand for various kinds of BESSEL functions as indicated in fig. 1.

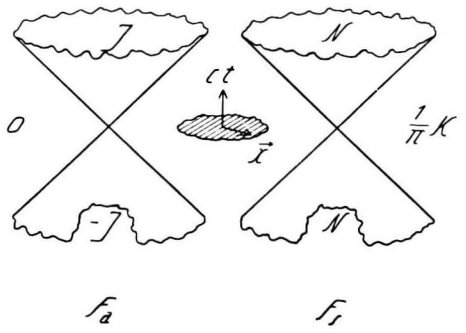

Fig. 1.
(Instead of $\pm J$ read $\pm iJ$.)

For $m = 0$ (27) degenerates into

$$D_a = -\frac{i}{hc}\,\delta(c^2t^2-x^2) \quad , \quad D_s = -\frac{1}{\pi hc}\frac{1}{c^2t^2-x^2}. \quad . \quad . \quad (27^*)$$

3.3 *Spin* 0. The 1-particle wave functions are scalars or pseudoscalars. We consider the scalar case. The wave equations can be derived from

$$\left.\begin{aligned}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\alpha_k}}+e\,\varphi_{\alpha_k}(x_k)\right)\Psi&=mc^2\,{}_{\alpha_k}\Psi,\\ \left(\frac{\hbar c}{i}\frac{\partial}{\partial x_{k\alpha_k}}+e\,\varphi^{\alpha_k}(x_k)\right){}_{\alpha_k}\Psi&=-mc^2\,\Psi.\end{aligned}\right\}\quad\ldots\quad(28)$$

This gives for the operator $\mathbf{K}\{xt\}$

$$\mathbf{K}\{xt\}=\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_\alpha}+e\,\varphi^\alpha(x)\right)\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}+e\,\varphi_\alpha(x)\right)+m^2c^4.\quad(29)$$

In 4-dimensional time-space the metric is indefinite

$$(-g_{00}=g_{11}=g_{22}=g_{33}=1),$$

but since the wave functions are scalars this has no direct consequence for the metric in HILBERT space. Meanwhile the 4-velocity operator

$$\left(\frac{\hbar c}{i}\frac{\delta}{\delta x^\alpha}-\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}-2e\,\varphi_\alpha(x)\right)$$

gives the density operator

$$\left(\frac{\hbar}{i}\frac{\delta}{\delta t}-\frac{\hbar}{i}\frac{\partial}{\partial t}-2e\,\varphi(x)\right)$$

which happens to be indefinite as well (∂ is meant to operate to the right, δ to the left). So we have to determine the operator η.

The conditions c_1, c_2, c_3 of 2.15 can readily be satisfied, but c_4 is somewhat knotty. In an external field the components of the 4-velocity operator commute neither with each other, nor with $\mathbf{K}\{xt\}$. Therefore there are no simultaneous eigenstates. If we brush aside this difficulty, we might say that, if acting on a solution of the wave equation, the 4-velocity operator behaves time-like with positive and negative eigenvalues of the time component separated by a gap (of $2mc^2$). If η acts on a solution of the wave equation (and that is all we need), c_4 is satisfied as long as there is such a distinct and invariant separation between positive and negative solutions. This holds exactly in zero external field, but is liable to break down in "hard" fields (hard enough for pair creation and annihilation).

For free particles a complete system of orthonormal solutions (satisfying (6)) of (20) is given by

$$(xt\,|\,\psi\,|\,\vec{\xi}_\pm)=e^{-\frac{i}{\hbar c}(\pm(\xi^2+m^2c^4)^{1/2}ct-\vec{\xi}\vec{x})}/(\xi^2+m^2c^4)^{1/4}(2h^3c^3)^{1/2}.\quad(30)$$

This gives for the right hand member of (22)

$$\sum_\pm\int(d\vec{\xi})(ys\,|\,\psi\,|\,\vec{\xi}_\pm)(\vec{\xi}_\pm\,|\,\psi^\dagger\,|\,y's')=D_s(y-y',s-s')\quad\ldots\quad(31)$$

3.4 *Spin* 1. 3.41 $m \neq 0$. The 1-particle wave functions are vectors or pseudovectors. We consider the vector case. The wave equations can be derived from

$$\left.\begin{aligned}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\alpha_k}}+e\varphi_{\alpha_k}(x_k)\right)\ldots{}_{\beta_k}\ldots\Psi-\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\beta_k}}+e\varphi_{\beta_k}(x_k)\right)\ldots{}_{\alpha_k}\ldots\Psi\\ =mc^2\ldots{}_{(\alpha_k\beta_k)}\ldots\Psi,\\ \left(\frac{\hbar c}{i}\frac{\partial}{\partial x_{k\alpha_k}}+e\varphi^{\alpha_k}(x_k)\right)\ldots{}_{(\alpha_k\beta_k)}\ldots\Psi=-mc^2\ldots{}_{\beta_k}\ldots\Psi.\end{aligned}\right\}\quad(32)$$

This gives for the operator $\mathbf{K}\{xt\}$ (operating on $\ldots{}_{\alpha}\ldots\Psi$ or $\Psi^{\dagger}\ldots{}_{\beta}\ldots$)

$$\mathbf{K}^{\alpha}_{\beta}\{xt\}=\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^{\nu}}+e\varphi^{\nu}(x)\right)\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^{\delta}}+e\varphi_{\delta}(x)\right)(\delta^{\delta}_{\gamma}\delta^{\alpha}_{\beta}-\delta^{\alpha}_{\gamma}\delta^{\delta}_{\beta})+m^2c^4\quad(33)$$

or the adjoint representation. (It should be observed that for $\gamma \neq \delta$ the factor operators cannot be commuted).

The 4-velocity operator (acting between $\Psi^{\dagger}\ldots{}_{\beta}\ldots$ and $\ldots{}_{\alpha}\ldots\Psi$ is

$$\left(\frac{\hbar c}{i}\frac{\delta}{\delta x_{\delta}}-\frac{\hbar c}{i}\frac{\partial}{\partial x_{\delta}}-2e\varphi^{\delta}(x)\right)(g_{\gamma\delta}g_{\beta\alpha}-g_{\gamma\alpha}g_{\beta\delta}-g_{\gamma\beta}g_{\alpha\delta})\quad.\quad(34)$$

The density operator ($\gamma = 0$) is indefinite like for zero spin. The indefinite g's even threaten to lead to further difficulties. In fact they do not for free particles (as we shall see), but they are liable to do so in "hard" fields.

For free particles the wave equations reduce to two sets

$$\left.\begin{aligned}\left(\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\beta_k}}\frac{\hbar c}{i}\frac{\partial}{\partial x_{k\beta_k}}+m_k^2c^4\right)\ldots{}_{\alpha_k}\ldots\Psi=0,\\ \frac{\hbar c}{i}\frac{\partial}{\partial x_{k\alpha_k}}\ldots{}_{\alpha_k}\ldots\Psi=0.\end{aligned}\right\}\quad\ldots\quad(35)$$

The second set of equations can be regarded as supplementary conditions to the first set. Owing to them the second and third term in the last factor of (34) can be dropped. That makes the density operator equal to

$$\left(\frac{\hbar}{i}\frac{\delta}{\delta t}-\frac{\hbar}{i}\frac{\partial}{\partial t}\right)g^{\beta\alpha}.$$

A complete system of orthonormal solutions (satisfying (6)) of (35) is given by

$$(xt|_{\alpha}\psi|\vec{\xi}_{\pm}r)={}_{\alpha}b|\vec{\xi}_{\pm}r)\,e^{-\frac{i}{\hbar c}(\pm(\xi^2+m^2c^4)^{1/2}ct-\vec{\xi}\vec{x})}/(\xi^2+m^2c^4)^{1/4}(2h^3c^3)^{1/2}\quad.\quad(36)$$

with 3 4-vectors ${}_{\alpha}b\,|\,\xi_{\pm}\,r)$ ($r = 1, 2, 3$) satisfying the supplementary conditions

$$\xi^{\alpha}_{\pm}\,{}_{\alpha}b|\vec{\xi}_{\pm}r)=0\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(37)$$

(where $\xi^0_{\pm}=\pm(\xi^2+m^2c^4)^{1/2}$) and for which

$$\left.\begin{aligned}(r\vec{\xi}_{\pm}|b^{\dagger\alpha}\cdot{}_{\alpha}b|\vec{\xi}_{\pm}s)=\delta_{rs},\\ \sum_r{}_{\alpha}b|\vec{\xi}_{\pm}r)(r\vec{\xi}_{\pm}|b^{\dagger}_{\beta}=g_{\alpha\beta}+\frac{\xi_{\alpha}\xi_{\beta}}{m^2c^4}.\end{aligned}\right\}\quad.\quad.\quad.\quad.\quad.\quad(38)$$

Because of (37) the b's are space-like, so that with regard to them $g_{\alpha\beta}$ behaves positive definite. That is why there are (for free particles) no further difficulties with the indefinite metric than for zero spin. The right hand member of (22) becomes

$$\left.\begin{aligned}\sum_r \sum_{\pm} \int (d\vec{\xi})\,(ys\,|_\alpha\psi\,|\vec{\xi}_\pm r)\,(r\vec{\xi}_\pm\,|\psi_\beta|\,y's') = \\ = \left(g_{\alpha\beta} + \frac{\hbar c}{i}\frac{\partial}{\partial y^\alpha}\frac{\hbar c}{i}\frac{\partial}{\partial y^\beta}\Big/ m^2c^4\right) D_s(y-y',\, s-s')\,.\end{aligned}\right\} \quad (39)$$

3.42 $m = 0$. Zero restmass forms a singular case for which the foregoing treatment breaks down after (37). ξ^α is now a zero-vector and not all b's satisfying the supplementary conditions are space-like. This gives difficulties with the indefinite $g_{\alpha\beta}$. Invariant relations similar to (38) cannot be formed. Instead of helping any longer, the supplementary conditions (37) only stand in the path. Now the supplementary conditions in (35) apply to the wave functions Ψ. They may be imposed on the complete system of individual reference functions ψ and on the wave operators ψ if such is possible and useful, but they need not if it is a nuisance or impossible. So we only impose them on the Ψ. If we like we can write them as

$$\frac{\partial}{\partial y_\beta}(ys\,|_\beta\psi^\alpha|\,xt\}\,(x_1 t_1, \ldots|_{\alpha_1}\ldots\,\Psi = 0 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in s_1 representation or as

$$\frac{\partial}{\partial y_\beta}(ys\,|_\beta\psi'^\alpha|\,xt_0\}\,(x_1 t_{10}, \ldots|_{\alpha_1}\ldots\,\Psi' = 0 \quad . \quad . \quad . \quad . \quad (41)$$

in s_2 representation. With the supplementary conditions imposed on the ψ, already the operators to the left of Ψ in (40) and (41) would themselves have been identically zero.

As we suppose the zero mass particles to be uncharged, the density operator is $\left(\frac{\hbar}{i}\frac{\delta}{\delta t} - \frac{\hbar}{i}\frac{\partial}{\partial t}\right) g^{\alpha\beta}$. The positive and negative states of the first factor are the same as for zero spin. In order to distinguish between positive and negative states of the second factor we choose an arbitrary time-like 4-vector j^α. As positive vectors we take those orthogonal to j^α, as negative vectors those parallel to j^α. The operator η is then the product of the corresponding operator for zero spin (which because of $e = 0$ exactly satisfies c_4) and the 4-tensor $(g_{\alpha\beta} + 2j_\alpha j_\beta)$. The latter factor does not satisfy c_4 because it depends on the choice of j^α.

As we drop (37), we now get in (36) (with $m = 0$) 4 4-vectors ${}_\alpha b\,|\vec{\xi}_\pm\, r)$ $(r = 1, 2, 3, 4)$ for which

$$\left.\begin{aligned}(r\vec{\xi}_\pm\,|b^{\dagger\alpha}\cdot{}_\alpha b\,|\vec{\xi}_\pm s) = \delta_{rs}, \\ \sum_r {}_\alpha b\,|\vec{\xi}_\pm r)\,(r\vec{\xi}_\pm\,|\,b^\dagger_\beta = g_{\alpha\beta} + 2j_\alpha j_\beta.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

3 b's are space-like, 1 is time-like. The right hand member of (22) becomes

$$\sum_r \sum_{\pm} \int (d\vec{\xi})\,(ys|_\alpha \psi|\vec{\xi}_\pm r)\,(r\vec{\xi}_\pm|\psi^\dagger_\beta|\,y's') = (g_{\alpha\beta} + 2j_\alpha j_\beta)\,D_s(y-y', s-s'). \quad (43)$$

4. **Present theories.**

Now we compare the results of our primitive form of "superquantization" with the starting point of the present theories.

4.1 *Notation.* In the present theories the functions on which the wave operators act are in general hardly taken into consideration. Consequently the variables on which they depend are usually not explicitly mentioned even in the wave operators. In our notation that would mean that in the expressions for the wave operators not the (xt), but only the (ys) are written down. This incomplete notation, which is quite sufficient for every-day use, is perhaps one of the main factors, which make that the meaning of the wave operators is not always clearly understood.

4.2 *Commutation relations.* If our wave operators shall be isomorphic to those of the present theories, they have to satisfy the same commutation relations.

4.21 *Field quantization.* In the present theories the field operators (more precisely the sum of creation and annihilation field operators, which is HERMITIAN) resulting from quantization of classical fields represent field observables. We have not considered this kind of observations. As soon as the desired isomorphy has been established, our field operators can be interpreted in the same way. The question how far field measurements can be interpreted by particle measurements belongs to problem Q_2.

For the moment we are only interested in the consequences with regard to the commutation relations. Because there can be no signals between two world points with a space-like connection, field observations in two such points cannot affect each other. Therefore the corresponding field operators in two such points must commute with each other.

Incidentally this also indicates that carrier particles obey B–E statistics. The problem whether that can be explained again belongs to Q_2.

4.22 *Superquantization.* More generally all wave operators of the present theories obey PAULI's postulate [7]) that in world points with a space-like connection they commute or anti-commute. In other words their (anti-)-commutators vanish outside the light cone.

4.23 *Discrepancies.* The commutation relations (31), (39) and (43) of the wave operators as we have preliminary defined them contain D_s, which according to fig. 1 does not vanish outside the light cone. Therefore our preliminary wave operators cannot be isomorphic with those of the present theories. We must try to modify the preliminary definition (10) of the wave operators in such a way, that they fit into the recognized commutation relations, without spoiling those properties, which are already all right. Now D_s in (31), (39) and (43) has to be replaced by D_a and moreover

$(g_{\alpha\beta} + j_\alpha j_\beta)$ in (43) by $g_{\alpha\beta}$. More generally the right hand member of (22) has to be replaced by

$$\sum_{(\mu)} \eta\{ys\}(ys|\psi_n|\mu)(\mu|\psi_n^\dagger|y's')\,\mathrm{S}_n = \sum_{(\mu)} (ys|\psi_n|\mu)(\mu|\psi_n^\dagger|y's')\eta\{y's'\}\,\mathrm{S}_n. \quad (44)$$

This modified expression satisfies the same wave equation as the original one. For $s = s'$ it has also the properties described in (6), which now only should be read in a different way.

4.3 *Modified wave operators.* We can make the modification in two different ways, which establish the required isomorphy with two different types of present theories: DIRAC's 1942 theory and the current hole theories.

4.31 DIRAC's *1942 theory.* One way to obtain the modification (44) is to define the modified wave operators $(ys|\psi_D|xt\}$ and $\{xt|\psi_D^\dagger|ys)$ by

$$\left.\begin{aligned} (ys|\psi_D|xt\} &= (ys|\psi|xt\}, \\ \{xt|\psi_D^\dagger|ys) &= \{xt|\psi^\dagger|ys)\eta\{ys\}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10\,D)$$

if they are operating to the right and

$$\left.\begin{aligned} (ys|\psi_D|xt\} &= \eta\{ys\}(ys|\psi|xt\}, \\ \{xt|\psi_D^\dagger|ys) &= \{xt|\psi^\dagger|ys), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10\,D^\dagger).$$

if they are operating to the left. They are HERMITIAN adjoint to each other. In s_2 representation they satisfy the "superquantized wave equations" (19) or (19′). Their commutation relations yield the required form (44). If (12), (12†) and (17), (17†) are written with the D-modified wave operators, the factors η are swallowed up by the creation wave operators; the resulting expressions remain unaltered. (If we let also the factors ϱ be swallowed up, we get a description with canonical conjugates).

If the substitution operators are correspondingly modified, we see that:

S_D The D-modified substitution operators are almost identical with the preliminary ones. The D-modified creation substitution operators only give an extra factor -1 wherever a negative density function in (ys) is replaced by the same function in $(x_k t_k)$.

The D-modified wave operators $(ys|\psi_D|xt\}$ and $\{xt|\psi_D^\dagger|ys)$ are now isomorphic with the fields $U(y)$ and $U^*(y)$ (in PAULI's notation [6]) of DIRAC's 1942 theory. A further discussion of the latter theory belongs to problem Q_3.

4.32 *Current theories.* 4.321 *Positive and negative states.* Up to now we could completely avoid to speak about positive and negative energy states and positive and negative particles. They are, however, so narrowly interwoven with already the wave operators of the current theories, that we have to deal with them in some extent. A complete discussion would lead into problem Q_3.

4.3211 *Energy states.* For all spin values the energy-momentum operator of a particle is $\left(\frac{\hbar c}{i}\frac{\delta}{\delta x^\alpha} - \frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha}\right)/2$. The kinetic energy-momentum operator is therefore $\left(\frac{\hbar c}{i}\frac{\delta}{\delta x^\alpha} - \frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha} - 2e\varphi_\alpha(x)\right)/2$. The kinetic energy operator ($\alpha = 0$) is indefinite in exactly the same way as the density operator for zero spin. We distinguish between positive and negative energy states in exactly the same way as between positive and negative density states for zero spin by means of an operator ζ, which is identical with η of the latter case and therefore also makes the same difficulties.

4.3212 *Charge conjugated states.* We consider two kinds of particles, which only differ in the sign of their charge e (for $e = 0$ the two kinds are identical). They can also be considered as particles of one kind with a charge operator $\mathbf{e}$ with eigenvalues $\pm e$. To each particle state corresponds another state (the charge conjugated state) with the opposite charge and energy-momentum vector (the charge conjugated state of ψ is e.g. the complex conjugate $\overline{\psi}$ in case of integer spin; in case of spin $\frac{1}{2}$ it is $\overline{\varrho_1\psi}$ in a representation in which $\vec{\alpha}$ and ϱ_3 have real matrixelements). Then also the kinetic energy-momentum is opposite. If one of the states is a positive energy state, the other is a negative energy state. The charge conjugate of $\psi\,|\,\mu)$ will be written as $\psi\,|\,\tilde{\mu})$.

The connection between creation of a particle in one state and annihilation of a particle in the charge conjugated state is one of the fundamental topics of problem Q_3.

4.322 *Hole theories.* We now turn to the current theories, but leave aside for a moment the photon case (spin 1, $m = 0$), which we shall deal with later on.

4.3221 *H-revision.* In order to obtain isomorphy with the current wave operators we have to replace in (10) the creation and annihilation operators of negative energy states respectively by the annihilation and creation operators of the charge conjugated states. This H-revision is a part of the hole trick. The other part is the omission of the interaction between the two opposite particles during a process of pair creation or annihilation, but that entirely belongs to Q_3. The H-revised wave operators become

$$\left.\begin{aligned}(ys|\psi_H|xt\} &= \sum_{(\mu)}(1+\zeta\{ys\})/2\ \ (ys\,|\,\psi\,|\,\mu)\,(\mu\,|\,\mathbf{a}^\dagger\,|\,xt\} \\ &\quad + \sum_{(\mu)}(1-\zeta\{ys\})/2\ \ (ys\,|\,\psi\,|\,\mu)\,\{xt\,|\,\mathbf{a}\,|\,\tilde{\mu}), \\ \{xt|\psi_H^\dagger|ys) &= \sum_{(\mu)}\{xt\,|\,\mathbf{a}\,|\,\mu)\,(\mu^\dagger\,|\,ys)\ \ (1+\zeta\{ys\})/2 \\ &\quad + \sum_{(\mu)}(\tilde{\mu}\,|\,\mathbf{a}^\dagger\,|\,xt\}\,(\mu\,|\,\psi^\dagger\,|\,ys)\ \ (1-\zeta\{ys\})/2.\end{aligned}\right\}\quad .\ (10H)$$

They are HERMITIAN adjoint to each other. In s_2 representation they

satisfy the "hyperquantized wave equations" (19) or (19′). In the same representation they satisfy the commutation relations

$$\left.\begin{aligned}[(ys|\psi'_H|xt_0\},\{xt_0|\psi'^{\dagger}_H|y's')]^{\pm} &= \sum_{(\mu)} \frac{1}{\zeta\{ys\}}(ys|\psi_n|\mu)(\mu|\psi^{\dagger}_n|y's')\mathrm{S}_n \\ &= \sum_{(\mu)}(ys|\psi_n|\mu)(\mu|\psi^{\dagger}_n|y's')\frac{1}{\zeta\{y's'\}}\mathrm{S}_n, \text{ etc.}\end{aligned}\right\}. \quad (45)$$

The upper factor refers to *F–D* statistics, the lower factor to *B–E* statistics.

4.3222 *Spin and statistics.* The right hand part of (45) is equal to the required form (44) for half-odd spin ($\eta = 1$) only in case of *F–D* statistics, for integer spin ($\eta = \zeta$) only in case of *B–E* statistics. The theoretical derivation [7]) of this connection between spin and statistics, which is due to PAULI and BELINFANTE, is based on

b_1 PAULI's postulate (cf. 4.22);

b_2 the *H*-revision (hole trick).

b_1 is not satisfied by our preliminary wave operators defined by (10). b_2 has neither been performed in our preliminary picture nor in its *D*-modified form, which is equivalent to DIRAC's 1942 theory. The necessity of b_1 and b_2 has not been unshakably established. b_1 and b_2 are sufficient but not necessary conditions for the connection between spin and statistics. The latter connection is the only point, which is directly backed by experimental evidence.

4.3223 *H-revised operators.* In order to make (17), (17†) isomorphic with the corresponding expressions in the current theories, they have to be written with the *H*-revised wave operators and the factors η have to be dropped. If the same is done with (12), (12†), the relation (16) remains unaltered. But the substitution operators and the particle operators are essentially changed.

4.32231 *Substitution operators.* The former are revised in such a way that:

S_H The *H*-revised creation/annihilation substitution operators differ from the preliminary ones in so far as, when taking out/inserting a negative energy function in (ys), the latter insert/take out the corresponding function in (xt) or (xt_0), but the former take out/insert the charge conjugated function in (xt) or (xt_0); further the *H*-revised creation operator gives an extra factor — 1 if it takes out a negative density function in (ys).

4.32232 *Particle operators.* The particle operators are essentially changed by the *H*-revision. That makes that the whole theory is essentially changed. This change is not a matter of quantization, it is only a result of the hole trick (which moreover omits the interaction between the two opposite

particles during a process of pair creation or annihilation). The discussion of the change belongs to problem Q_3.

4.3224 *Photons.* In the photon case there is a slight complication, because both factors of the density operator $\left(\frac{\hbar}{i}\frac{\delta}{\delta t} - \frac{\hbar}{i}\frac{\partial}{\partial t}\right) g_{\alpha\beta}$ in 3.42 are indefinite. One way to meet with these difficulties is to perform the interchange of creation and annihilation operatore of "charge" conjugated states for those negative energy functions, which are positive vectors, and for those positive energy functions, which are negative vectors. There are other ways. None of these ways has been followed in the current theories, for the current theories do not ask for an explicit realization of the wave operators. In fact they need not and that is also why the theory is ultimately independent of the choice of j^a in 3.42. As a starting point of quantum electrodynamics one can take the commutation relations and they do not depend on j^a. Then the usual course is to choose a time axis and to perform a transformation, which eliminates the scalar and longitudinal field operators, so that only the transverse ones are left. Because they are space-like, there are no further difficulties with the indefinite metrical tensor. (In our way of reasoning we might say that j^a is chosen in the direction of the time axis). Though this representation depends on the choice of the time axis, the processes which it describes do not. So the theory is ultimately (in its observable consequences) invariant and independent of j^a.

4.4 *Half-odd and integer spin.* One remark might be added about the characteristic difference between half-odd and integer spin in problem Q_1. The differential operator $\mathbf{K}\{xt\}$ is (in particular in the time coordinate) of 1st order for half-odd spin and of 2nd order for integer spin. The density operator is a zero order differential operator, which is even equal to 1, in the first case. It is a 1st order differential operator in the time coordinate and even indefinite in the second case. The different density operators can be regarded as characteristic for the difference between the two cases. The 2nd order wave equations for integer spin can be reduced to 1st order equations of more complicated wave functions as (28) and (32). This gives for various purposes a simpler description indeed. But it does not reduce the density operator to 1. It is this density operator, which can be hold responsible for many of the complications in case of integral spin.

5. Conclusion.

5.1 *Plus and minus troubles.* Not all difficulties mentioned in 1.3 could be shifted to problem Q_3 and none of them has been solved. Those of 1.311 appeared already in our preliminary picture in case of integral spin, those of 1.312 and 1.32 only in its H-revised form.

5.11 *Distinction of negative states* (d_1). We have postulated an operator η, which is $+1$ for positive and -1 for negative density states and an operator ζ, which is $+1$ for positive and -1 for negative energy states.

These operators have only been determined in case of free particles and their existence in presence of hard external fields has not even been warranted.

5.12 *Elimination of negative states* (d_2). In the ordinary quantum theory of particles we have multiplied the original indefinite density operator by η. In this way we have provisionally brought about an artificially definite metric in HILBERT space.

In the H-revision the negative energy states have been attended by means of the hole trick.

5.2 *"Superquantization"*. 5.21 *Wave operators.* By introducing creation and annihilation operators and wave operators we have brought the ordinary quantum theory of particles in "superquantized" form. The wave operators form an indispensable tool for describing interaction processes in which particles are created or annihilated (e.g. P_2 and P_3).

The wave operators have first been defined preliminary by (10).

5.22 *Present theories.* 5.221 DIRAC*'s 1942 theory.* In order to obtain DIRAC's 1942 theory we had to redefine the wave operators according to the D-modification:

D the operators $\eta\{ys\}$ are taken up in the creation wave operators.

Then the original indefinite density operator is left unaccompanied by η. That makes the artificially definite metric look indefinite.

Though DIRAC's 1942 theory is the youngest of the present theories, its starting point is the most primitive. The discussion of its consequences belongs to Q_3.

5.222 *Hole theories.* In order to obtain the older current theories we had to apply the H-revision:

H_1 creation/annihilation operators of negative energy states are replaced by annihilation/creation operators of the charge conjugated states;

H_2 the operators $\eta\{ys\}$ are replaced by 1.

Again the original indefinite density operator is left unaccompanied by η. Contrary to the D-modification, the H-revision makes a real change in the theory. This change is not a quantization process.

5.23 *Quantization processes.* Thus our picture of the quantization processes is:

PQ Starting from classical particle theory, ordinary particle quantization is the first and only step of quantization. The ordinary quantum theory of particles is equivalent to DIRAC's 1942 theory. If afterwards the hole trick is performed we get the hole theories.

FQ Starting from classical field theory, field quantization is the first and only step of quantization. The quantized field theory is equivalent to ordinary quantum theory of particles.

Therefore all present theories are equivalent with ordinary quantum theory of particles in which in some cases (hole theories) the hole trick has been performed. None of them contains a second step of quantization, which goes beyond the first step.

Summary.

It often appears that one is not always clearly conscious of the relations between ordinary quantization of classical particle theory, quantization of classical field theory and superquantization of ordinary quantum theory of particles. In this paper the situation has been looked at from a perhaps unorthodox point of view. All present quantum theories can without a further process of quantization be derived from ordinary quantum theory of particles. The latter is already equivalent with DIRAC's 1942 theory. The older current theories can be obtained by performing a trick, which is not a matter of quantization and which is characteristic for hole theories.

REFERENCES.

1. V. FOCK, Z. Phys. **75**, 622 (1932).
2. W. PAULI, Rev. Mod. Phys. **13**, 203 (1941).
3. P. A. M. DIRAC, Proc. Roy. Soc. London, A **126**, 360 (1930).
4. W. PAULI und V. WEISSKOPF, Helv. Phys. Acta, **7**, 709 (1934).
5. P. A. M. DIRAC, Proc. Roy. Soc. London, A **180**, 1 (1942).
6. W. PAULI, Rev. Mod. Phys., **15**, 175 (1943).
7. W. PAULI, Phys. Rev. (2) **58**, 716 (1940).