Mechanics. — *Damped oscillations of a spherical mass of an elastic fluid.* By J. M. BURGERS. (Mededeling No. 59 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

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1. *Introduction.* — Professor BUNGENBERG DE JONG had asked me if a theoretical treatment could be given which would throw light on the results of his investigations on the oscillatory movements presented by certain soap solutions [1]). Since the most striking features of his beautiful experiments are the regularity of the observed oscillations and the geometrical pattern of the motion, it is natural that one should turn to the theoretical work of LAMB on the oscillations of a viscous spheroid and on the vibrations of an elastic sphere [2]). It is the object of the following lines to consider in which way the theory developed by LAMB can be applied in a discussion of BUNGENBERG DE JONG's experimental results, and to get information concerning the phenomena responsible for the damping of the motion.

Although it is not possible to extend LAMB's classical invèstigations, it may be of help to the reader to substitute for his highly mathematical and rather abstract deductions a more simple and direct treatment, adapted to the particular cases investigated by BUNGENBERG DE JONG. These cases are:

a) motion in concentric spherical layers or shells;
b) axially symmetric motions in meridian planes.

It will be assumed that the amplitude of the oscillations is small, so that velocities and accelerations can be calculated by means of the partial derivatives with respect to the time. Hence if u is any component of displacement, the corresponding velocity will be $\partial u/\partial t$, the acceleration $\partial^2 u/\partial t^2$, where these two quantities refer to the same point of space as does u itself.

A few general remarks may precede the deduction of the equations.

The fact that isochronous oscillations are obtained, proves that elastic forces are operative which are linear functions of the deformations.

The general relation between the dimensions of the field of motion, that

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[1]) H. G. BUNGENBERG DE JONG, Elastic-viscous oleate systems, containing KCl (Part I), these Proceedings **51**, 1197—1210 (1948). Parts II and III will appear in the next issues.

[2]) H. LAMB, On the oscillations of a viscous spheroid, Proc. London Mathem. Soc., **13**, 51 (1881); On the vibrations of an elastic sphere, ibidem **13**, 189 (1882); On the motion of a viscous liquid contained in a spherical vessel, ibidem **16**, 27 (1884); Hydrodynamics (Cambridge 1932), art. **354** (p. 637) and **356** (p. 642). The elastic vibrations of a sphere are also treated in A. E. H. LOVE's Theory of Elasticity (Cambridge 1920), Ch. XII (p. 281).

is in the case considered the radius R of the spherical vessel, and the period T of the oscillation, can be deduced from the following argument. If all linear dimensions of a given field of motion are changed in the same ratio, so that angles and angular displacements remain unaltered, the magnitude of the deformations will remain unaltered likewise; consequently the elastic stresses per unit area will remain the same. The resulting moment, e.g. over a spherical surface, changes proportionally with R^3. As the moment of inertia of a spherical mass is proportional with R^5, it follows that the angular accelerations produced by the elastic reactions will change proportionally with R^{-2}. On the other hand, as angular displacements are not changed, angular accelerations must be proportional with T^{-2}. Hence we must conclude that the period of the elastic oscillation will be proportional to the radius R, as was found in the experiments.

If we keep to the case of oscillations which are not heavily damped, it will be evident that when viscous forces are present, depending on the rate of deformation, the viscous stresses per unit area will be proportional with T^{-1}. The moment of the frictional stresses will be proportional to $R^3\,T^{-1}$; and the angular accelerations or decelerations produced by them will be proportional to $R^{-2}\,T^{-1}$, that is, to R^{-3}. Hence with increasing radius R the decelerations due to viscosity will decrease in comparison with the accelerations due to the elastic reactions, and it follows that the damping per period will decrease with increasing R. This has the consequence that the logarithmic decrement Λ of the oscillations will be proportional to R^{-1}.

Damping can also be due to relaxation of the elastic stresses. This phenomenon is characterised by a constant of the nature of a time, the relaxation time, which is a property of the fluid and does not depend on the period of the motion or the dimensions of the field. It follows that the effects produced by relaxation will increase with the period of the motion, and it is found that in such a case the logarithmic decrement becomes proportional to T, that is, to R.

Damping finally can be a consequence of slipping of the fluid (more accurately: of the elastic system in the fluid) along the wall of the vessel. As the angular displacements are supposed to be the same, the linear displacements are proportional with R and the velocity of slipping will be proportional to $R\,T^{-1}$. If we suppose that the frictional force per unit area called into play by the slipping is proportional to the latter quantity, its moment will be proportional with $R^4\,T^{-1}$; the angular deceleration produced by it will be proportional with $R^{-1}\,T^{-1}$, that is, with R^{-2}. Hence in this case the deceleration will be proportional to the elastic accelerations and it is found that the logarithmic decrement becomes independent of R.

As will be seen from BUNGENBERG DE JONG's account of his observations, a logarithmic decrement independent of the radius of the vessel has been found in the case of certain dilute soap solutions, while more concentrated solutions show a decrement proportional to the radius.

2. *Motion in concentric spherical shells.* — In order to show that a motion in concentric spherical layers, performing rotational oscillations about a common axis, is possible, we denote the angular displacement of a particular layer by $\phi(r, t)$. The shear along a parallel circle at the angular distance θ from the pole of the axis is then given by:

$$\gamma = r \sin\theta \cdot \partial\phi/\partial r \quad \quad (1)$$

The shearing stress τ, acting across a spherical layer, in the direction of the parallel circles, will be a function of γ. When ordinary elastic behaviour is present we must take:

$$\tau = G\gamma \quad \quad (2a)$$

where G is the shear modulus. In the case where the elastic reaction is accompanied by viscous friction we may take:

$$\tau = G\gamma + \eta\,(\partial\gamma/\partial t) \quad \quad (2b)$$

with $\eta =$ viscosity. If, instead of viscosity, relaxation of the elastic stresses makes itself felt, we must write the relation between τ and γ in the form [3]:

$$\partial\tau/\partial t = G\,(\partial\gamma/\partial t) - \tau/\lambda \quad \quad (2c)$$

where λ is the relaxation time.

In the case of harmonic, or damped harmonic, oscillations we can write:

$$\phi = e^{\nu t} \cdot \Phi(r) \quad \quad (3)$$

with ν imaginary or complex. In that case: $\partial\phi/\partial t = \nu\phi$, from which $\partial\gamma/\partial t = \nu\gamma$. Equations $(2a)$—$(2c)$ can then be brought into the general form:

$$\tau = L\gamma \quad \quad (4)$$

with either:

$$L = G \quad \quad (5a)$$

or:

$$L = G + \nu\eta \quad \quad (5b)$$

or:

$$L = G\,(1 + 1/\nu\lambda)^{-1} \cong G\,(1 - 1/\nu\lambda) \quad \quad (5c)$$

corresponding to the three cases represented by $(2a)$, $(2b)$, $(2c)$ respectively. The expression for L can also be adapted to more complicated cases; L will always be an algebraic function of ν, independent of r.

Introduction of (1) into (4) gives:

$$\tau = L\,r \sin\vartheta \cdot \partial\phi/\partial r \quad \quad (6)$$

We now consider the motion of a ring-shaped mass of fluid, contained between two concentric spherical shells with radii r and $r + dr$, and two

[3]) Compare *e.g.* "First Report on Viscosity and Plasticity", Verhand. Kon. Ned. Akad. v. Wetensch., Amsterdam (1e sectie) vol. **15**, no. 3, p. 18 (1939).

conical surfaces with semi-angles θ and $\theta + d\theta$ (compare fig. 1). The

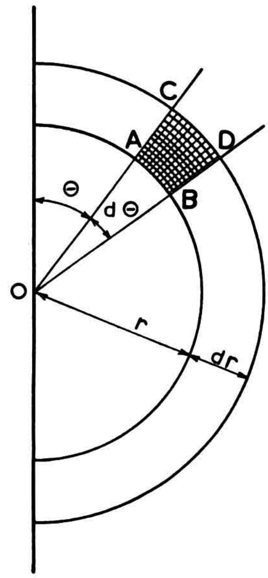

Fig. 1.

resulting moment of the shear stress acting on the exterior and interior surfaces (described by the arcs AB and CD) of this ring is:

$$\partial\,(2\pi r^3 \sin^2\theta \cdot \tau)/\partial r \cdot dr\, d\theta.$$

As the moment of inertia of the ring amounts to: $2\pi r^4 \sin^3\theta \cdot \varrho\, dr\, d\theta$, where ϱ is the density of the fluid, its equation of motion takes the form:

$$2\pi r^4 \sin^3\theta \cdot \varrho\,(\partial^2 \Phi/\partial t^2) = \partial\,(2\pi r^3 \sin^2\theta \cdot \tau)/\partial r.$$

When use is made of (3) and eq. (6) is substituted for τ, it will be seen that both the exponential factor and the factor $2\pi \sin^3\theta$ drop out, so that we obtain:

$$\varrho r^4 \nu^2 \Phi = L\, d\,(r^4\, d\Phi/dr)/dr. \quad \ldots\ldots \quad (7)$$

The fact that this equation does not contain the angle θ proves that the angular displacement Φ of each spherical shell can be independent of the polar distance, so that each shell can move as a whole.

We write:

$$a^2 = -\varrho \nu^2/L \quad \ldots\ldots\ldots \quad (8)$$

which makes it possible to bring (7) into the form:

$$\frac{d}{dr}\left(r^4 \frac{d\Phi}{dr}\right) + a^2 r^4 \Phi = 0. \quad \ldots\ldots\ldots \quad (9)$$

The solutions of this equation have been given by LAMB; with the omission of an arbitrary constant factor the solution applicable to the present case (field extending to $r = 0$) is:

$$\Phi = \frac{\sin ar}{(ar)^3} - \frac{\cos ar}{(ar)^2} \quad \ldots \ldots \ldots \quad (10)$$

We assume that the wall of the spherical vessel (radius R) is at rest. When there is no slipping at the boundary, the function Φ must vanish for $r = R$. When slipping is possible, the angular displacement at the wall will be given by ϕ, so that the linear displacement of the fluid relative to the wall is equal to $\phi R \sin \theta$ and the velocity of slipping will be given by: $(\partial \phi / \partial t)\, R \sin \theta = \nu \phi\, R \sin \theta$. Introducing a friction coefficient $\varkappa$, we assume the relation:

$$(\tau)_R = -\varkappa \nu \phi R \sin \theta. \quad \ldots \ldots \ldots \quad (11)$$

Making use of (6) we find:

$$L\,(d\Phi/dr)_{r=R} + \varkappa \nu\, \Phi(R) = 0. \quad \ldots \ldots \ldots \quad (12)$$

Inserting the expression (10) for Φ and writing, for shortness, $aR = \zeta$, this equation can be transformed into:

$$\operatorname{tg} \zeta - \zeta = -(L/\varkappa \nu R)\,(\zeta^2 \operatorname{tg} \zeta - 3 \operatorname{tg} \zeta + 3\zeta). \quad \ldots \quad (13)$$

We suppose that $\varkappa$ is large, so that slipping will be no more than a small disturbing effect. The solution of (13) will then differ only slightly from the solution of $\operatorname{tg} \zeta = \zeta$, the first root of which is 4,493. This first root corresponds to the most simple type of motion, which was observed by BUNGENBERG DE JONG in his "rotational oscillations". We shall distinguish the values of ζ, a, ν and T for this type of motion by the subscript $_0$. We then write $\zeta_0 = 4{,}493 + \triangle\zeta_0$ on the left hand side of (13) and $\zeta_0 = 4{,}493$ on the right hand side which has the large value of $\varkappa$ in the denominator; this gives:

$$\triangle \zeta_0 = -L\zeta_0/\varkappa \nu_0 R. \quad \ldots \ldots \ldots \quad (13a)$$

With the aid of this result we obtain the following expression for the root of (13):

$$a_0 R = \zeta_0 = 4{,}493\,(1 - L/\varkappa \nu_0 R). \quad \ldots \ldots \quad (14)$$

Now equation (8) gives us:

$$\frac{\varrho \nu_0^2}{L} = -a_0^2 = -\frac{(4{,}49)^2}{R^2}\left(1 - \frac{L}{\varkappa \nu_0 R}\right)^2. \quad \ldots \ldots \quad (15)$$

Three cases will be considered.

(I) *Damping through viscous forces* ($\lambda = \infty$; $\varkappa = \infty$), in which case L is given by (5*b*). Equation (15) becomes:

$$\nu_0^2 = -\frac{(4{,}49)^2}{R^2}\,\frac{G}{\varrho}\left(1 + \frac{\nu_0 \eta}{G}\right).$$

from which the following approximate value of ν_0 is obtained:

$$\nu_0 = i\,\frac{4{,}49}{R}\sqrt{\frac{G}{\varrho}} - \frac{(4{,}49)^2\,\eta}{2R^2\varrho}.$$

The period T_0 and the logarithmic decrement Λ_0 become:

$$T_0 = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}}\,;\quad \Lambda_0 = \frac{4{,}49\,\pi\eta}{R\sqrt{G\varrho}}. \quad \dots\dots \quad (16)$$

(II) *Damping through relaxation* ($\eta = 0$; $\varkappa = \infty$), in which case L is given by (5c). Equation (15) becomes:

$$\nu_0^2 = -\frac{(4{,}49)^2}{R^2}\,\frac{G}{\varrho}\left(1 - \frac{1}{\nu_0\lambda}\right),$$

giving the approximate solution:

$$\nu_0 = i\,\frac{4{,}49}{R}\sqrt{\frac{G}{\varrho}} - \frac{1}{2\lambda}.$$

The period T_0 and the logarithmic decrement Λ_0 become [4]):

$$T_0 = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}}\,;\quad \Lambda_0 = \frac{\pi R}{4{,}49\,\lambda}\sqrt{\frac{\varrho}{G}}. \quad \dots\dots \quad (17)$$

(III) *Damping through slipping.* We take $\eta = 0$; $\lambda = \infty$ but retain $\varkappa$, and use eq. (5*a*) for L. In this case eq. (15) takes the form:

$$\nu_0^2 = -\frac{(4{,}49)^2}{R^2}\,\frac{G}{\varrho}\left(1 - \frac{G}{\varkappa\nu_0 R}\right)^2,$$

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[4]) The accurate equation for the calculation of ν_0 in the case of damping through relaxation has the form:

$$\nu_0^2 + \frac{\nu_0}{\lambda} + \frac{(4{,}49)^2\,G}{R^2\varrho} = 0.$$

Its roots are:

$$\nu_0 = \pm\, i\sqrt{\frac{(4{,}49)^2\,G}{R^2\varrho} - \frac{1}{4\lambda^2}} - \frac{1}{2\lambda},$$

from which:

$$T_{\mathrm{obs}} = 2\pi\left\{\frac{(4{,}49)^2\,G^2}{R^2\varrho} - \frac{1}{4\lambda^2}\right\}^{-1/2};\ \Lambda_{\mathrm{obs}} = T_{\mathrm{obs}}/2\lambda.$$

If we write:

$$T_{\mathrm{corr}} = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}},$$

the following relation exists between T_{corr} and T_{obs}:

$$T_{\mathrm{corr}} = T_{\mathrm{obs}}\left\{1 + (\Lambda_{\mathrm{obs}}/2\pi)^2\right\}^{-1/2}.$$

This formula has been applied by BUNGENBERG DE JONG in those cases where a more accurate determination of the shear modulus G is desired.

giving the approximate value:

$$\nu_0 = i\,\frac{4{,}49}{R}\sqrt{\frac{G}{\varrho}} - \frac{G}{\varkappa R}.$$

The period T_0 and the logarithmic decrement Λ_0 now become [5]):

$$T_0 = \frac{2\pi R}{4{,}49}\sqrt{\frac{\varrho}{G}}\;;\quad \Lambda_0 = \frac{2\pi}{4{,}49}\,\frac{\sqrt{G\varrho}}{\varkappa}\quad \ldots\ldots \quad (18)$$

3. *Motion in meridian planes.* — The discussion of this case is less simple than that of the former one. However, as the fluid can be considered as incompressible (the shear modulus proves to be very much lower than the modulus of compressibility can be expected to be), we may take the equations of motion of an elastic body in the form [6]):

$$\varrho\,(\partial^2 \omega_i/\partial t^2) = L \,\triangle\, \omega_i \quad \ldots\ldots\ldots \quad (19)$$

where the ω_i are the components of the rotation (not of the rotational velocity, or vorticity, which is given by $\partial\omega_i/\partial t$) of an element of volume of the fluid, with respect to rectangular coordinates; $\triangle$ is the Laplacian; and L is the same quantity as in (4) and (5*a*)—(5*c*).

Motion in meridian places can be described with the aid of a function $\psi(r, \theta, t)$, analogous to STOKES' stream function used in hydrodynamical problems with axial symmetry. The components of the linear displacement, defined with respect to spherical polar coordinates, are given by:

$$u_r = \frac{1}{r^2 \sin\theta}\,\frac{\partial\psi}{\partial\theta}\;;\quad u_\theta = \frac{-1}{r\sin\theta}\,\frac{\partial\psi}{\partial r}\quad \ldots\ldots \quad (20)$$

and the only component of rotation (rotation in the meridian plane about an axis perpendicular to that plane, that is, tangential to a parallel circle) becomes:

$$\omega = \frac{1}{r}\left\{\frac{\partial}{\partial r}(r\,u_\theta) - \frac{\partial u_r}{\partial\theta}\right\} = -\frac{1}{r\sin\theta}\left\{\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\,\frac{\partial\psi}{\partial\theta}\right)\right\}. \quad (21)$$

We assume:

$$\omega = e^{\nu t}\cdot \Omega\,(r, \theta). \quad \ldots\ldots\ldots \quad (22)$$

5) The deduction of more accurate formulae for this case leads to certain complications which require careful inspection. It is hoped to come back to this point in connection with the IIIrd part of BUNGENBERG DE JONG's paper (to be published in one of the following issues of these Proceedings).

6) This form corresponds to that used in hydrodynamics for a number of problems of slow motion, when terms of the second degree in the velocities can be neglected. The hydrodynamical equations for that case are obtained from (19) if L is replaced by $\eta\,(\partial/\partial t)$.

For several of the formulae used in the text the reader may be referred to the analogous hydrodynamical equations as given in S. GOLDSTEIN, Modern Developments of Fluid Dynamics (Oxford 1938), vol. I, pp. 103—105 and 114—115.

In applying eq. (19) to the system of spherical polar coordinates used here, we must keep in mind that the ω defined by (21) is directed along the tangent to a parallel circle and thus has a different meaning from the rectangular components ω_i used in (19). Whereas in the case of a scalar quantity ω the Laplacian would be given by:

$$\triangle\,\omega = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\omega}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\omega}{\partial\theta}\right),$$

we must now add to this expression the amount $-\omega/r^2\sin^2\theta$, corresponding to the derivative of the second order taken in a fixed direction normal to a particular meridian plane, of the component of ω along that normal [7]. With this addition eq. (19) takes the form:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Omega}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right) - \frac{\Omega}{\sin^2\theta} + \alpha^2 r^2\,\Omega = 0 \quad . \qquad (23)$$

where α^2 is the same quantity as defined by (8).

We look for solutions of the type: $\Omega = h(r)\cdot k(\theta)$. Introducing a number m to be determined afterwards, it is found that the function $k(\theta)$ must satisfy the equation:

$$k'' + k'\cot\theta - k/\sin^2\theta + m\,k = 0.$$

If we put: $k = dl/d\theta$, where $l(\theta)$ is another function of θ, we find that l must be a solution of the equation of the Legendre functions:

$$l'' + l'\cot\theta + m\,l = 0.$$

The constant m must have one of the values $n(n+1)$, n being an integer, if we desire solutions which are regular in the domain $0 \leq \theta \leq \pi$ with the endpoints included. The cases corresponding to BUNGENBERG DE JONG's "meridional oscillations" and "quadrantal oscillations" respectively, are obtained with:

$$n=1 \;;\; m=2 \;:\; l_1 = \cos\theta \;;\; k_1 = -\sin\theta$$
$$n=2 \;;\; m=6 \;:\; l_2 = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2} \;;\; k_2 = -3\sin\theta\cos\theta.$$

The corresponding equation for the function h becomes:

$$r^2 h'' + 2\,r\,h' + (\alpha^2 r^2 - m)h = 0,$$

and its solutions for the cases mentioned are:

$$n=1 \;;\; m=2 \;:\; h_1 = \frac{\sin\alpha r}{(\alpha r)^2} - \frac{\cos\alpha r}{\alpha r}$$

$$n=2 \;;\; m=6 \;:\; h_2 = \frac{(3-\alpha^2 r^2)\sin\alpha r}{(\alpha r)^3} - \frac{3\cos\alpha r}{(\alpha r)^2}.$$

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[7]) When the meridian plane is taken as x,-y-plane, we can write $\omega_z = \omega\cos\varphi$ (φ being the position angle of a point, measured from this plane). We then have:

$$\triangle\,\omega_z = \cos\varphi\cdot\triangle\,\omega - \frac{1}{r^2\sin^2\theta}\left(2\sin\varphi\frac{\partial\omega}{\partial\varphi} + \omega\cos\varphi\right).$$

which reduces to the terms given in the text when φ is taken zero.

We must now find ψ from (21). To this end we write:

$$\psi = e^{\nu t} \cdot \Psi(r, \theta) = e^{\nu t} H(r) K(\theta). \quad \ldots \ldots \quad (24)$$

After some calculations the following expressions are obtained:

$$\left. \begin{array}{lll} n=1: & K_1 = \sin^2\theta & ; \quad H_1 = -a r h_1 + C_1 (a r)^2 \\ n=2: & K_2 = 3\sin^2\theta \cos\theta; & \quad H_2 = -a r h_2 + C_2 (a r)^3 \end{array} \right\} \quad . \quad . \quad (25)$$

where C_1 and C_2 are integration constants.

The following boundary conditions must be observed. In the first place the radial velocity of the fluid must be zero at the wall of the spherical vessel; this requires Ψ to become a constant for $r = R$, which necessitates that $H(R)$ shall be zero. This condition fixes the values of the constants C_1, C_2.

The velocity of slipping along the wall is then given by:

$$\left(\frac{\partial u_\theta}{\partial t}\right)_{r=R} = -\left(\frac{\nu}{r \sin\theta} \frac{\partial \psi}{\partial r}\right)_{r=R},$$

and the equivalent of equation (11) takes the form:

$$(\tau_{r\theta})_{r=R} = -\varkappa \left(\frac{\partial u_\theta}{\partial t}\right)_{r=R}. \quad \ldots \ldots \ldots \quad (26)$$

The shearing stress $\tau_{r\theta}$ appearing in this equation is given by the formula:

$$\tau_{r\theta} = G \left\{ r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right\} = G \left\{ -\frac{r}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^3}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right) \right\}.$$

When the expression (24) is inserted for ψ and attention is given to the fact that the function $H(r)$ vanishes for $r = R$, the following equation is obtained, which takes the place of eq. (12) in the case of section **2**:

$$\frac{d^2 H}{d(ar)^2} - \frac{2}{ar}\frac{dH}{d(ar)} + \frac{\varkappa \nu R}{L}\frac{1}{ar}\frac{dH}{d(ar)} = 0 \text{ (for } r = R). \quad . \quad . \quad (27)$$

The case of no slipping is obtained by making $\varkappa$ infinite, in which case the condition becomes: $dH/d(ar) = 0$. This gives:

for $n = 1$ (meridional oscillation) $\zeta_1 = 5{,}76$
for $n = 2$ (quadrantal oscillation) $\zeta_2 = 6{,}99$.

In the case of a finite (but large) value of $\varkappa$ we write: $\zeta_1 = 5{,}76 + \Delta\zeta_1$; $\zeta_2 = 6{,}99 + \Delta\zeta_2$. It is found that for both values of n the correction is given by the expression: $\Delta\zeta = -L\zeta/\varkappa\nu R$, so that the roots of eq. (27) become:

$$\left. \begin{array}{l} \text{for } n=1 \ : \ a_1 R = \zeta_1 = 5{,}76\,(1 - L/\varkappa \nu_1 R) \\ \text{for } n=2 \ : \ a_2 R = \zeta_2 = 6{,}99\,(1 - L/\varkappa \nu_2 R) \end{array} \right\}. \quad . \quad . \quad (28)$$

We can now calculate the values of ν_1 and ν_2 and the corresponding periods and logarithmic decrements, in the same way as was done at the end of section **2**. It will be seen that the only difference is the substitution of the numerical factor 5,76 (for the meridional oscillation) or 6,99 (for the quadrantal oscillation) in the place of the factor 4,49. It follows that the period of the oscillation is decreased in such a way that:

$$T_{\text{rot}}/T_{\text{mer}} = \alpha_1/\alpha_0 = 1{,}282$$
$$T_{\text{rot}}/T_{\text{quadr}} = \alpha_2/\alpha_0 = 1{,}556.$$

Having regard to (17) and (18) it is further seen that both in the case of damping through relaxation and in the case of damping through slipping the logarithmic decrement changes in the same ratio as the period. The way in which the decrement depends on the radius is not changed when we pass from the rotational oscillations to the meridional or the quadrantal oscillations.

4. *Magnitude of the shear stress.* — Numerical data concerning the shear modulus G, the relaxation time λ and the coefficient of friction $\varkappa$ operative in slipping will be given in BUNGENBERG DE JONG's papers.

It may be of interest to have an estimate of the magnitude of the elastic stresses active in the system. This can easily be obtained for the case of the rotational oscillation. The angular displacement is given by the formula:

$$\Phi = A \frac{\sin \alpha r - \alpha r \cos \alpha r}{(\alpha r)^3} e^{-t/2\lambda} \cos \frac{2\pi t}{T},$$

A being a coefficient determining the amplitude. Leaving aside the time factors, the linear displacement at $\theta = 90^\circ$ is determined by:

$$r\Phi = \frac{A}{\alpha} \frac{\sin \alpha r - \alpha r \cos \alpha r}{(\alpha r)^2}.$$

The maximum of this expression is found in the neighbourhood of $r = \frac{1}{2} R$, that is $\alpha r = 2{,}25$, giving $0{,}433\ A/\alpha = 0{,}096\, A\, R$. Hence if we write a for the maximum deviation actually observed, we shall have: $A = 10{,}4\ a/R$.

The maximum value of the shearing stress is found at the wall of the vessel, at $\theta = 90^\circ$. Equation (2a) gives (when the time factor is again left aside):

$$\tau_{\max} = GR \frac{d}{dr} \left\{ A \frac{\sin \alpha r - \alpha r \cos \alpha r}{(\alpha r)^3} \right\}_{r=R} = GA \frac{\sin \alpha R}{\alpha R} = 0{,}217\ GA.$$

With the value of A given above there results: $\tau_{\max} \cong 2{,}25\ Ga/R$.

According to a footnote to BUNGENBERG DE JONG's second paper, the deviation from the equilibrium position at the moment the determination of the damping ratio was started, amounted to ca. 3 mm, but larger deviations had been observed before that instant. If we choose $a = 5$ mm $= 0{,}5$ cm in a vessel of 7,5 cm radius, we find:

$$\tau_{\max} \cong 0{,}15\ G.$$

Résumé.

Afin d'obtenir des formules qui puissent élucider les résultats obtenus par BUNGENBERG DE JONG dans ses expériences sur les oscillations élastiques présentées par certaines solutions d'oléates (v. l'article précédent), il a été donné dans l'article ci-dessus une déduction directe de certaines formules de LAMB pour les oscillations d'un fluide élastique, contenu dans un réservoir sphérique. En même temps on a calculé le degré d'amortissement, provoqué soit par une résistance visqueuse, soit par rélaxation des tensions élastiques, soit par un glissement du fluide le long du paroi du réservoir.

On trouve que la période des oscillations est toujours proportionelle au rayon du réservoir. D'autre part le décrément logarithmique dans les trois cas se comporte différemment: le décrément est inversement proportionnel au rayon dans le cas d'une résistance visqueuse, directement proportionnel au rayon dans le cas de rélaxation, et indépendant du rayon dans le cas de glissement. Le deuxième cas est présenté par les résultats obtenus par BUNGENBERG DE JONG avec des solutions d'oléates à concentration supérieure à ca. 1,1 %, le troisième pour des solutions à concentration inférieure à ca. 0,9 %.

Les calculs ont trait aux trois formes d'oscillations observées: rotationelles, méridionales et quadrantales.

Resumo.

Por trovi formulojn kiuj povos klarigi la rezultojn de la esploroj de BUNGENBERG DE JONG rilate al la elastaj osciloj kiujn montras iuj solvaĵoj de oleatoj (vidu la antaŭan artikolon), oni donas en la ĉi-supra artikolo rektan derivadon de kelkaj formuloj de LAMB por la osciloj de elasta fluidaĵo entenata en sfera vazo. Samtempe oni kalkulas la gradon de amortizo, kaŭzitan ĉu per viskozeca rezisto, ĉu per perdo de elastaj streĉoj, ĉu per glito de la fluidaĵo laŭ la pario de la vazo.

Oni trovas ke la periodo de la osciloj ĉiam estas rekte proporcia al la radio de la vazo. Kontraŭe, la logaritma dekremento kondutas diference en la tri kazoj: la dekremento estas inverse proporcia al la radio en la kazo de viskozeca rezisto, rekte proporcia al la radio en la kazo de streĉoperdo, kaj nedependa de la radio en la kazo de glito. La duan kazon prezentas la rezultoj akiritaj de BUNGENBERG DE JONG kun solvaĵoj de oleatoj de koncentriteco pli ol cirkaŭ 1,1 % a, la trian tiuj kun solvaĵoj de koncentriteco malpli ol ĉirkaŭ 0,9 % a.

La kalkuloj rilatas al la tri observitaj formoj de osciloj: rotaciaj, meridianaj kaj kvadrantaj.