Mathematics. — *On Differentiable Linesystems of one Dual Variable.* I.

By N. H. KUIPER (Princeton N.Y.) (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 30, 1948.)

CONTENTS.

1. *Introduction.*

Lines, combined with one of their two directions, in a Euclidean three-dimensional space, can be represented by unitvectors with three components over the ring of dual numbers. STUDY [1], BLASCHKE [3] [1]).

Dual numbers are pairs of real numbers with two laws of composition: $A + B \equiv (a + \varepsilon\bar{a}) + (b + \varepsilon\bar{b}) = (a + b) + \varepsilon(\bar{a} + \bar{b})$; $A \cdot B = ab + \varepsilon(\bar{a}b + a\bar{b})$; $(\varepsilon^2 = 0)$. A dual element will be denoted by a capital; the real part by the same letter small, the other part by the same small letter with a bar. Vectors will be denoted by German letters. A unit dual vector is for example $\mathfrak{A} = \mathfrak{a} + \varepsilon\bar{\mathfrak{a}} \equiv (A_1, A_2, A_3) = (a_1 + \varepsilon\bar{a}_1, a_2 + \varepsilon\bar{a}_2, a_3 + \varepsilon\bar{a}_3)$; $\mathfrak{A}^2 = \mathfrak{a}^2 + 2\varepsilon\, \mathfrak{a}\bar{\mathfrak{a}} = 1$. $\mathfrak{a}$ represents the direction of a line. $\bar{\mathfrak{a}}$ is the moment of a univector in the directed line, with respect to the origin of the coordinate system.

Distance, angle and *orientation* of two directed lines $\mathfrak{A}$ and $\mathfrak{B}$ $(\mathfrak{A}^2 = \mathfrak{B}^2 = 1)$ are determined by the scalar product

$$\cos \Phi = \mathfrak{A}\mathfrak{B}, \; \Phi = \varphi + \varepsilon\bar{\varphi} = \text{angle} + \varepsilon \text{ distance}, \; 0 \leq \varphi \leq \pi \qquad (1)$$

The orientation (positive or negative) is the sign of φ. $\varphi\bar{\varphi}/|\varphi\bar{\varphi}|$ determines the orientation of $\mathfrak{A}$ with respect to $\mathfrak{B}$.

Cos Φ is an example of a differentiable function of a dual variable, which can be defined in analogy to a differentiable function of a

[1]) Bibliography is found at the end of this paper.

complex variable. A differentiable function of a dual variable $X = x + \varepsilon \bar{x}$ has the form

$$F(X) = F(x) + \varepsilon \bar{x} F'(x), F'(x) = dF(x)/dx \quad . \quad . \quad . \quad (2)$$

e.g. $\cos X = \cos x - \varepsilon \bar{x} \sin x$.
$F(X)$ in (2) is the unique differentiable continuation of $F(x)$ Formels for differentiation and integration are

$$\left.\begin{array}{l} dF(X)/dX \equiv F'(X) = F'(x) + \varepsilon \bar{x} F''(x) \\ \quad \text{(in particular for } X = x + \varepsilon \cdot 0 : F'(X) = F'(x)\text{)} \\ \int_A^X F(Y)\, dY = \int_a^x F(y)\, dy + \varepsilon (\bar{x} \cdot F(x) - \bar{a} \cdot F(a)) \end{array}\right\} . \quad (3)$$

The important properties of vectoranalysis are valid for the vectorspace over the ring of dual numbers. Moreover, identities in real variables induce identities in dual variables, obtained by differentiable continuation (2). The Euclidean threedimensional linegeometry, expressed with the help of dual unitvectors, is therefore closely analogous to the spherical geometry, expressed with the help of real unitvectors. Properties of elementary spherical geometry can be carried over to linegeometry by some simple translationrules. For example the theorem on the perpendiculars in a triangle and the theorem of Desargues. KUIPER [7] Ch. 1.

2. *The triangle-inequality in the linegeometry* [2]).

A property, which is not expressible in the form of identities in vectors, but which also admits an analogy in linegeometry, is the triangle-inequality.

The dual numbers are *ordered* as follows

$$A = a + \varepsilon \bar{a} > B = b + \varepsilon \bar{b} \leftrightarrows \left\{ \begin{array}{ll} a > b & \text{in case } a \neq b \\ \bar{a} > \bar{b} & \text{in case } a = b \end{array} \right\} . \quad . \quad (4)$$

We define the norm of a dual number A by $|A| = |a| + \varepsilon |\bar{a}|$ (5)

Now it is a matter of simple geometrical considerations, to show that the dual angles between three directed lines ($0 \leqq \Phi_{12}, \Phi_{23}, \Phi_{31} \leqq \pi$) obey

$$|\Phi_{13}| \leqq |\Phi_{12}| + |\Phi_{23}| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Necessary and sufficient conditions for equality in (6) are

a) The three lines are in parallel planes. No two lines are parallel (same direction). If $\mathfrak{A}$ is the line which lies in the middle plane, and if the directions of the lines are represented on a unitsphere, then lies the direction of $\mathfrak{A}$ on the geodesic arc $\leqq \pi$ between the other directions (on the unitsphere).

Or b) Two, not three, lines are parallel. The plane of these lines contains the common perpendicular of the three lines. The third line intersects the plane of the other two in a point not between these lines.

Or c) The three lines are parallel and are in one plane.

---

[2]) R. DEBEVER proposed the problem of a metric in the space of threedimensional lines to me.

3. *D-systems* [3]). *Introduction continued.*

The differential geometry of sphere-curves can also be carried over by analogy into the geometry of lines in Euclidean threespace. The theory of reguli (ruled surfaces) was developped with this analogy in mind (BLASCHKE [3]). Those properties of reguli which are analogous to properties of spherical curves, can be expressed by the same formulas (compare for example BIRAN [6] and SABAN [9]).

However, reguli are not the linesystems with the closest analogy to spherical curves. *A better analogy is obtained from D-systems* [3]), *differentiable linesystems of one dual parameter.* A *D*-system in general is a very particular linecongruence (two real parameters). That is why *D*-systems have not had much attention. We want to study *D*-systems because of their simple character due to the mentioned "close analogy", and because of their relation to reguli. Every regulus determines by differentiable continuation ((2)) a unique *D*-system in which it is contained. *The differential invariants and the properties, which the regulus has in analogy to the sphere-curve, are the differential invariants and the properties of the related D-system.*

A complete system of differential invariants of a regulus can be obtained in two parts: 1. the invariants of a complete system of the *D*-system determined by the regulus. Properties analogous to properties of sphere-curves are expressed in *these* invariants. 2. the invariants which determine the position of the regulus in the related *D*-system (parameter of distribution, § 6).

It is our aim to study *D*-systems in relation with their reguli, and to show some properties of reguli to be properties of the related *D*-systems.

4. *The first invariant orthogonal system at a line of a D-system.*

A regulus is determined by a dual unitvector, function of one real variable: $\mathfrak{A} = \mathfrak{A}(t)$, $\mathfrak{A}^2 = 1$. The unique dual differentiable continuation of this function is given by (2):

$$\mathfrak{A}(T) = \mathfrak{A}(t) + \varepsilon \bar{t}\, \mathfrak{A}'(t),\ \mathfrak{A}^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$\mathfrak{A}(T)$ is the *D*-system "related to" the regulus $\mathfrak{A}(t)$.

In this paper all real functions of real variables will be assumed to be analytic, though the greater part of the paper is also valid under weaker differentiability conditions.

W. BLASCHKE determined an invariant orthogonal system at a line of a regulus. We obtain the first invariant orthogonal system at a line ($T = 0$) of a *D*-system $\mathfrak{A}(T)$ in formally the same way. The three mutually orthogonal and intersecting lines $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ of this system obey

$$d\mathfrak{A}/dT = \dot{\mathfrak{A}} = P\mathfrak{A}_1,\ \mathfrak{A}_2 = \mathfrak{A} \times \mathfrak{A}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

[3]) STUDY [1] p. 305 called these systems *"Synektische Strahlensysteme"*.

When $P^2 = \dot{\mathfrak{A}}^2$ is not a zerodivisor of the ring of dual numbers, then has (8) two trivially related solutions $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}, \mathfrak{A}_1^0 = -\mathfrak{A}_1$, $\mathfrak{A}_2^0 = -\mathfrak{A}_2$ (N is zerodivisor, if $M \neq 0$ exists, such that $N \cdot M = 0$). Usually we choose the solution for which $P > 0 \cdot P = +\sqrt{\dot{\mathfrak{A}}^2}$. We can decompose any dual vector with respect to the system $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ and we find in particular with the help of

$$\mathfrak{A}^2 = 1, \; \mathfrak{A}\dot{\mathfrak{A}} = 0, \; \mathfrak{A}\mathfrak{A}_1 = 0, \; \dot{\mathfrak{A}}\mathfrak{A}_1 + \mathfrak{A}\dot{\mathfrak{A}}_1 = 0, \text{ etc.}$$

$$\left.\begin{array}{lll} \dot{\mathfrak{A}} = & P\mathfrak{A}_1 & P^2 = \dot{\mathfrak{A}}^2 \\ \dot{\mathfrak{A}}_1 = -P\mathfrak{A} & + Q\mathfrak{A}_2; & P^2 Q = (\mathfrak{A}\,\dot{\mathfrak{A}}\,\ddot{\mathfrak{A}}) \\ \dot{\mathfrak{A}}_2 = & -Q\mathfrak{A}_1 & \end{array}\right\} \quad \ldots \quad (9)$$

From (9) and (3) follows (compare BLASCHKE [3] or KUIPER [7]).

Theorem 1. *The Blaschke-system at a line of a regulus (not cylinder) coincides with the first invariant orthogonal system of the related D-system at that line:*

$$T = t + \varepsilon \cdot 0 \quad ; \quad \dot{\mathfrak{A}} = \frac{d\mathfrak{A}(T)}{dT} = \frac{d\mathfrak{A}(t)}{dt}.$$

If $\dot{\mathfrak{A}}^2$ *is a zerodivisor* we get, because

$$\dot{\mathfrak{A}}(T) = \dot{\mathfrak{A}}(t) + \varepsilon \bar{t}\ddot{\mathfrak{A}}(t) = \dot{\mathfrak{a}}(t) + \varepsilon\,[\dot{\bar{\mathfrak{a}}}(t) + \bar{t}\ddot{\mathfrak{a}}(t)] \quad ((3)),$$
$$\dot{\mathfrak{a}}(t) = 0.$$

The direction $\mathfrak{a}(t)$ of lines in a first order neighbourhood of the considered line is constant.

From $\mathfrak{A}(T) = \mathfrak{A}(t) + \varepsilon\bar{t}\dot{\mathfrak{A}}(t) = \mathfrak{A}(t) + \varepsilon\bar{t}\dot{\mathfrak{a}}(t) \quad ((2))$
we also conclude that the D-system, function of the real variables t, $\bar{t}$, does not essentially depend on the variable $\bar{t}$ at the considered line ($T = 0$):

$$\left[\frac{d}{d\bar{t}}\mathfrak{A}(t, \bar{t})\right]_{T = t + \varepsilon\bar{t} = 0} = 0.$$

The D-system is *degenerated* to a *cylinder*. If this is the case at all lines of the D-system, it is a cylinder in the usual sense.

A D-system is *completely degenerated* when $\dot{\mathfrak{A}}(T) = 0$.

The first invariant orthogonal system is not determined ((8)) at a line where a D-system is degenerated, but not completely. Only the directions of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are then uniquely defined. We will restrict to D-systems for which it is possible to choose $\mathfrak{A}_1(T)$ as *differentiable* function of T such that (8) holds. With this choice also (9) holds.

From (7) and the remarks above we conclude to

Theorem 2. *The D-system related to a regulus, which is not a cylinder,* *is a congruence* (linesystem depending essentially on two real parameters). *The D-system related to a cylinder is the cylinder itself.*

The geometrical construction of the D-system related to a regulus (not cylinder) is easily obtained, once we know the geometrical meaning of $\mathfrak{A}_2(T)$. $\mathfrak{A}_2(T)$ is the common perpendicular of $\mathfrak{A}(T)$ and any line of the D-system in the first-order-neighbourhood of $\mathfrak{A}(T)$:

$$\mathfrak{A}_2(T)\,\mathfrak{A}(T) = 0,\ \mathfrak{A}_2(T)\,[\mathfrak{A}(T) + \dot{\mathfrak{A}}(T)\,\triangle T] = 0 \quad ((9))$$

The lines $\mathfrak{A}(t + \varepsilon \bar{t})$ (t fixed, $\bar{t}$ variable) belong to the first-order-neighbourhood of $\mathfrak{A}(t)$

$$\mathfrak{A}(t + \varepsilon \bar{t}) = \mathfrak{A}(t) + \dot{\mathfrak{A}}(t) \cdot \varepsilon \bar{t}$$

Hence these lines $\mathfrak{A}(T)$, *which have the same direction* $\mathfrak{a}(t)$ *as* $\mathfrak{A}(t)$, *intersect* $\mathfrak{A}_2(t)$ *perpendicular.* From these properties follows, given $\mathfrak{A}(t)$, the construction of $\mathfrak{A}(T)$.

> The third axis $\mathfrak{A}_2(t)$ of the Blaschke-system at a line of a regulus (not cylinder) meets the rule $\mathfrak{A}(t)$ perpendicular and in the strictionpoint, and is perpendicular to the asymtotic tangentplane of the rule (BLASCHKE [3]).

Notice that $\mathfrak{A}_2(T)$ is common perpendicular of $\mathfrak{A}(T)$ and a line in the first-order-neighbourhood of $\mathfrak{A}(T)$ in *any* regulus in the D-system containing the line $\mathfrak{A}(T)$. Hence all these reguli have at $\mathfrak{A}(T)$ the same strictionpoint. The reguli of a *general* congruence have at a line a variable strictionpoint.

5. *The invariant dual parameter.*

The dual angle $\triangle \Psi$ between two nearby lines of a D-system

$$\mathfrak{A}(T) \text{ and } \mathfrak{A}(T + \triangle T) = \mathfrak{A} + \dot{\mathfrak{A}} \cdot \triangle T + \ldots = \mathfrak{A} + P\,\mathfrak{A}_1 \cdot \triangle T + \ldots$$

is, *modulo* $(\triangle T)^2$, determined by

$$\begin{cases} \mathfrak{A}(T + \triangle T) \cdot \mathfrak{A}(T) = \cos \triangle \Psi = 1 \\ \mathfrak{A}(T + \triangle T) \cdot \mathfrak{A}_1(T) = \sin \triangle \Psi = \triangle \Psi = P \cdot \triangle T \\ \mathfrak{A}(T + \triangle T) \cdot \mathfrak{A}_2(T) = 0 \end{cases}$$

The infinitesimal dual angle between two infinitesimally near lines is therefore in the usual notation

$$d\Psi = P\,dT. \quad \ldots \ldots \ldots \quad (10)$$

$S = \int P\,dT = S(T)$, defined but for a constant, is called *the dual arclength of the D-system.* The equation $S = S(T)$ can be solved for T, if the D-system is nondegenerate [4]) ($dS/dT = P$ is not a zerodivisor). Non-

---

[4]) Proof: $S = S(T) = s + \varepsilon \bar{s} = \int p(t)dt + \varepsilon\,[\int \bar{p}(t)dt + \bar{t}\,p(t)] + K$ ((3)).

$$\therefore \left| \frac{\partial (s, \bar{s})}{\partial (t, \bar{t})} \right| = (p(t))^2 \neq 0.$$

Hence t, $\bar{t}$ can be solved as functions of s, $\bar{s}$; and dT/dS exists because $dT/dS = P^{-1}$.

degenerate D-systems can therefore be represented with the help of this *invariant dual parameter* S. The equations (9) simplify with the invariant parameter to

$$\left.\begin{aligned} \dot{\mathfrak{A}} &= \mathfrak{A}_1 \\ \dot{\mathfrak{A}}_1 &= -\mathfrak{A} + \operatorname{cotg}\Phi\cdot\mathfrak{A}_2 \qquad \operatorname{cotg}\Phi = (\mathfrak{A}\,\dot{\mathfrak{A}}\,\ddot{\mathfrak{A}}) \\ \dot{\mathfrak{A}}_2 &= -\operatorname{cotg}\Phi\cdot\mathfrak{A}_1 \end{aligned}\right\} \quad (11)$$

6. *The reguli in a D-system.*

A regulus, contained in a given nondegenerate D-system $\mathfrak{A} = \mathfrak{A}(T)$, resp. $\mathfrak{A} = \mathfrak{A}(S)$, is defined by a function

$$T = T(u) = t(u) + \varepsilon\,\bar{t}\,(u), \text{ resp. } S = S(u) = s(u) + \varepsilon\,\bar{s}\,(u).$$

The dual angle between two infinitely near lines of the regulus is

$$d\,\Psi = d\psi + \varepsilon d\bar{\psi} = P\frac{dT}{du}\,du = (p + \varepsilon\bar{p})\,(t_u + \varepsilon\,\bar{t}_u)\,du \quad \left(t_u = \frac{dt}{du}\right)$$

$$= [pt_u + \varepsilon\,(\bar{p}t_u + p\bar{t}_u)]\,du$$

$$= \frac{dS}{du}\,du = (s_u + \varepsilon\,\bar{s}_u)\,du$$

The *parameter of distribution of the regulus* is (BLASCHKE [3])

$$\delta = \frac{d\bar{\psi}}{d\psi} = \frac{\bar{p}t_u + p\bar{t}_u}{pt_u} = \frac{\bar{s}_u}{s_u} = \frac{d\bar{s}}{ds}. \quad \ldots\ldots \quad (12)$$

The *cylinders* in the D-system $\mathfrak{A}(T)$, resp. $\mathfrak{A}(S)$, obey $t_u = s_u = 0$. The rules of these reguli are the lines $\mathfrak{A}(t + \varepsilon\,\bar{t})$ (t fixed) mutually parallel, and contained in the plane $\mathfrak{A}(t) — \mathfrak{A}_2(t)$. The other peculiar reguli in the D-system are the *torsi* generated by the *tangents to a curve.* They can also be developed over the plane and have a parameter of distribution $\delta = 0$:

$$\bar{p}t_u + p\bar{t}_u = \bar{s}_u = 0 \quad , \quad \bar{s} \text{ constant.}$$

The reguli in the D-system with a constant parameter of distribution are given by linear equations in s and $\bar{s}$: $a \cdot s + b \cdot \bar{s} + c = 0$.

A regulus, not a cylinder, contained in a given D-system $\mathfrak{A}(S)$, can also be defined by $\bar{s} = \bar{s}\,(s)$. This function again is determined by $\delta = \delta(s)$ and $\bar{s}\,(0)$ ((12)):

$$\bar{s} = \int_0^s \delta\,(s)\,ds + \bar{s}\,(0).$$

Theorem 3. *A regulus, not cylinder, in a given D-system with invariant parameter,* $\mathfrak{A}(S)$, *is determined by one rule* ($\mathfrak{A}(S)$, $S = 0\ (+\ \varepsilon\,\bar{s}(0))$) *and the parameter of distribution* $\delta(s)$, *function of the invariant parameter of direction* s (s is the arclength on a unitsphere, on which the directions of the rules of the regulus may be represented).

7. *Geometrical classification of non-degenerate D-systems.*

$\mathfrak{A}(S)$ be a given non-degenerate D-system with invariant parameter S. Then is $\mathfrak{A}(s) = \mathfrak{A}(s + \varepsilon \cdot 0)$ a *torsus*, i.e. a regulus generated by tangents to a curve (§ 6).

$$\mathfrak{A}_2(S) = \mathfrak{A}_2(s) + \varepsilon \bar{s}\, \dot{\mathfrak{A}}_2(s) = \mathfrak{A}_2(s) - \varepsilon \bar{s} \operatorname{cotg} \Phi \cdot \mathfrak{A}_1(s) \qquad ((11))$$

(s fixed, $\bar{s}$ variable) are lines that intersect the line $\mathfrak{A}(s)$, and are perpendicular to the constant (is asymtotic!) tangent plane along the rule $\mathfrak{A}(s)$ of the torsus. $\mathfrak{A}_2(S)$ (S variable) consists of the normals to this torsus.

In case $\mathfrak{A}_2(S)$ is not degenerated, the relation between $\mathfrak{A}(S)$ and $\mathfrak{A}_2(S)$ is symmetric ((9) (11)), and hence also $\mathfrak{A}(S)$ consists of the normals to a torsus.

When $\mathfrak{A}_2(S)$ is degenerated, but not completely, then is $\mathfrak{A}_2(S)$ a cylinder. $\mathfrak{A}(S)$ intersects $\mathfrak{A}_2(S)$, and also the lines in first order neighbourhood of $\mathfrak{A}_2(S)$, perpendicular ((11)). $\mathfrak{A}(S)$ consists of the perpendicular tangents of a cylinder.

When $\mathfrak{A}_2(S)$ is completely degenerated, consists $\mathfrak{A}(S)$ of the perpendiculars to the line $\mathfrak{A}_2 = \mathfrak{A}_2(S)$.

Theorem 4. *A non-degenerate D-system consists of a) the normals to a torsus, b) the perpendicular tangents of a cylinder, or c) the normals to a line. A degenerate D-system is a cylinder. A completely-degenerated D-system is a line.*

(Compare STUDY [1] p. 305).

If a regulus has a related D-system on which a) of theorem 4 is applicable, then a torsus as mentioned can be constructed as follows: $(\mathfrak{A}\, \mathfrak{A}_1\, \mathfrak{A}_2\, X_u)$ be the invariant (Blaschke-)system of axes at the line $t = u$ of the regulus $\mathfrak{A}(t) \cdot x(t)$ be an orthogonal trajectory of the rules of $\mathfrak{A}(t) \cdot \mathfrak{B}(u)$ is the line parallel to $\mathfrak{A}_2(u)$ and passing through the point $x(u)$. Then is $\mathfrak{B}(t)$ the required torsus.

8. *The momentanous axes of a D-system.*

In the theory of D-systems we can very well use the terminology of cinematics. We call the dual parameter in $\mathfrak{A}(T)$: time. The moments of time are ordered by (4). $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2) = (\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)\,(T)$ is a Motion. The momentanous axis of this Motion and hence of the D-system is defined by

$$\mathfrak{S} = \Lambda\,\mathfrak{A} + \Lambda_1\,\mathfrak{A}_1 + \Lambda_2\,\mathfrak{A}_2\,,\ \dot{\mathfrak{S}} = \dot{\Lambda} = \dot{\Lambda}_1 = \dot{\Lambda}_2 = 0\,,\ \mathfrak{S}^2 = 1, \Lambda_2 > 0. \quad (13)$$

With (9) follows

$$\Lambda_1 P = \Lambda_1 Q = \Lambda P - \Lambda_2 Q = 0.$$

These equations have a unique solution if $P^2 + Q^2 \neq 0$ ($P^2 + Q^2$ is never a zerodivisor different from zero; $p > 0$)

$$\mathfrak{S} = \frac{Q\,\mathfrak{A} + P\,\mathfrak{A}_2}{V(P^2 + Q^2)} = \cos \Phi \cdot \mathfrak{A} + \sin \Phi \cdot \mathfrak{A}_2. \quad . \quad . \quad . \quad (13)$$

Φ, also in (11), is the dual angle between a line of the D-system and its momentanous axis: $\mathfrak{S}\,\mathfrak{A} = \cos \Phi$.
The system of momentanous axes of a D-system is another D-system, eventually degenerated. The dual arclength of $\mathfrak{S}(T)$ is obtained from (13)

$$\dot{\mathfrak{S}} = \dot{\Phi}\,(-\sin \Phi \cdot \mathfrak{A} + \cos \Phi \cdot \mathfrak{A}_2) = \dot{\Phi} \cdot \mathfrak{S}_1.$$

$$\text{Dual arclength of } \mathfrak{S}(T) = \int \dot{\Phi}\, dT = \Phi(+\text{ a dual constant})! \quad . \quad (14)$$

In words

Theorem 5. *The dual angle Φ between $\mathfrak{A}(T)$ and its momentanous axis $\mathfrak{S}(T)$ is dual arclength of the D-system of momentanous axes $\mathfrak{S}(T)$. $\mathfrak{A}(T)$ is a "developed D-system", developed from the "enveloped D-system" $\mathfrak{S}(T)$.*

It is easily seen that if $\mathfrak{A}(T)$ and $\mathfrak{B}(T)$ are developed from the same non-degenerate D-system $\mathfrak{S}(T)$, then is (compare KUIPER [7] ch. 3)

$$\mathfrak{S}_2 = -\mathfrak{A}_1 = -\mathfrak{B}_1$$
$$\mathfrak{B}(T) = \cos K \cdot \mathfrak{A}(T) + \sin K \cdot \mathfrak{A}_2(T)$$
$$\Phi_b(T) - \Phi_a(T) = K \text{ (dual constant).}$$

The dual angle between $\mathfrak{A}(T)$ and $\mathfrak{B}(T)$ is constant: K.
(13) and (14) are also valid when $\mathfrak{A}(T)$ is degenerated but $q \neq 0$:

$$\mathfrak{S} = \frac{Q\mathfrak{A} + \varepsilon \bar{p}\mathfrak{A}_2}{Q} = \mathfrak{A} + \varepsilon \frac{\bar{p}}{q}\mathfrak{A}_2$$
$$\sin \Phi = \sin \varphi + \varepsilon \bar{\varphi} \cos \varphi = \varepsilon \bar{p}/q$$
$$\Phi = 0 + \varepsilon \bar{p}/q, \ \varphi = 0, \ \bar{\varphi} = \bar{p}/q.$$

The momentanous axis at a line where a D-system is a cylinder, is parallel to that line. In particular:

Theorem 6. *The enveloped D-system of a cylinder is another cylinder. $\bar{p}/q$ is the radius of curvature of an orthogonal trajectory of the rules of the first cylinder.*

To get the definition and computation of *the dual velocity of a Motion* $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T)$, we consider a line perpendicular to the momentanous axis $\mathfrak{S}(T)$, namely $\mathfrak{A}_1(T)$. We consider the dual angle $\triangle \chi$ between $\mathfrak{A}_1(T)$ and $\mathfrak{A}_1(T + \triangle T)$. *Modulo* $(\triangle T)^2$ we have (see (9))

$$\mathfrak{A}_1(T + \triangle T) \cdot \mathfrak{S} = [\mathfrak{A}_1 + (-P\mathfrak{A} + Q\mathfrak{A}_2)\triangle T] \cdot \frac{Q\mathfrak{A} + P\mathfrak{A}_2}{V(P^2 + Q^2)} = 0$$
$$\mathfrak{A}_1(T + \triangle T) \cdot \mathfrak{A}_1 = 1 = \cos \triangle\chi$$
$$\mathfrak{A}_1(T + \triangle T) \cdot (\mathfrak{S} \times \mathfrak{A}_1) = \mathfrak{A}_1(T + \triangle T) \cdot \frac{-P\mathfrak{A} + Q\mathfrak{A}_2}{V(P^2 + Q^2)} = \sqrt{P^2 + Q^2}\,\triangle T$$
$$= \sin \triangle\chi = \triangle\chi.$$

The dual velocity is

$$\frac{d\chi}{dT} = V(P^2 + Q^2) > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The dual velocity is positive with respect to the momentanous axis $\mathfrak{S}$. It has the orientation of the rotation over the smallest angle which sends $\mathfrak{A}_1$ into $\mathfrak{S} \times \mathfrak{A}_1$; $\mathfrak{A}_1 \times (\mathfrak{S} \times \mathfrak{A}_1) = \mathfrak{S}$.

From (13) we get an important interpretation of the dual motion of the first invariant orthogonal system of a D-system $\mathfrak{A}(T)$ with non-degenerate enveloped D-system $\mathfrak{S}(T) = \mathfrak{S}(\Phi(T))$. $\mathfrak{S}(\Phi)$ be therefore considered as a D-system in an *immobile* space. The system of lines

$$\mathfrak{R}(\chi) = \cos\chi \cdot \mathfrak{A} + \sin\chi \cdot \mathfrak{A}_2, \; \chi \text{ variable},$$

is a "geodesic D-system" (§ 11) in a *mobile* space attached to the orthogonal system $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T)$.

The dual motion $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T)$ *is then obtained by "developing the mobile D-system* $\mathfrak{R}$ *along the immobile D-system* $\mathfrak{S}$*", i.e. such that for any* Φ, $\mathfrak{S}(\Phi)$, $\mathfrak{S}_1(\Phi)$ *coincides with* $\mathfrak{R}(\Phi)$, $\mathfrak{R}_1(\Phi)$.

Remark. A *general* dual motion, represented by

$$\mathfrak{A}(T),\ \mathfrak{B}(T),\ \mathfrak{C}(T),\ \mathfrak{A}^2 = \mathfrak{B}^2 = \mathfrak{C}^2 = 1,\ \mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{C} = \mathfrak{C}\mathfrak{A} = 0,$$

is obtained by development of a D-system $\mathfrak{S}^*(\Phi)$ in a mobile space (here $\mathfrak{S}^*(\Phi)$ need not be geodesic!) along another D-system $\mathfrak{S}(\Phi)$ in an immobile space.

The results of this § can easily be specialised to reguli; namely by restriction of the dual parameters to values $T = T(u)$, $S = S(u)$, $\Phi = \Phi(u)$. $\mathfrak{A}(T(u))$ for example be a regulus in the D-system $\mathfrak{A}(T)$ with non-degenerate enveloped D-system $\mathfrak{S}(T)$. The motion of the invariant orthogonal system of Blaschke $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T(u))$ (theorem 1), can be obtained by gliding development of the regulus

$$\mathfrak{R}(\Phi(u)) = \cos\Phi(u) \cdot \mathfrak{A} + \sin\Phi(u) \cdot \mathfrak{A}_2$$

in the mobile space attached to $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(T(u))$, along the regulus $\mathfrak{S}(\Phi(u))$ in the immobile space, such that at the moments u, $(\mathfrak{S}, \mathfrak{S}_1)(\Phi(u))$ coincides with $(\mathfrak{R}, \mathfrak{R}_1)(\Phi(u))$. Notice that the reguli $\mathfrak{S}(\Phi(u))$ and $\mathfrak{R}(\Phi(u))$ have the same parameter of distribution

$$\delta = \frac{d\overline{\varphi}}{d\varphi} = \frac{d\overline{\varphi}}{du} \bigg/ \frac{d\varphi}{du}.$$

The "enveloped regulus" of a regulus and the dual velocity of the invariant orthogonal system of Blaschke of a regulus were studied by BIRAN [6 b, c]. Analogues of the formulas and constructions of Euler–Savary were studied by DISTELI [2], GARNIER [5], VAN HAASTEREN [8], BIRAN [6 d]. Compare § 13.

(To be continued.)